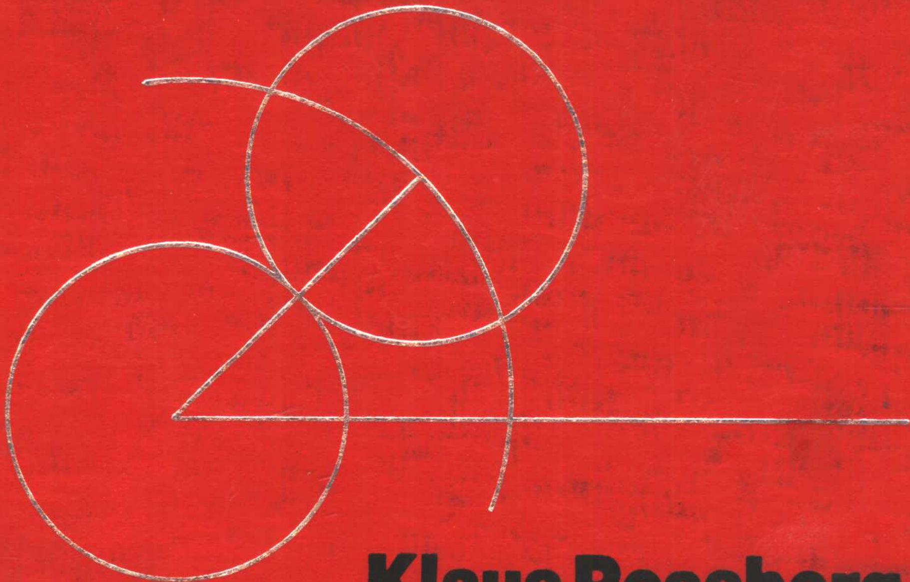

Klaus Rossberg

A FIRST COURSE IN ANALYTICAL MECHANICS

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preface

Many good books on analytical mechanics have been written, and most of them are still available. Therefore, the publication of another book on this subject must be well justified.

The plan for writing this book grew out of the search for a textbook that would be an introduction to analytical mechanics on the sophomore or junior level, would cover the essentials of mechanics without being too lengthy, and would have a well-balanced treatment of the foundations of mechanics and of important examples. Moreover, the book should present the necessary background material for other advanced physics courses, especially for modern physics and quantum mechanics.

Although these conditions impose strong limitations on the scope and the content of the book, they still could be satisfied in several different ways. The material selected for this book and its presentation reflect my personal views. Other physicists might include additional material and/or delete certain sections depending on their philosophies, interests, and teaching experiences.

The difficult problem of the selection of the material from the enormously large field of classical mechanics had to be attacked from both ends: Which topics are essential for the understanding of all of physics, and which areas of mechanics can be omitted from a first course in analytical mechanics without losing the broad overview? Concerning the size of the book, the second question was of greater importance. Because most sophomores have little working knowledge of partial differential equations, it was easy to decide against the inclusion of fluid dynamics, potential theory, and wave propagations. Topics that are mathematically too involved, such as the three body problem, were also omitted. Chapters on noninertial frames and Hamilton's principle were not included because these topics usually are discussed in the second semester of a sequence of mechanics courses. There remained the controversial issue of the treatment of special relativity. I decided against the inclusion of this field because a survey shows that most textbooks on introduction to physics, modern physics, and electromagnetic fields cover special relativity in approximately thirty pages. In my opinion, special relativity should be presented to physics majors in a separate one-semester course because of its importance and its broad range of applicability outside mechanics.

