Birger Iversen

Cohomology of Sheaves

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Introduction

This text exposes the basic features of cohomology of sheaves and its applications. The general theory of sheaves is very limited and no essential result is obtainable without turning to particular classes of topological spaces. The most satisfactory general class is that of locally compact spaces and it is the study of such spaces which occupies the central part of this text.

The fundamental concepts in the study of locally compact spaces is cohomology with compact support and a particular class of sheaves, the so-called soft sheaves. This class plays a double role as the basic vehicle for the internal theory and is the key to applications in analysis. The basic example of a soft sheaf is the sheaf of smooth functions on $\mathbb{R}^n$ or more generally on any smooth manifold. A rather large effort has been made to demonstrate the relevance of sheaf theory in even the most elementary analysis. This process has been reversed in order to base the fundamental calculations in sheaf theory on elementary analysis.

The central theme of the text is Poincaré duality or rather its generalizations by Borel and Verdier. In its first form this appears as a duality between cohomology and cohomology with compact support. A more general Poincaré duality theory is developed for a continuous map between locally compact spaces. The important special case of a closed imbedding admits generalization to arbitrary topological spaces and is best understood in the framework of local cohomology. This theory is used for construction of characteristic classes of all sorts: Chern classes, Stiefel-Whitney classes, ...

For further applications to algebraic topology, a homology theory is developed for locally compact spaces and proper maps. This allows one to express Poincaré duality as an isomorphism between homology and cohomology. Applications are given to the classical theory of topological manifolds: fundamental class, diagonal class, Lefschetz fixed point formula ...

This homology theory is particularly suited for the study of algebraic varieties and a detailed introduction to (co)homology classes of algebraic cycles is given, including a topological definition of the local intersection symbol. It is a rather remarkable feature that this homology theory more or less automatically grinds out algebraic cycles.

A word about homological algebra. The first chapter of the text gives an introduction to homological algebra sufficient for most of the text. The last chapter, or appendix if you wish, gives an introduction to derived categories used in the more advanced parts of the text and in the proofs of the basic cup product formulas. It is my hope that this will give some readers motivation for Verdier's rather difficult text (1) on triangulated categories.

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Contents

IV. COHOMOLOGY AND ANALYSIS

V DUALITY WITH COEFFICIENT IN A FIELD

VI. POINCARÉ DUALITY WITH GENERAL COEFFICIENTS

VII. DIRECT IMAGE WITH PROPER SUPPORT

VIII. CHARACTERISTIC CLASSES

I. Homological Algebra

I.1 Exact categories

Consider a category with zero object 0, that is for every object A there is precisely one morphism $A \to 0$ and precisely one $0 \to A$.

A zero morphism $A \to B$ is one which can be factored $A \to 0 \to B$.

A kernel, $\operatorname{Ker} f$ for a morphism $f: A \to B$ is a pair (K,i) where $i: K \to A$ is a monomorphism with $fi = 0$ and such that any morphism $g: X \to A$ with $fg = 0$ factors through $i: K \to A$.

A cokernel, $\operatorname{Cok} f$ for f is a pair (C,p) where $p: B \to C$ is an epimorphism with $pf = 0$ such that any morphism $h: B \to Y$ with $hf = 0$ factors through p.

We shall assume that every morphism has kernel and cokernel.

An image, $\operatorname{Im} f$ is a kernel for a cokernel.

A coimage, $\operatorname{Coim} f$ is a cokernel for a kernel.

Every morphism f has a canonical factorization

$$A \to \operatorname{Coim} f \xrightarrow{f'} \operatorname{Im} f \to B$$

<u>Definition 1.1</u>. An <u>exact category</u> is a category with zero objects, kernels, cokernels and such that $\mathrm{Coim}\, f \xrightarrow{f'} \mathrm{Im}\, f$ always is an isomorphism.

In the remaining part of this section we shall work in an exact category.

<u>Definition 1.2</u>. A sequence of morphisms

$$\cdots A^{n-1} \xrightarrow{f^{n-1}} A^{n} \xrightarrow{f^{n}} A^{n+1} \xrightarrow{f^{n+1}} \cdots$$

is called <u>exact</u> if $\mathrm{Im}(f^{n-1}) = \mathrm{Ker}(f^{n})$, for all n.

<u>Proposition 1.3</u>. Consider the exact, commutative diagram

$$\begin{array}{ccccccc} A & \rightarrow & B & \rightarrow & C & \rightarrow & D \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d \\ A' & \rightarrow & B' & \rightarrow & C' & \rightarrow & D' \\ \downarrow & & & & & & \\ 0 & & & & & & \end{array}$$

The induced sequence $\mathrm{Ker}\, b \rightarrow \mathrm{Ker}\, c \rightarrow \mathrm{Ker}\, d$ is exact.

