

Birger Iversen

Cohomology of Sheaves



Birger Iversen

Cohomology of Sheaves

**Springer-Verlag
Berlin Heidelberg New York Tokyo**

Birger Iversen
Mathematisk Institut, Aarhus Universitet
Ny Munkegade, DK-8000 Aarhus C, Denmark

AMS MOS (1980) Classification numbers:
14C17, 18E30, 18F20, 18G35, 32A27, 55N30, 55U30, 57R20

ISBN 3-540-16389-1 Springer-Verlag Berlin Heidelberg New York Tokyo
ISBN 0-387-16389-1 Springer-Verlag New York Heidelberg Berlin Tokyo

Library of Congress Cataloging in Publication Data. Main entry under title:
Iversen, Birger. Cohomology of sheaves. (Universitext). Bibliography: p. Includes index.
1. Sheaves, Theory of. 2. Homology theory. I. Title. QA612.36.I93 1986 514'.224
86-3789

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically those of translation, reprinting, re-use of illustrations, broadcasting, reproduction by photocopying machine or similar means, and storage in data banks. Under § 1 of the German Copyright Law where copies are made for other than private use, a fee is payable to "Verwertungsgesellschaft Wort", Munich.

© Springer-Verlag Berlin Heidelberg 1986
Printed in Germany

Printing and bookbinding: Beltz Offsetdruck, Hemsbach, Bergstraße
2141/3140-543210

Introduction

This text exposes the basic features of cohomology of sheaves and its applications. The general theory of sheaves is very limited and no essential result is obtainable without turning to particular classes of topological spaces. The most satisfactory general class is that of locally compact spaces and it is the study of such spaces which occupies the central part of this text.

The fundamental concepts in the study of locally compact spaces is cohomology with compact support and a particular class of sheaves, the so-called soft sheaves. This class plays a double role as the basic vehicle for the internal theory and is the key to applications in analysis. The basic example of a soft sheaf is the sheaf of smooth functions on \mathbb{R}^n or more generally on any smooth manifold. A rather large effort has been made to demonstrate the relevance of sheaf theory in even the most elementary analysis. This process has been reversed in order to base the fundamental calculations in sheaf theory on elementary analysis.

The central theme of the text is Poincaré duality or rather its generalizations by Borel and Verdier. In its first form this appears as a duality between cohomology and cohomology with com-

compact support. A more general Poincaré duality theory is developed for a continuous map between locally compact spaces. The important special case of a closed imbedding admits generalization to arbitrary topological spaces and is best understood in the framework of local cohomology. This theory is used for construction of characteristic classes of all sorts: Chern classes, Stiefel-Whitney classes, ...

For further applications to algebraic topology, a homology theory is developed for locally compact spaces and proper maps. This allows one to express Poincaré duality as an isomorphism between homology and cohomology. Applications are given to the classical theory of topological manifolds: fundamental class, diagonal class, Lefschetz fixed point formula ...

This homology theory is particularly suited for the study of algebraic varieties and a detailed introduction to (co)homology classes of algebraic cycles is given, including a topological definition of the local intersection symbol. It is a rather remarkable feature that this homology theory more or less automatically grinds out algebraic cycles.

A word about homological algebra. The first chapter of the text gives an introduction to homological algebra sufficient for most of the text. The last chapter, or appendix if you wish, gives an introduction to derived categories used in the more advanced parts of the text and in the proofs of the basic cup product formulas. It is my hope that this will give some readers motivation for Verdier's rather difficult text (1) on triangulated categories.

It remains for me to thank W. Fulton and R. MacPherson for their encouragement to publish the text, to thank a number of colleagues, who read part of the manuscript, H.H. Andersen, J.P. Hansen, A. Kock, O. Kroll, O.A. Laudal and H.A. Nielsen and to thank Else Yndgaard for excellent typing and cooperation.

