Communications and Control Engineering

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Liapunov Functions and Stability in Control Theory

2nd Edition

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With 20 Figures

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Preface

We are interested in mathematical models of input systems, described by continuous-time, finite dimensional ordinary differential equations

$$\dot{x} = f(t, x, u) \tag{1}$$

where $t \geq 0$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ represents the state variables, $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ represents the input variables and $f = (f_1, \ldots, f_n) : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Together with (1), we will often consider the *unforced associated system*

$$\dot{x} = f(t, x, 0) \ . \tag{2}$$

Basically, (2) accounts for the "internal" behavior of the system. More precisely, (2) describes the natural dynamics of (1) when no energy is supplied through the input channels. The analysis of the "external" behavior is rather concerned with the effect of the inputs (disturbances or exogenous signals) on the evolution of the state response of (1).

Physical systems are usually expected to exhibit a "stable" behavior. A primary aim of this book is to survey some possible mathematical definitions of internal and external stability in a nonlinear context and to discuss their characterizations in the framework of the Liapunov functions method.

We will also consider the problem of achieving a more desirable stability behavior (both from the internal and the external point of view) by means of properly designed feedback laws. To this end, it is convenient to think of the input as a sum $u = u_{\mathrm{e}} + u_{\mathrm{c}}$. The term u_{e} represents external forces, while u_{c} is actually available for control action. Roughly speaking, (1) is said to be "stabilizable" if there exists a map $u_{\mathrm{c}} = k(t, x)$ such that the closed loop system

$$\dot{x} = f(t, x, k(t, x) + u_{\mathrm{e}}) \tag{3}$$

exhibits improved (internal and/or external) stability performances.

Intimate relationships among all these aspects of systems analysis emerge with some evidences from classical linear systems theory. In particular, as we shall see at the beginning of Chapter 2, the external behavior of a linear system is strongly related to its internal structure. On the contrary, dealing with nonlinear systems these connections become weaker and need a more delicate treatment.

We shall see in particular that the approach to stability and stabilizability of nonlinear systems rests much more heavily on the method of Liapunov functions. Thus, we are led to emphasize the interest in a variety of theorems which state, under minimal assumptions, the existence of Liapunov functions with suitable properties. These theorems are usually called "converse Liapunov theorems". A secondary aim of this book is to illustrate the state of the art on this subject, and to present some recent developments.

We have not yet specified what kind of assumptions should be made about the map f which appears at the right hand side of (1) and about the admissible inputs.

The class of admissible inputs should be so large to include representations of all signals commonly used in engineering applications. To this purpose, it is well known that in certain circumstances, a discontinuous function often is more suited than a continuous one. Thus, throughout these notes, we shall adopt the following agreement:

(I) the class of *admissible inputs* is constituted by all measurable, essentially bounded functions $u : [0, +\infty) \to \mathbb{R}^m$.

To establish the assumptions about f is a more delicate task. In a classical "smooth" setting, it seems natural to ask that f is time invariant, namely $f(t, x, u) = f(x, u)$, and at least continuous as a function of x, u, though additional regularity could be required for certain purposes[1]. This is actually the point of view we intend to adopt at the beginning but, as long as we proceed in our exposition, it will become clear that the smooth setting is too conservative for certain developments. This occurs in particular when we seek Liapunov functions of (Liapunov or Lagrange) stable systems or when we aim to design internally asymptotically stabilizing feedback laws. Indeed, the solution of

[1] Recent results of the so-called geometric control theory apply to systems whose right hand side can be represented as a family of C^∞ or real analytic vector fields (see [79], [80], [155]).

these problems cannot be found in general within a pre-assigned class of time invariant smooth functions, unless severe restrictions are made on the system under consideration. We will be so led to introduce in our treatment nondifferentiable functions and differential equations with discontinuous right hand side.

We remark that differential equations with discontinuous right hand side arise in many engineering and physical applications. Historically, one of the main motivation was the study of the motion of a body with one degree of freedom subject to an elastic force, in presence of both viscous and dry friction ([58], [54]). This is modelled by the second order equation

$$\ddot{x} + kx + b\dot{x} + a \operatorname{sgn} \dot{x} = 0$$

or, equivalently, by the two dimensional system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -kx - by - a \operatorname{sgn} y \end{cases} \tag{4}$$

(here, k, b and a are positive constants). For $y \neq 0$, the motion is correctly represented by the solutions of the system. But if the body reaches a position $(x, 0)$ with $-\frac{a}{k} < x < \frac{a}{k}$, our intuition suggests that the elastic force is too weak. It cannot overcome the dry friction, and the body remains at rest. This intuition is easily confirmed by physical observation, but it is not reflected by system (4), at least as far as the solutions are intended in the usual sense.