<u>Proof</u>. Break the diagram into two pieces

$$\begin{array}{ccccccc} 0 & \rightarrow & E & \rightarrow & C & \rightarrow & D \\ & & \downarrow e & & \downarrow c & & \downarrow d \\ 0 & \rightarrow & E' & \rightarrow & C' & \rightarrow & D' \end{array} \qquad \begin{array}{ccccccc} A & \rightarrow & B & \rightarrow & E & \rightarrow & 0 \\ \downarrow a & & \downarrow b & & \downarrow e & & \\ A' & \rightarrow & B' & \rightarrow & E' & \rightarrow & 0 \\ \downarrow & & & & & & \\ 0 & & & & & & \end{array}$$

We have to prove that

α) $0 \rightarrow \mathrm{Ker}\, e \rightarrow \mathrm{Ker}\, c \rightarrow \mathrm{Ker}\, d$ is exact

β) $\mathrm{Ker}\, b \rightarrow \mathrm{Ker}\, e$ is surjective α) Check that $\operatorname{Ker} e \to \operatorname{Ker} c$ is a kernel for $\operatorname{Ker} c \to D$.

β.1) Check that $\operatorname{cok} b \to \operatorname{Cok} e$ is an isomorphism (use the dual statement to α , if necessary).

β.2) The exact commutative diagram

$$\begin{array}{ccccccc}
A' & \longrightarrow & B' & \longrightarrow & E' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Cok} b & \longrightarrow & \operatorname{Cok} e & \longrightarrow & 0
\end{array}$$

shows that $A' \to \operatorname{Im} b \to \operatorname{Im} e$ is exact: replace A' by a kernel for $B' \to E'$ and use α).

β.3) This gives an exact commutative diagram

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & E & \longrightarrow & 0 \\
\downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle e'} & & \\
A' & \longrightarrow & \operatorname{Im} b & \longrightarrow & \operatorname{Im} e & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Check that e' is a cokernel for $\operatorname{Ker} b' \to E$.

The dual statement is Q.E.D.

Proposition 1.4. Consider the exact, commutative diagram

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D'
\end{array}$$

The induced sequence $\operatorname{Cok} a \to \operatorname{Cok} b \to \operatorname{Cok} c$ is exact.

<u>Corollary 1.5</u>. Consider morphisms $f: X \to Y$ and $g: Y \to Z$. The following sequence is exact

$$0 \to \mathrm{Ker}\, f \to \mathrm{Ker}\, gf \to \mathrm{Ker}\, g \to \mathrm{Cok}\, f \to \mathrm{Cok}\, gf \to \mathrm{Cok}\, g \to 0.$$

<u>Proof</u>. Apply 1.3 and 1.4 to the two diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}\, f & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & \mathrm{Cok}\, f \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle gf} & & \downarrow{\scriptstyle g} & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & Z & \longrightarrow & Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & & & \\
0 & & 0 & & & & & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & 0 \\
 & & & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & X & \xrightarrow{id} & X & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle gf} & & \downarrow & & \downarrow \\
\mathrm{Ker}\, g & \longrightarrow & Y & \xrightarrow{g} & Z & \longrightarrow & \mathrm{Cok}\, g & \longrightarrow & 0
\end{array}$$

Q.E.D.

<u>Snake Lemma 1.6</u>. Consider the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & & & 0 \\
 & & & & & & & & \downarrow \\
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \\
\downarrow & & & & & & & & \\
0 & & & & & & & &
\end{array}$$

There is an exact sequence

$$\mathrm{Ker}\, b \to \mathrm{Ker}\, c \to \mathrm{Ker}\, d \xrightarrow{\partial} \mathrm{Cok}\, b \to \mathrm{Cok}\, c \to \mathrm{Cok}\, d\,.$$

More precisely

1) Put $K = \mathrm{Ker}(C \to D')$. $K \to \mathrm{Ker}\, d$ is an epimorphism.

2) Put $K' = \mathrm{Cok}(B \to C')$. $\mathrm{Cok}\, b \to K'$ is a monomorphism.

3) There exists a unique map

$$\partial : \mathrm{Ker}\, d \to \mathrm{Cok}\, b$$

such that the two composites

$$K \to C \overset{c}{\to} C' \to K'$$
$$K \to \mathrm{Ker}\, d \overset{\partial}{\to} \mathrm{Cok}\, b \to K'$$

are the same.

4) The six-term sequence above is exact.