Aarhus, Denmark

September 1985

Birger Iversen

Contents

I. HOMOLOGICAL ALGEBRA

1. Exact categories	1
2. Homology of complexes	7
3. Additive categories	11
4. Homotopy theory of complexes	16
5. Abelian categories	34
6. Injective resolutions	40
7. Right derived functors	51
8. Composition products	60
9. Résumé of the projective case	64
10. Complexes of free abelian groups	68
11. Sign rules	71

II. SHEAF THEORY

0. Direct limits of abelian groups	74
1. Presheaves and sheaves	80
2. Localization	83
3. Cohomology of sheaves	91
4. Direct and inverse image of sheaves. f_*, f^*	96
5. Continuous maps and cohomology	100
6. Locally closed subspaces. $h_!, h^!$	106
7. Cup products	113
8. Tensor product of sheaves	118
9. Local cohomology	123
10. Cross products	130
11. Flat sheaves	140
12. $\text{Hom}(E, F)$	145

III. COHOMOLOGY WITH COMPACT SUPPORT

1. Locally compact spaces	146
2. Soft sheaves	149
3. Soft sheaves on \mathbb{R}^n	157
4. The exponential sequence	162
5. Cohomology of direct limits	173
6. Proper base change and proper homotopy	176
7. Locally closed subspaces	183
8. Cohomology of the n -sphere	187
9. Dimension of locally compact spaces	195
10. Wilder's finiteness theorem	200

IV. COHOMOLOGY AND ANALYSIS

1. Homotopy invariance of sheaf cohomology	202
2. Locally compact spaces, countable at infinity	206
3. Complex logarithms	210
4. Complex curve integrals. The monodromy theorem	216
5. The inhomogenous Cauchy-Riemann equations	228
6. Existence theorems for analytic functions	231
7. De Rham theorem	237
8. Relative cohomology	249
9. Classification of locally constant sheaves	250

V DUALITY WITH COEFFICIENT IN A FIELD

1. Sheaves of linear forms	254
2. Verdier duality	259
3. Orientation of topological manifolds	266
4. Submanifolds of \mathbb{R}^n of codimension 1	271
5. Duality for a subspace	276
6. Alexander duality	279
7. Residue theorem for $n-1$ forms on \mathbb{R}^n	284

VI. POINCARÉ DUALITY WITH GENERAL COEFFICIENTS

1. Verdier duality	289
2. The dualizing complex D'	294
3. Lefschetz duality	297
4. Algebraic duality	299
5. Universal coefficients	303
6. Alexander duality	307

VII. DIRECT IMAGE WITH PROPER SUPPORT

1. The functor $f_!$	313
2. The Künneth formula	319
3. Global form of Verdier duality	324
4. Covering spaces	327
5. Local form of Verdier duality	330

VIII. CHARACTERISTIC CLASSES

1. Local duality	332
2. Thom class	337
3. Oriented microbundles	340
4. Cohomology of real projective space	347
5. Stiefel-Whitney classes	352
6. Chern classes	358
7. Pontrjagin classes	372

IX. BOREL MOORE HOMOLOGY

1. Proper homotopy invariance	374
2. Restriction maps	377
3. Cap products	378
4. Poincaré duality	380
5. Cross products and the Künneth formula	382
6. Diagonal class of an oriented manifold	386
7. Gysin maps	390
8. Lefschetz fixed point formula	394
9. Wu's formula	397
10. Preservation of numbers	398
11. Trace maps in homology	399

X. APPLICATION TO ALGEBRAIC GEOMETRY

1. Dimension of algebraic varieties	400
2. The cohomology class of a subvariety	402
3. Homology class of a subvariety	406
4. Intersection theory	409
5. Algebraic families of cycles	414
6. Algebraic cycles and Chern classes	420

XI. DERIVED CATEGORIES

1. Categories of fractions	424
2. The derived category $D(A)$	430
3. Triangles associated to an exact sequence	441
4. Yoneda extensions	446
5. Octahedra	453
6. Localization	458

BIBLIOGRAPHY

461

I. Homological Algebra

I.1 Exact categories

Consider a category with zero object 0 , that is for every object A there is precisely one morphism $A \rightarrow 0$ and precisely one $0 \rightarrow A$.