Returning to the first question, namely, what to include in this book, I decided to present first some background material in mathematics (Chapter 2) needed for the precise formulation of mechanics. This material is intended as a quick reference, and it should not be discussed at great length in the lecture for reasons of time. Kinematics (Chapter 3) is the logical starting point in mechanics. Dynamics (Chapter 4) is most naturally approached in Newton's formalism. Oscillations (Chapter 5) and central forces (Chapter 6) provide important examples of broad applicability, many of which are exactly solvable. Many particle systems (Chapter 7) are included because all systems consist of more than one particle in reality. Lagrange's and Hamilton's formalisms (Chapter 8) are important for the foundations of advanced mechanics and other areas of physics, for example, statistical mechanics, geometrical optics, quantum mechanics, and quantum field theory. It would be nice to introduce these formalisms at an earlier stage, but students need to have some working knowledge of Newtonian dynamics before they are exposed to more general ideas of mechanics. Although the chapter on rigid body mechanics (Chapter 9) should logically be placed right after many particle systems, some equations of rigid body mechanics are more easily derived in Lagrange's formalism, which justifies the placement of this chapter. The last chapter (Chapter 10) on coupled oscillations (which are also best treated in Lagrange's formalism) provides the basis for the transition to the theory of wave motion.

What is presented in this book is still more than what can be covered efficiently in a one-semester course. It is left to the instructor to make additional cuts.

Appendix 1 contains a list of some vector relations. The other appendixes contain material that usually is not found in elementary texts but that I considered appropriate here.

The selected problems are all of a moderate degree of difficulty, and the excessive use of mathematics has been avoided. Too much mathematics at the beginner's level may obscure the ideas of physics or frighten the students. Some questions concerning the philosophy of science and the foundations of mechanics have been included. Physics majors should not merely know that $ma = F$, they should also have some understanding of how the building of mechanics is constructed.

Analytical mechanics is logically and traditionally the first course in the sequence of theoretical physics courses because basic concepts and principles are introduced that are absolutely essential for the understanding of the other areas of physics. It is a good sign that many students are interested in the exciting frontiers of physics, but these students are advised to take the studies of the fundamentals of mechanics very seriously. Only then is it possible to fully understand the modern developments of physics that have their ultimate roots in classical mechanics. Wherever possible, I have mentioned in the text the points of departure of modern physics from classical mechanics and their relationships.

I have tried to present all material in a clear form and to avoid ambiguities, jumps, and statements such as "... it can easily be seen that..." It is my opinion, however, that a textbook should not contain every detail of a calculation or a derivation. Spoon feeding does not help a student master his or her field of study. All physics and mathematics textbooks should not merely be read but studied with pencil and paper.

Klaus Rossberg

acknowledgments

Many people have given me their support, encouragement, and constructive criticism during the long years of preparing this book. I have learned from them in many ways, professionally and personally, and I express my gratitude to them all.

First, I thank my students who were exposed to the earlier versions of the manuscript in my mechanics courses and who continually reminded me that a textbook should be written for the students and not for the experts. I thank especially Karen Mares, Mary Tyner, and Stewart Meyer.

I thank Professors Somdev Tyagi (Drexel University), Jerry P. Draayer (Louisiana State University), Lester M. Clendenning (Humboldt State University), David B. Tribble (Loyola University of Chicago), David A. Lind (University of Colorado), Leon Blitzer (University of Arizona), and graduate student R. S. Robinson (University of Arizona), who read parts or all of the book at various stages. They made numerous helpful suggestions and also pointed out several errors. Special thanks are extended to Professors John A. Wheeler (University of Texas) and Barry N. Taylor (National Bureau of Standards) for their replies to specific questions.

It is my pleasure to express my sincerest and warmest thanks to my colleagues and friends, Professors Leonard Flansburg and Ronald Olowin, with whom I discussed details of the book almost daily, and to Professor Kurt Just (University of Arizona), who carefully examined the whole manuscript. His countless suggestions and corrections were invaluable, especially his clarifying remarks on the foundation of Newton's laws and on inertial frames. He also suggested several problems that are included in this book.

I extend my thanks to the members of the editing, designing, and production staff of John Wiley for their excellent work. I enjoyed the cordial atmosphere with the many persons I had contact with, directly or via long distance.

