Progress in Mathematics 4

Edited by
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S. Helgason

Complex Approximation

Proceedings, Quebec, Canada
July 3-8, 1978

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To John Wermer

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INTRODUCTION. This book contains the texts of the lectures given by the invited lecturers at the Conference on Complex Approximation held at Quebec on July 3-8, 1978. It contains also supplementary papers resulting from discussion which took place during this meeting.

The three main subjects were: approximation in $\mathbb{C}^n$ and function algebras, analytic and harmonic approximation in $\mathbb{C}$, exponential approximation and approximation in L^p.

We received substantial financial help from the National Research Council of Canada and the Ministry of Education of the Province of Quebec, and from Laval University.

In the name of all participants we sincerely thank these institutions for having made this successful meeting possible.

Bernard Aupetit

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INVITED TALKS

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ON THE AREA OF THE SPECTRUM OF AN ELEMENT OF A UNIFORM ALGEBRA

by

H. Alexander

INTRODUCTION. In classical function theory, the area of the image of a holomorphic function was usually computed *with multiplicity*. In [5], Alexander, Taylor and Ullman obtained an estimate for the area, *without multiplicity*, of the image of a function holomorphic in the unit disc. This had applications to function theory. Here we shall discuss an area theorem in the context of uniform algebras where an estimate for the planar area of the spectrum of an element of the uniform algebra will be obtained. The proof which we shall give will depend on a quantitative version of the classical Hartogs-Rosenthal theorem on rational approximation in the complex plane. Applied to certain polynomial algebras, this "area theorem" yields properties of analytic subvarieties of $\mathbb{C}^n$. Hartogs' theorem, the separate analyticity implies analyticity, and an analogous result of Nishino, that separate normality implies normality, are consequences.

To fix some notation, $C(X)$ will denote the Banach algebra of all continuous complex-valued functions on a compact Hausdorff space X, normed with the supremum norm. When X is a compact subset of $\mathbb{C}^n$, $C(X)$ has subalgebras $P(X)$ and $R(X)$ which are the closures in $C(X)$ of the polynomials (in the coordinates) and the rational functions, holomorphic on a neighborhood of X, respectively. The maximal ideal space of $P(X)$ can be identified with the polynomially convex hull $\hat{X}(\equiv\{p\in\mathbb{C}^n : |f(p)| \le |f|_X$ for every polynomial f$\})$ of X. Thus, for a polynomial f, the plane set

$f(\hat{X})$ is the spectrum of f, considered as an element of the Banach algebra P(X). It is this set whose area we shall estimate. For the fundamentals on uniform algebras and pòlynomial convexity we refer to the books of Stout [11] ans Wermer [12].

AREA THEOREM. First recall the classical area formula. Let f be holomorphic in the open unit disc U with $f(0) = 0$ and Taylor series $f(z) = \sum_{n=1}^{\infty} a_n z^n$. Then, as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$, the Jacobian determinant of f is $|f'|^2$ and so the planar area of the image of f, *counting multiplicity*, is given by

$$\int_U |f'|^2 \, dxdy = \pi \sum_{n=1}^{\infty} |a_n|^2 . \tag{1}$$

the integral being easily evaluated in polar coordinates. Now suppose that f has L^2 boundary values, also denoted by f; i.e., take f to be in the Hardy space H^2.. Then, for the measure $dm = \frac{1}{2\pi} d\theta$ on the unit circle T,

$$\int_T |f|^2 dm = \sum_{n=1}^{\infty} |a_n|^2 . \tag{2}$$

The obvious estimate in (1) and (2) gives

$$\text{area, with mult., of } f(U) \geq \pi \int_T |f|^2 dm. \tag{3}$$

EXAMPLE: Take $f(z) = z^5$. On the left side of (3) we have 5π, the area, *with multiplicity*, of the image of f, while the integral on the "right side of (3) equals π, which, in fact, is the area of the image of f *without* counting multiplicity.

The estimate (3) will be generalized as follows:

$$\text{area, } \textit{without} \text{ mult., of } f(U) \geq \pi \int_T |f|^2 dm \tag{4}$$

Observe that in the above example, (4) becomes an equality. The estimate (4) was obtained by Alexander, Taylor and Ullman [5]. Here we shall give a general version which is valid for uniform algebras; the following proof (see [2] and [3]) is not the original one of [5] .

Let A be a uniform algebra with maximal ideal space M, let $x \in M$, and let σ be a (positive) representing for x supported on M. (In applications, we usually take σ to live on the Shilov boundary.) Planar Lebesque measure will be denoted be λ below. We can now state the generalization to the setting of uniform algebras of (4) which itself is the special case of A the disc algebra, x the origin and σ the measure $\frac{1}{2\pi}d\theta$.