A zero morphism $A \rightarrow B$ is one which can be factored $A \rightarrow 0 \rightarrow B$.

A kernel, $\text{Ker } f$ for a morphism $f: A \rightarrow B$ is a pair (K, i) where $i: K \rightarrow A$ is a monomorphism with $fi = 0$ and such that any morphism $g: X \rightarrow A$ with $fg = 0$ factors through $i: K \rightarrow A$.

A cokernel, $\text{Cok } f$ for f is a pair (C, p) where $p: B \rightarrow C$ is an epimorphism with $pf = 0$ such that any morphism $h: B \rightarrow Y$ with $hf = 0$ factors through p .

We shall assume that every morphism has kernel and cokernel.

An image, $\text{Im } f$ is a kernel for a cokernel.

A coimage, $\text{Coim } f$ is a cokernel for a kernel.

Every morphism f has a canonical factorization

$$A \rightarrow \text{Coim } f \xrightarrow{f'} \text{Im } f \rightarrow B$$

Definition 1.1. An exact category is a category with zero objects, kernels, cokernels and such that $\text{Coim } f \xrightarrow{f'} \text{Im } f$ always is an isomorphism.

In the remaining part of this section we shall work in an exact category.

Definition 1.2. A sequence of morphisms

$$\dots A^{n-1} \xrightarrow{f^{n-1}} A^n \xrightarrow{f^n} A^{n+1} \xrightarrow{f^{n+1}} \dots$$

is called exact if $\text{Im}(f^{n-1}) = \text{Ker}(f^n)$, for all n .

Proposition 1.3. Consider the exact, commutative diagram

$$\begin{array}{ccccccc} A & \rightarrow & B & \rightarrow & C & \rightarrow & D \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d \\ A' & \rightarrow & B' & \rightarrow & C' & \rightarrow & D' \\ \downarrow & & & & & & \\ 0 & & & & & & \end{array}$$

The induced sequence $\text{Ker } b \rightarrow \text{Ker } c \rightarrow \text{Ker } d$ is exact.

Proof. Break the diagram into two pieces

$$\begin{array}{ccccccc} 0 & \rightarrow & E & \rightarrow & C & \rightarrow & D \\ & & \downarrow e & & \downarrow c & & \downarrow d \\ 0 & \rightarrow & E' & \rightarrow & C' & \rightarrow & D' \end{array}$$

$$\begin{array}{ccccccc} A & \rightarrow & B & \rightarrow & E & \rightarrow & 0 \\ \downarrow a & & \downarrow b & & \downarrow e & & \\ A' & \rightarrow & B' & \rightarrow & E' & \rightarrow & 0 \\ \downarrow & & & & & & \\ 0 & & & & & & \end{array}$$

We have to prove that

a) $0 \rightarrow \text{Ker } e \rightarrow \text{Ker } c \rightarrow \text{Ker } d$ is exact

b) $\text{Ker } b \rightarrow \text{Ker } e$ is surjective

a) Check that $\text{Ker } e \rightarrow \text{Ker } c$ is a kernel for $\text{Ker } c \rightarrow D$.

β.1) Check that $\text{Cok } b \rightarrow \text{Cok } e$ is an isomorphism (use the dual statement to α, if necessary).

β.2) The exact commutative diagram

$$\begin{array}{ccccccc} A' & \longrightarrow & B' & \longrightarrow & E' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{Cok } b & \longrightarrow & \text{Cok } e & \longrightarrow & 0 \end{array}$$

shows that $A' \rightarrow \text{Im } b \rightarrow \text{Im } e$ is exact: replace A' by a kernel for $B' \rightarrow E'$ and use α).

β.3) This gives an exact commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & E & \longrightarrow & 0 \\ \downarrow a' & & \downarrow b' & & \downarrow e' & & \\ A' & \longrightarrow & \text{Im } b & \longrightarrow & \text{Im } e & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array}$$

Check that e' is a cokernel for $\text{Ker } b' \rightarrow E$.

The dual statement is

Q.E.D.