Finally, I thank all of my children for their great understanding of my time-consuming work and for their unselfish patience. Their love has given me the strength to finish this book.

K. R.

Oklahoma City
February 1983

contents

The more alternatives,
the more difficult the choice.

ABBE D'ALLAINVAL

chapter 1 Introduction

The begynnynge of every thynge is the hardiste.

SIXTEENTH CENTURY ENGLISH PROVERB

1.1 GENERAL CONSIDERATIONS ON PHYSICS AND MECHANICS

Since their emergence, humans have tried to explore and understand nature. Prehistoric humans sought to satisfy their basic need for survival; later, they strove to satisfy their curiosity and their philosophical needs. Part of this human endeavor is natural science.

Physics is the basis of natural science. It deals with the laws of matter over a wide range of its forms—from elementary particles and atoms to macroscopic objects, planetary systems, and the whole of our universe. It does not come as a surprise that the laws of nature are different for the many forms of matter, but it is one of the physicist's strong beliefs that a unifying principle exists which would allow him or her to derive from it the laws for all forms of matter. In spite of the progress made in recent years a complete unifying theory has not yet been found.

Physics, like all branches of natural science, is founded upon observations and experiments. A system of principles and relationships constructed to explain the observed phenomena is called a hypothesis. As a result of creative inductive reasoning, it is merely an unproven guess. A hypothesis becomes a scientific theory if it has the ability to predict new phenomena. A scientific theory is true, by definition, when it is consistently corroborated by experimental facts. It cannot be *proven* true, but it can be proven wrong. Agreement between conclusions derived from a theory and experimental confirmation can only lend support to the theory. If the conclusions disagree with an experiment, then the theory must be abandoned or amended.[1] Examples of the creation of new, enlarged theories are numerous in physics. Two of the most striking examples are the development of the theory of relativity and that of quantum mechanics.

Theoretical physics deals with the mapping of natural processes and properties of matter on a set of mathematical relations among well-defined quantities. The language of mathematics is used because it is more precise than any other

[1]For a short but comprehensive discussion of the philosophy of science and the foundations of physics see P. J. Brancazio, *The Nature of Physics*, Macmillan, New York, 1975.

known language and also because it allows us to express even complicated relationships in a useful shorthand notation. The mathematical frame consists of two parts, namely, the set of all definitions of the physical quantities, whether observable or not, and the set of fundamental laws and axioms from which other equations and rules can be derived. Mathematical simplicity is one of the guidelines for developing a physical theory. Simplicity is most clearly expressed in the forms of conservation principles (e.g., energy, momentum), symmetry principles[2] (e.g., the behavior of systems under certain space-time transformations), and variational principles (extremum principles). Early in this century, it became clear that the three types of principles are related to each other. The application of various symmetry principles has led to insights in elementary particle physics where dynamical laws have not yet been found.

Physics may be called an exact science in the sense that correct mathematical reasoning allows us to derive not only qualitative statements but also quantitative statements from the set of assumed fundamental laws formulated in mathematical language. This is in contrast to common sense which may be directly applied to some natural phenomena to derive qualitative statements only. But common sense may play tricks. Seemingly "obvious" assumptions can lead to conclusions that are not in agreement with the observed facts. (See Questions 1.4, 1.5, and 1.6.)

Physical quantities must be well defined, that is, they must have one and only one meaning. A definition is the reduction of a concept to be defined to other previously defined concepts. But the process of defining cannot be continued indefinitely. Some fundamental (or primitive) quantities, notions, or concepts must be accepted a priori, even in physics. For example, the concepts of space, time, and matter cannot be defined. The fundamental concepts may seem fuzzy at first, but they become clear when used repeatedly. They may also become subject to revision; for example, the notion of a flat space-time continuum turned out to be inappropriate in general relativity.