THEOREM 1 [3]. *Let $f \in A$ and $f(x) = 0$, then*

$$\lambda(f(M)) \geq \pi \int |f|^2 d\sigma \quad . \tag{5}$$

REMARK: Since this requires the functions to be continuous on the maximal ideal space, one can apply the theorem on discs of radius less than one and take a limit to get (4) form (5).

The proof will be based on the following quatitative form of the Hartogs-Rosenthal theorem.

THEOREM [2]. *Let K be a compact subset of the complex plane. Considering $\bar{z}$ as a function in C(K), one has the following estimate for the distance from $\bar{z}$ to the subset R(K) of C(K):*

$$dist\ (\bar{z}, R(K)) \leq \left(\frac{\lambda(K)}{\pi}\right)^{1/2} .$$

PROOF: Let ψ be a C^∞ function with compact support in the plane such that $\psi(z) \equiv \bar{z}$ on a neighborhood of K. By the generalized Cauchy integral formula,

$$\psi(z) = -\frac{1}{\pi}\iint \frac{\partial\psi}{\partial\bar{\zeta}} \frac{dudv}{\zeta - z}, \quad \zeta = u + iv \ ,$$

for all $z \in \mathbb{C}$. Restricting ψ to K and using $\partial\psi/\partial\bar{\zeta} \equiv 1$ on K we get

$$\bar{z} = -\frac{1}{\pi}\iint_K \frac{dudv}{\zeta - z} - \frac{1}{\pi}\iint_{C\setminus K} \frac{\partial\psi}{\partial\bar{\zeta}} \frac{dudv}{\zeta - z}$$

for $z \in K$. Since the second integral represents a function in R(K), we have

$$dist(\bar{z}, R(K)) \leq \sup_{z \in K} \left|\frac{1}{\pi}\iint_K \frac{dudv}{\zeta - z}\right| \tag{6}$$

By an elegant computation, Ahlfors and Beurling ([1] , p. 106-107) have shown that the right side of (6) is dominated by $(\lambda(K)/\pi)^{1/2}$. □

REMARK: If $\lambda(K) = 0$, Lemma 2 says that $\bar{z} \in R(K)$. Then, by the Stone-Weierstrass theorem, R(K) = C(K). This is the Hartogs-Rosenthal theorem.

PROOF OF THEOREM 1: Let $\epsilon > 0$ and put $K = f(M)$. Use the lemma to obtain a rational function r(z) with poles off K such that

$$|\bar{z}-r(z)|_K < \left(\frac{\lambda(K) + \epsilon}{\pi}\right)^{\frac{1}{2}}.$$

Since r is is holomorphic on a neighborhood of the spectrum K of f, it follows from the Gelfand theory that $g = r \circ f \epsilon A$ and $|\bar{f}-g|_M < [(\lambda(K)+\epsilon)/\pi]^{\frac{1}{2}}$. We have $|f|^2 = f(\bar{f}-g) + fg$. Since $g \epsilon A$, we get $\int fg d\sigma = f(x) \times g(x) = 0$ and so $\int |f|^2 d\sigma = \int f(f-g)d\sigma$. Thus $\int |f|^2 d\sigma \le |\bar{f}-g|_M \int |f| d\sigma \le [(\lambda(K)+\epsilon)/\pi]^{\frac{1}{2}} \int |f| d\sigma$. Now letting $\epsilon \to 0$ yields

$$\int |f|^2 d\sigma \le \left(\frac{\lambda(K)}{\pi}\right)^{\frac{1}{2}} \int |f| d\sigma \tag{7}$$

This is somewhat more general than (5) (and can be used to study the case of equality in (5)). An application of Hölder's inequality, $\int |f| d\sigma \le (\int |f|^2 d\sigma)^{\frac{1}{2}}$, in (7) gives (5).

Our first application is to the area of an analytic variety in $\mathbb{C}^n$ of complex dimension one -" a Riemann surface with singularities". 1-varieties have a naturel Lebesque area when viewed as real two-dimensional surfaces of $\mathbb{R}^{2n} = \mathbb{C}^n$. In fact, if we put $\omega = i/2 \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$, then for any analytic 1-variety V in C^n, we have

$$\text{area}(V) = \int_V \omega , \tag{8}$$

where the right side indicates integration of the real differential 2-form ω over the real oriented 2-dimensional surface V. The verification [7] of (8) amounts to two observations: (a) (8) is clearly valid if V is linear space and (b)ω is invariant under translations and unitary transformations. Now writing $\int_V \omega = \sum_{k=1}^{n} \frac{i}{2} \iint_V dz_k \wedge d\bar{z}_k$ and observing that $\frac{i}{2} dz \wedge d\bar{z} = dx \wedge dy$ for $z = x + iy$, we see that the k^{th} term in this sum is just the area of $z_k(V)$, counted with multiplicity, where $z_k(V)$ is the planar image of V under the k^{th} coordinate function z_k. Thus, for any 1-variety V in $\mathbb{C}^n$,

$$\text{area}(V) = \sum_{k=1}^{n} [\text{area, with mult., of } z_k(V)] \tag{9}$$