Proof. Let f denote the morphism $C' \to \mathrm{Ker}(D' \to E')$. Consider the exact commutative diagram

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle fc} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Im}\, f & \longrightarrow & D' & \longrightarrow & E' \\
\downarrow & & \downarrow & & & & & & \\
0 & & 0. & & & & & &
\end{array}$$

It follows from 1.3 applied twice that

$$B \longrightarrow K \longrightarrow \mathrm{Ker}\, d \longrightarrow 0$$

is exact, and similar, that

$$0 \longrightarrow \mathrm{Cok}\, b \longrightarrow K' \longrightarrow D'$$

is exact. This proves 1), 2), 3). By 1.3, 1.4 and duality it suffices to prove that

$$\operatorname{Ker} c \longrightarrow \operatorname{Ker} d \longrightarrow \operatorname{Cok} b$$

is exact. It suffices to prove exactness of

$$\operatorname{Ker} c \longrightarrow \operatorname{Ker} d \longrightarrow K'.$$

Consider the diagram

$$\begin{array}{ccccccc}
B & \longrightarrow & K & \longrightarrow & \operatorname{Ker} d & \longrightarrow & 0 \\
\downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \downarrow \\
B & \longrightarrow & C' & \longrightarrow & K' & \longrightarrow & 0 \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}$$

Conclusion by 1.3.

Q.E.D.

Let us record a much used special case

<u>Five lemma 1.7</u>. Given an exact commutative diagram

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

If a,b,d,e are isomorphism, then c is an isomorphism.

I.2 Homology of complexes

We shall discuss the concept, homology in the framework of an exact category.

By a complex we understand a sequence $C^{\cdot} = (C^n, \partial^n)_{n\in\mathbb{Z}}$ of objects and morphisms

$$\cdots \longrightarrow C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \xrightarrow{\partial^{n+1}} C^{n+2} \longrightarrow \cdots$$

with $\partial^{n+1}\partial^n = 0$ for all $n \in \mathbb{Z}$. The ∂'s are called differentials or boundary operators.

A morphism of complexes $f: C^{\cdot} \longrightarrow D^{\cdot}$ is a sequence $f = (f^n)_{n\in\mathbb{Z}}$ of morphisms $f^n: C^n \longrightarrow D^n$ with

$$f^{n+1}\partial^n = \partial^n f^n \quad \text{for all} \quad n \in \mathbb{Z}.$$

For a complex $C^{\cdot}$ we define for $n \in \mathbb{Z}$ the n'th homology object

2.1
$$H^n(C^{\cdot}) = \operatorname{Ker} \partial^n / \operatorname{Im} \partial^{n-1}$$

A morphism $f: C^{\cdot} \longrightarrow D^{\cdot}$ of complexes will induce a morphism on homology

2.2
$$H^n(f) = H^n(C^{\cdot}) \longrightarrow H^n(D^{\cdot})$$

Consider a sequence of complexes

$$0 \longrightarrow P^{\cdot} \xrightarrow{f} Q^{\cdot} \xrightarrow{g} R^{\cdot} \longrightarrow 0$$

which is a chainwise exact, i.e. with

$$0 \longrightarrow P^n \longrightarrow Q^n \longrightarrow R^n \longrightarrow 0$$

exact for all $n \in \mathbb{Z}$. We shall construct the so called connecting morphism

2.3 $$c^n \colon H^n(R^\cdot) \to H^{n+1}(P^\cdot)$$

and derive a long exact sequence

2.4 $$H^n(P^\cdot) \xrightarrow{H^n(f)} H^n(Q^\cdot) \xrightarrow{H^n(g)} H^n(R^\cdot) \xrightarrow{c^n} H^{n+1}(P^\cdot) \xrightarrow{H^{n+1}(f)} H^{n+1}(Q^\cdot)$$

<u>Construction</u>. For a complex $C^\cdot$ we put

2.5 $$Z^{n+1}(C^\cdot) = \operatorname{Ker} \partial^{n+1}, \qquad 'Z^n(C^\cdot) = \operatorname{Cok} \partial^{n-1}$$

The boundary $\partial^n \colon C^n \to C^{n+1}$ induces

$$d^n \colon {}'Z^n(C^\cdot) \to Z^{n+1}(C^\cdot)$$

As is easily seen we have an exact sequence

2.6 $$0 \to H^n(C^\cdot) \to {}'Z^n(C^\cdot) \xrightarrow{d^n} Z^{n+1}(C^\cdot) \to H^{n+1}(C^\cdot) \to 0$$

We can now derive a commutative diagram

$$\begin{array}{ccccccccc} & & 'Z^n(P^\cdot) & \longrightarrow & 'Z^n(Q^\cdot) & \longrightarrow & 'Z^n(R^\cdot) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z^{n+1}(P^\cdot) & \longrightarrow & Z^{n+1}(Q^\cdot) & \longrightarrow & Z^{n+1}(R^\cdot) & & \end{array}$$

whose rows are exact as one easily derives from 1.3 and 1.4. We can now conclude the construction by appealing to the snake lemma 1.6.

The connecting morphism 2.3 has the following functorial property. Given

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P^\cdot & \longrightarrow & Q^\cdot & \longrightarrow & R^\cdot & \longrightarrow & 0 \\ & & \downarrow u & & \downarrow v & & \downarrow w & & \\ 0 & \longrightarrow & U^\cdot & \longrightarrow & V^\cdot & \longrightarrow & W^\cdot & \longrightarrow & 0 \end{array}$$

a commutative diagram of complexes whose rows are chainwise exact.