Much of our knowledge of nature is expressed in terms of the observable quantities mass, position, momentum, energy, and so on. Observable quantities are defined operationally by a measuring device, that is, by a prescription of how the quantity shall be measured. Classical physics deals directly with these observable quantities, and it is assumed that they can be measured in principle with unlimited precision. However, every attempt to apply the methods of classical mechanics to the atom failed.[3] With the work of Heisenberg, Schroedinger, and other pioneers of modern quantum mechanics it became clear that the observable quantities do not directly describe microscopic systems, and that certain combinations of observable quantities cannot simultaneously be measured with unlimited accuracy. Also, some quantities needed to be introduced that were not observables at all. Although the realm of applicability of classical physics is limited, quantities such as mass, position, momentum, and energy also have a meaning in quantum mechanics. This is because all measuring devices from which we obtain our information of nature are macroscopic systems that are subject to the laws of

[2]See Section 6.3.

[3]Max Born in a letter to Albert Einstein (October 21, 1921): "Die Quanten sind eine hoffnungslose Schweinerei." (The quanta are a hopeless mess.)

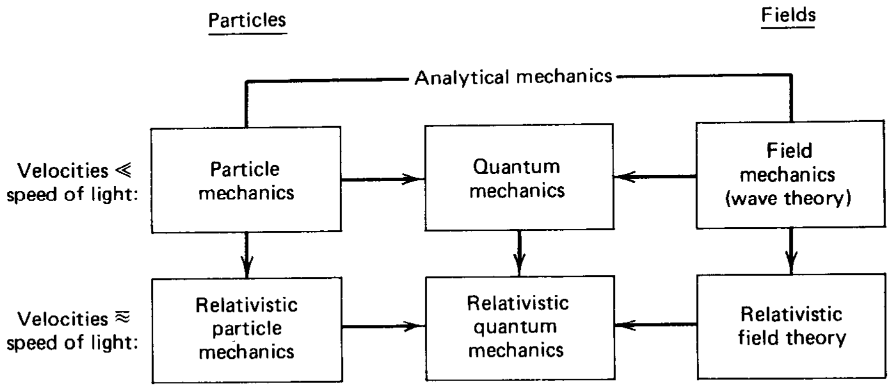

Figure 1.1 Diagram showing the relations between analytical mechanics, quantum mechanics, and relativity.

classical physics. The examples cited should alert us not to take all physical concepts for granted or to apply them uncritically to all areas of physics.

Analytical mechanics is logically and traditionally the first course in the series of theoretical physics. The name originates from Lagrange's book *Mechanique Analytique*. Following Newton's *Principia*, Lagrange derived analytically from Newton's axioms of motion the main parts of the theory of mechanics. Analytical mechanics encompasses both particle mechanics and wave theory. The relationship of analytical mechanics to other branches of physics is shown in Figure 1.1.

1.2 REFERENCE FRAMES AND COORDINATE SYSTEMS

Three important concepts need to be introduced. A physical system is a distribution of matter on which observations are made and experiments are performed. A reference frame is any part of the universe relative to which observations of a physical system are made. The existence of matter and/or radiation is necessary for a meaningful definition of the reference frame. A coordinate system is any mentally constructed system attached to a reference frame relative to which the physical system under investigation is described. Usually that particular coordinate system is chosen which yields the simplest possible description of the system and/or exhibits the largest possible degree of symmetry of the physical system. Observers in two different reference frames may or may not measure the observable quantities of a physical system with the same outcome. Even the physical laws (e.g., force laws) may not be identical, which may be the case if at least one reference frame is not inertial. (For the definition of inertial frame see Section 4.2.) However, two different coordinate systems, which are attached to the same reference frame, may still yield different descriptions of the physical system, but the physical laws must be the same in these coordinate systems because the laws of nature cannot be altered by a construction of the human mind. One often uses the term *coordinate system* very loosely. Actually, it means a "reference frame provided with a coordinate system." The reader is asked to distinguish the term *coordinate frame* from the term *reference frame* in order to avoid possible confusion.