Proposition 1.4. Consider the exact, commutative diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' \end{array}$$

The induced sequence $\text{Cok } a \rightarrow \text{Cok } b \rightarrow \text{Cok } c$ is exact.

Corollary 1.5. Consider morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$.

The following sequence is exact

$$0 \rightarrow \text{Ker } f \rightarrow \text{Ker } g \rightarrow \text{Ker } gf \rightarrow \text{Cok } f \rightarrow \text{Cok } gf \rightarrow \text{Cok } g \rightarrow 0.$$

Proof. Apply 1.3 and 1.4 to the two diagrams

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \text{Ker } f & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & \text{Cok } f \\
 \downarrow & & \downarrow & & \downarrow gf & & \downarrow g & & \downarrow \\
 0 & \longrightarrow & 0 & \longrightarrow & Z & \longrightarrow & Z & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & & & & & \\
 0 & & 0 & & & & 0 & & 0 \\
 & & & & & & \downarrow & & \downarrow \\
 0 & \longrightarrow & X & \xrightarrow{\text{id}} & X & \longrightarrow & 0 & \longrightarrow & 0 \\
 \downarrow & & \downarrow f & & \downarrow gf & & \downarrow & & \downarrow \\
 \text{Ker } g & \longrightarrow & Y & \xrightarrow{g} & Z & \longrightarrow & \text{Cok } g & \longrightarrow & 0
 \end{array}$$

Q.E.D.

Snake Lemma 1.6. Consider the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & & & 0 \\
 & & & & & & & & \downarrow \\
 A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
 \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d & & \downarrow e \\
 A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \\
 \downarrow & & & & & & & & \\
 0 & & & & & & & &
 \end{array}$$

There is an exact sequence

$$\text{Ker } b \rightarrow \text{Ker } c \rightarrow \text{Ker } d \xrightarrow{\beta} \text{Cok } b \rightarrow \text{Cok } c \rightarrow \text{Cok } d.$$

More precisely

- 1) Put $K = \text{Ker}(C \rightarrow D')$. $K \rightarrow \text{Ker } d$ is an epimorphism.
- 2) Put $K' = \text{Cok}(B \rightarrow C')$. $\text{Cok } b \rightarrow K'$ is a monomorphism.
- 3) There exists a unique map

$$\partial: \text{Ker } d \rightarrow \text{Cok } b$$

such that the two composites

$$\begin{aligned} K &\rightarrow C \xrightarrow{c} C' \rightarrow K' \\ K &\rightarrow \text{Ker } d \xrightarrow{\partial} \text{Cok } b \rightarrow K' \end{aligned}$$

are the same.

- 4) The six-term sequence above is exact.

Proof. Let f denote the morphism $C' \rightarrow \text{Ker}(D' \rightarrow E')$.

Consider the exact commutative diagram

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow f & & \downarrow d & & \downarrow e \\ 0 & \longrightarrow & 0 & \longrightarrow & \text{Im } f & \longrightarrow & D' & \longrightarrow & E' \\ \downarrow & & \downarrow & & & & & & \\ 0 & & 0 & & & & & & \end{array}$$

It follows from 1.3 applied twice that

$$B \longrightarrow K \longrightarrow \text{Ker } d \longrightarrow 0$$

is exact, and similar, that

$$0 \longrightarrow \text{Cok } b \longrightarrow K' \longrightarrow D'$$

is exact. This proves 1), 2), 3). By 1.3, 1.4 and duality it suffices to prove that

$$\text{Ker } c \longrightarrow \text{Ker } d \longrightarrow \text{Cok } b$$

is exact. It suffices to prove exactness of

$$\text{Ker } c \longrightarrow \text{Ker } d \longrightarrow K'.$$

Consider the diagram

$$\begin{array}{ccccccc} B & \longrightarrow & K & \longrightarrow & \text{Ker } d & \longrightarrow & 0 \\ \downarrow \text{id} & & \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & C' & \longrightarrow & K' & \longrightarrow & 0 \\ \downarrow & & & & & & \\ 0 & & & & & & \end{array}$$

Conclusion by 1.3.