Differential equations with discontinuous right hand side play an important role also in *variable structure control* methodologies. Consider, for simplicity, a time-invariant system

$$\dot{x} = f(x, u) \ .$$

In variable structure control theory, the goal is to track a path lying on a hypersurface Σ defined by an equation $s(x) = 0$, where $s(x)$ is a smooth function. To this purpose, it is often convenient to use discontinuous feedback, say for instance

$$u = k(x) = \begin{cases} 1 & \text{if } s(x) > 0 \\ -1 & \text{if } s(x) < 0 \ . \end{cases}$$

Clearly, the closed loop system

$$\dot{x} = f(x, k(x))$$

turns out to be discontinuous even if $f(x,u)$ is a smooth function. The desired motion is given by a trajectory sliding on Σ; in general, it is not a solution in the usual sense of the closed loop system.

We finally remark that discontinuities of the velocity and sometimes also of the state evolution are a typical feature of the so-called *hybrid dynamical systems*: the book [152] provides a nice introduction on this subject, with many practical examples (manual transmission, temperature control, electric circuits with diodes, and many others).

These remarks point out that the treatment of differential equations with discontinuous right hand side requires a generalization of the classical notion of solution.

To be prepared for this extension, in Chapter 1 we recall some preliminary material about existence of solutions for ordinary differential equations and differential inclusions.

The main subject will be addressed starting from Chapter 2. As already mentioned, in Chapter 2 we focus more precisely on the case where the right hand side of (1) is time invariant and continuous with respect to both x, u. The reason why we prefer to begin with such a restricted class of systems is twofold. First, the more general approach could be felt at that point unmotivated and too abstract. Second, the main notions, methods and achievements available in the literature about stability and stabilizability theory of control systems have been mostly obtained, in the last few years, just for this class of systems. Of course, the choice of proceeding from the simplest situation to the more general one, implies also a few of complications (for instance, the need of a progressive updating of definitions and results when we shall undertake certain extensions) but gives a clearer perspective of problems and theoretical difficulties.

A first attempt to re-interpret our problems in a more general context is made in Chapter 3, where we consider time varying systems. We focus in particular on possible notions of internal stability and on their relationships. Although we are able to give some more precise results about existence of Liapunov functions and of stabilizing feedback, we shall see that the picture of the situation is not yet completely satisfactory.

The goal of replacing the classical smooth setting by a more general time dependent and "nonsmooth" one, will be fully pursued in Chapter 4, where we finally consider systems of the general form (1), and f is allowed to be discontinuous with respect to x. More precisely, in Chapter 4 we discuss direct and converse theorems about stability and asymptotic stability, together with their applications to external stabilization. We present also a new approach which allows us to prove in a unified manner several recent results. The proof given here is considerably shorter and easier than other proofs available in the original papers.

Certain additional properties of Liapunov functions will be discussed in Chapter 5. Here, we consider again the case of systems of ordinary differential equations, with time invariant and smooth right hand side. The topics include existence of analytic or homogeneous Liapunov functions and their symmetries, and relationship between Liapunov functions and decay of trajectories.

Finally, in Chapter 6 we review some tools from nonsmooth analysis which can be useful in the investigation of nondifferentiable systems with discontinuous Liapunov functions.

Contents

Chapter 1

Differential equations

In what follows, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ represent respectively the sets of natural, integer, rational and real numbers. Sometimes, we may use also the notation $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

Let $N \in \mathbb{N}^*$. The norm of a vector $v = (v_1, \ldots, v_N) \in \mathbb{R}^N$ is denoted by $||v||$. As is well known, for finite dimensional vector spaces all the norms are equivalent. Actually, the choice of the norm does not matter in the first three chapters. However, in view of the developments of Chapter 4, it is convenient to take the sup-norm

$$||v|| = \max\{|v_i| : 1 \leq i \leq N\} .$$

The Hausdorff distance between nonempty, compact subsets of $\mathbb{R}^N$ will be denoted by h. We recall that

$$h(A, B) = \max\{\sup_{a \in A} \text{dist}\,(a, B), \sup_{b \in B} \text{dist}\,(b, A)\}$$

where $\text{dist}\,(a, B) = \inf_{b \in B} ||a - b||$.