Now suppose that V is a subvariety of a ball of radius R in $\mathbb{C}^n$ which passes through the center of the ball which we take to be the origin. Then it is known[10] that the area of V is bounded below and that the extremal case occurs when V is a linear space; namely, area $(V) \geq \pi R^2$. In view of (9), the following estimate, first obtained in [5], generalizes this.

$$\sum_{k=1}^{n} [\text{area, } \textit{without} \text{ mult., of } z_k(V)] \geq \pi R^2. \tag{10}$$

We shall prove a version of this in the more general setting of polynomial hulls.

THEOREM 3 [2]: *Let Y be a compact subset of the sphere, centered at the origin, of radius R in* $\mathbb{C}^n$ *and put* $X = \hat{Y}$. *Suppose that X contains the origin. Then*

$$\sum_{k=1}^{n} \lambda(z_k(X)) \geq \pi R^2 \tag{11}$$

REMARK: To obtain (10), fix $r < R$ and let Y be the intersection of V with the sphere of radius r. Then X is the intersection of V with the closed ball of radius r. Now apply (11) and let $r \uparrow R$. In the same way, by invoking the local maximum modulus principle, one can improve (11) by replacing X with $X \setminus Y$.

PROOF: In Theorem 1 take A to be P(X) (with maximal ideal space X) and x to be the origin. Let σ be a representing measure for the origin $(0 \in X)$ which has its support in Y. Then, as $z_k \in P(X)$, we have

$$\lambda(z_k(X)) \geq \pi \int |z_k|^2 d\sigma.$$

Summing over k gives

$$\sum_{k=1}^{n} \lambda(z_k(X)) \geq \pi \int \{ \sum_{k=1}^{n} |z_k|^2 \} d\sigma.$$

But $\sum_{1}^{n} |z_k|^2 \equiv R^2$ on Y and so (11) follows. □

For our second application of (5) we shall consider a one-dimensional subvariety V of the unit polydisc U in $\mathbb{C}^2$. Let z and w be the coordinate functions in $\mathbb{C}^2$.

THEOREM 4. *Let* $p = (a,b)$ *be a fixed point of V. Suppose that there exists a* $\delta > 0$ *such that* $|w| > \delta$ *on V. Then*

$$\lambda(z(V)) \geq \pi \frac{Log \frac{1}{|b|} (1-|a|)^2}{log (\frac{1}{\delta})} \qquad (12)$$

REMARKS: While this estimate on the area of the z-projection of V is not usually best possible, asymptotically this is the case when $a = 0$ and $|b| \downarrow \delta$; for then both sides of (12) approach π. The "vertical" variety $z = a$ and its small pertubations, which have small projections into the z-plan, are ruled out by the condition $|w| > \delta$ on V.

Moreover, the strict positivity of δ is needed to force a lower bound on the area of z(V). Indeed, given $0 < b < 1$, one can construct a subvariety V of U^2 containing the point (0,b), on which $w \neq 0$, and such that z(V) has arbitrarily small area ϵ. To obtain V, let K be a compact subset of the unit disc U which is disjoint form the closed unit interval [0,1] and such that the area of U\K is less than ϵ. By Runge's theorem, there exists a polynomial g such that $|g-1| < 1/2$ on K and $g(0) = \text{Log } b$. Now put $f = \exp(g)$ and $V = \{(z,f(z)) : z \in U\} \cap U^2$. Then V is a 1-subvariety of U^2, $(0,b) \in V$, $w \neq 0$ on V, and $\lambda(z(V)) < \epsilon$, because $|f(z)| > \sqrt{e}$ for $z \in k$ implies $z(V) \subseteq U \backslash K$.

PROOF: Without loss of generality we may assume that V extends to be a subvariety of a neighborhood of the closed unit polydisc; this is because we can work with small dilations of the original variety and take a limit. This means that $\bar{V} \cap \bar{U}^2$ is the maximal ideal space of the algebra $A = P(\bar{V} \cap \bar{U}^2)$ and that the Shilov boundary Σ of A is contained in ∂U^2. Let σ be a Jensen measure supported on Σ which represents the point p for the algebra A. Then, since the coordinate function w is an invertible element of A, we have

$$\log\left(\frac{1}{|b|}\right) = \int \log\left(\frac{1}{|w|}\right) d\sigma = \int_{\{z=1\}} \log\left(\frac{1}{|w|}\right) d\sigma \qquad (13)$$

$$\leq \log\left(\frac{1}{\delta}\right) \cdot \sigma\{|z| = 1\}$$

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