Q.E.D.

Let us record a much used special case

Five lemma 1.7. Given an exact commutative diagram

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d & & \downarrow e \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

If a, b, d, e are isomorphism, then c is an isomorphism.

I.2 Homology of complexes

We shall discuss the concept, homology in the framework of an exact category.

By a complex we understand a sequence $C^* = (C^n, \partial^n)_{n \in \mathbb{Z}}$ of objects and morphisms

$$\dots \longrightarrow C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \xrightarrow{\partial^{n+1}} C^{n+2} \longrightarrow \dots$$

with $\partial^{n+1} \partial^n = 0$ for all $n \in \mathbb{Z}$. The ∂ 's are called differentials or boundary operators.

A morphism of complexes $f: C^* \rightarrow D^*$ is a sequence $f = (f^n)_{n \in \mathbb{Z}}$ of morphisms $f^n: C^n \rightarrow D^n$ with

$$f^{n+1} \partial^n = \partial^n f^n \text{ for all } n \in \mathbb{Z}.$$

For a complex C^* we define for $n \in \mathbb{Z}$ the n 'th homology object

$$2.1 \quad H^n(C^*) = \text{Ker } \partial^n / \text{Im } \partial^{n-1}$$

A morphism $f: C^* \rightarrow D^*$ of complexes will induce a morphism on homology

$$2.2 \quad H^n(f) = H^n(C^*) \rightarrow H^n(D^*)$$

Consider a sequence of complexes

$$0 \longrightarrow P^* \xrightarrow{f} Q^* \xrightarrow{g} R^* \longrightarrow 0$$

which is a chainwise exact, i.e. with

$$0 \longrightarrow P^n \longrightarrow Q^n \longrightarrow R^n \longrightarrow 0$$

exact for all $n \in \mathbb{Z}$. We shall construct the so called connecting morphism

$$2.3 \quad c^n: H^n(R^*) \rightarrow H^{n+1}(P^*)$$

and derive a long exact sequence

$$2.4 \quad H^n(P^*) \xrightarrow{H^n(f)} H^n(Q^*) \xrightarrow{H^n(g)} H^n(R^*) \xrightarrow{c^n} H^{n+1}(P^*) \xrightarrow{H^{n+1}(f)} H^{n+1}(Q^*)$$

Construction. For a complex C^* we put

$$2.5 \quad Z^{n+1}(C^*) = \text{Ker } \partial^{n+1}, \quad 'Z^n(C^*) = \text{Cok } \partial^{n-1}$$

The boundary $\partial^n: C^n \rightarrow C^{n+1}$ induces

$$d^n: 'Z^n(C^*) \rightarrow Z^{n+1}(C^*)$$

As is easily seen we have an exact sequence

$$2.6 \quad 0 \rightarrow H^n(C^*) \rightarrow 'Z^n(C^*) \xrightarrow{d^n} Z^{n+1}(C^*) \rightarrow H^{n+1}(C^*) \rightarrow 0$$

We can now derive a commutative diagram

$$\begin{array}{ccccccc} 'Z^n(P^*) & \longrightarrow & 'Z^n(Q^*) & \longrightarrow & 'Z^n(R^*) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & Z^{n+1}(P^*) & \longrightarrow & Z^{n+1}(Q^*) & \longrightarrow & Z^{n+1}(R^*) & \end{array}$$

whose rows are exact as one easily derives from 1.3 and 1.4. We can now conclude the construction by appealing to the snake lemma 1.6.

The connecting morphism 2.3 has the following functorial property. Given

$$\begin{array}{ccccccc} 0 & \longrightarrow & P^* & \longrightarrow & Q^* & \longrightarrow & R^* \longrightarrow 0 \\ & & \downarrow u & & \downarrow v & & \downarrow w \\ 0 & \longrightarrow & U^* & \longrightarrow & V^* & \longrightarrow & W^* \longrightarrow 0 \end{array}$$

a commutative diagram of complexes whose rows are chainwise exact.