1.3 STANDARD UNITS OF MECHANICS

All quantities in mechanics are based upon the three fundamental quantities: length, time interval, and mass. The standard international units of these quantities are the meter, second, and kilogram, respectively, defined as follows:

1 meter = 1 m = 1,650,763.73 times the wavelength of light emitted from the krypton isotope Kr^{86} in the transition from the state $2p_{10}$ to the state $5d_5$.[4]

1 second = 1 s = 9,129,631,770 times the period of oscillation between the two hyperfine levels of the ground state of the caesium isotope Cs^{133}.

1 kilogram = 1 kg = the mass of a platinum-iridium prototype cylinder which is kept at the International Bureau of Weights and Measures in Sevres, France.

Practically, it is sufficient to define the standard of length by the standard meter which is also kept in Sevres, and to define the second as 1/86 400 of the mean solar day.

1.4 ON STUDYING PHYSICS

It may be appropriate here to take a closer look at the reasons why many students experience difficulties when studying physics, especially during the first semesters. One of the difficulties is directly related to the act of learning itself. Here, learning is understood as the conscious process of acquiring knowledge or skills with the purpose that the acquired knowledge may be actively reproduced and applied when necessary. Pure memorization of equations or vocabularies is only a form of passive reproduction. A person is able to actively reproduce if the object under study has been viewed, investigated, and analyzed under several possible angles. Learning, therefore, is a process that involves time; it does not occur in an instant. All phases that a person goes through during a creative process[5] (e.g., painting a picture) from the first vague idea to the completion of his or her work also appear during the process of learning. In order to acquire certain technical skills or to understand ideas or concepts, two seemingly contradicting requirements are essential—hard work, at times even accompanied by intense feelings of frustration, followed by complete relaxation. If the facts of the learning process are well understood, the active engagement in learning becomes more enjoyable and the outcome of studying more gratifying and rewarding.

Other difficulties originate at the three steps involved in the mapping process of the phenomena and the laws of nature on the language of mathematics, and vice versa. The mapping process is outlined in Figure 1.2.

[4]The Comite Consultatif pour la Definition du Metre of the Comite International des Poids e Mesures is currently discussing the exact wording of a redefinition of the meter as *the distance traveled by light in 1/299,792,458 of a second.*

[5]See Rollo May, *The Courage to Create*, W. W. Norton & Company, Inc., New York, 1975.

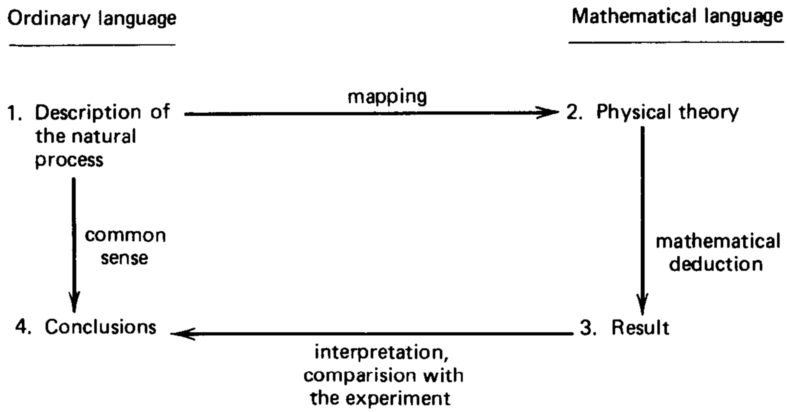

Figure 1.2 Diagram showing the path of the thought process in theoretical physics.

Step 1 to 2 is a creative process of the human mind. This is certainly true if a new theory is developed. It is also true to a lesser degree if the theory is recreated (learned), or if a set of equations needs to be found to solve an assigned homework problem. Because a creative process requires insight and intuition (which are products of the mind of the individual), a general prescription as to how to establish the mapping from nature to the physical theory (or parts thereof) cannot be given. J. A. Wheeler's "first moral principle"[6] expresses these ideas in a humorous way: "Never make a calculation until you know the answer."

Mathematical difficulties that may appear in step 2 to 3 are usually overcome as the student progresses in mathematics.

Step 3 to 4 is the comparison of the derived mathematical result with the observed data. The translation from the "foreign" mathematical language to ordinary language is usually much simpler than the inverse step 1 to 2.

The finding of the mapping 1 to 2 can be facilitated if one heeds the following suggested study hints:

1. The definitions of physical quantities should not merely be memorized mechanically. It is essential that the student understand all implications and ramifications of the defined quantity. This very important advice is usually underestimated or entirely ignored by students.

2. The student should learn to distinguish between the physical and mathematical aspects of a physical problem. The progress from known equations to needed ones is sometimes determined by physical arguments; sometimes it is purely a mathematical derivation.

3. The student should learn to approach any physical problem in an economical way, namely, from the general to the specific. It is not necessary to remember a vast amount of equations and relations applicable to special cases only. The clearer understanding of a physical theory would be negatively affected by too much unnecessary ballast.

[6] E. F. Taylor, J. A. Wheeler, *Space-Time Physics*, W. H. Freeman & Co., San Francisco, 1966.

4. All fundamental laws are simple because they express certain symmetries of nature. (The physical theory becomes more and more complicated when the laws are applied to larger and more complex systems, because the degree of symmetries decreases.) Usually, the fundamental laws are easily learned. The student needs to see however *how* these laws are applied to special cases. Problem solving, therefore, is a necessity. It helps the student master all steps from 1 to 4. The sentence "I really do understand the material—but I just cannot solve the problems" is self-contradictory.

QUESTIONS

1.1 The *modus tollens* (way of reasoning) in logic is an argument of the form:

If *H* is true, then so is *I*.
As evidence shows, *I* is not true.
Therefore, *H* is not true.

The *fallacy of affirming the consequent* is an invalid argument of the form:

If *H* is true, then so is *I*.
As evidence shows, *I* is true.
Therefore, *H* is true.

Discuss the importance of both arguments for the foundation of a scientific theory.

1.2 Comment on "Physics is too difficult for the physicist." (David Hilbert)

1.3 Ponder on "Science, like art, is not a copy of nature but a re-creation of her"[7] and on "A physical theory, being an amalgam of inventions, definitions, and laws, is regarded as a *model* for a certain part of nature, asserting not so much what nature *is*, but rather what it *is like*."[8]

For the following "common sense" questions state your answer within 5 seconds. Do not use force diagrams.

1.4 A rope is hanging over a pulley. A person attaches himself to one end of the rope and pulls on the other side to move himself up. Clearly, the force on the pulley is equal to the weight of the person (if the weights of the rope and of the pulley are neglected). The person now fastens the other end of the rope to the wall. Does the force on the pulley remain the same?

1.5 In 1654, Otto von Guericke, inventor of the air pump, demonstrated the existence of air pressure by evacuating two brass hemispheres and having two teams of eight horses pull on each side of the hemispheres. Assume that the air pressure inside the sphere is low enough that the hemispheres could just be pulled apart by the 16 horses. Assume now that the same air pressure exists as

[7] J. Bronowski, *Science and Human Values*, Harper & Row, New York, 1965.
[8] W. Rindler: *Essential Relativity*, Van Nostrand-Reinhold, New York, 1969.

before, but that now one of the hemispheres is fastened to a rigid wall. Then, eight horses cannot pull the hemispheres apart. True or false?

1.6 A wound up yo-yo is attached to a scale. The scale records the weight of the yo-yo. If this is released to unwind itself downward, the scale records a force (tension) that is smaller than the weight. Common sense would predict this result also. After passing through the lowest point the yo-yo is moving upward. Does the scale now indicate a force (tension) which is smaller than, equal to, or larger than the weight?