For $x \in \mathbb{R}^N$ and $r > 0$, the open ball of center x and radius r is denoted by

$$B_r(x) = \{y \in \mathbb{R}^N : \ ||y - x|| < r\} .$$

Of course, $\overline{B_r(x)}$ denotes the closed ball. When $x = 0$, we shall write simply B_r instead of $B_r(0)$. We shall also use the symbol B^r for the complement of a closed ball, namely

$$B^r = \{y \in \mathbb{R}^N : \ ||y|| > r\} = \mathbb{R}^N \setminus \overline{B_r} .$$

Finally, let $g : \Omega \to \mathbb{R}^M$, where $\Omega \subseteq \mathbb{R}^N$. The function g is said to be *locally Lipschitz continuous* on Ω if for each $\bar{x} \in \Omega$ there exist positive real numbers L, δ such that

$$x', x'' \in B_\delta(\bar{x}) \cap \Omega \Longrightarrow ||g(x') - g(x'')|| \leq L||x' - x''|| .$$

1.1 Recall about existence results

The first natural question about a system of the form (1) concerns of course the existence of (local) solutions corresponding to any admissible input. Throughout this chapter we assume that $u(t)$ is fixed, so that we can adopt the simplified notation $f(t, x) = f(t, x, u(t))$. We are therefore led to consider a system of ordinary differential equations of the form

$$\dot{x} = f(t, x) \tag{1.1}$$

where $f(t, x)$ is defined for all $x \in \mathbb{R}^n$ and $t \geq 0$. As is well known, Peano's Theorem states that if $f(t, x)$ is continuous on $[0, +\infty) \times \mathbb{R}^n$, then for each initial pair $(t_0, x_0) \in [0, +\infty) \times \mathbb{R}^n$ there exists at least one local *classical solution* $x(t) : I \to \mathbb{R}^n$ such that $x(t_0) = x_0$. Here, I is an interval of real numbers such that $t_0 \in I \subseteq [0, +\infty)$. The qualifier "classical" emphasizes that $x(t)$ is of class C^1 and

$$\dot{x}(t) = f(t, x(t)) \qquad \forall t \in I .$$

The continuity assumption required by Peano's Theorem is too restrictive for applications to control theory. Indeed, in general admissible inputs are assumed to be only measurable and essentially bounded. Therefore, even if the right hand side of (1) is continuous, we cannot hope that the resulting map $f(t, x) = f(t, x, u(t))$ is continuous.

The following set of assumptions for (1.1) seems to be more appropriate:

($\mathbf{A}_1$) the function $f(t, x)$ is locally essentially bounded on $[0, +\infty) \times \mathbb{R}^n$

($\mathbf{A}_2$) for each $x \in \mathbb{R}^n$, the function $t \mapsto f(t, x)$ is measurable

($\mathbf{A}_3$) for a.e. $t \geq 0$, the function $x \mapsto f(t, x)$ is continuous.

A function $x(t)$ is called a local *Carathéodory solution* of (1.1) on the interval I if it is absolutely continuous on every compact subinterval of I and satisfies

$$\dot{x}(t) = f(t, x(t)) \qquad \text{a.e. } t \in I \ .$$

Carathéodory's Theorem states that if assumptions ($\mathbf{A}_1$), ($\mathbf{A}_2$), ($\mathbf{A}_3$) are fulfilled, then for each initial pair $(t_0, x_0) \in [0, +\infty) \times \mathbb{R}^n$ there exists an interval I with $t_0 \in I$ and a Carathéodory solution $x(t)$ defined on I.

For a system of the form (1.1), the set of all local Carathéodory solutions corresponding to a given initial pair (t_0, x_0) will be denoted by $\mathcal{S}_{t_0,x_0}$. When we need to emphasize the dependence of a particular solution $x(t) \in \mathcal{S}_{t_0,x_0}$ on the initial time and state, we shall use the notation $x(t) = x(t; t_0, x_0)$.

Moreover, when (1.1) results from an input system like (0.1) and we want to emphasize the dependence of solutions on the input $u(t)$, we shall write respectively $\mathcal{S}_{t_0,x_0,u(\cdot)}$ and $x(t) = x(t; t_0, x_0, u(\cdot))$.

Remark 1.1 Of course, any classical solution is also a Carathéodory solution. To show that the converse is false, consider the following simple one-dimensional equation

$$\dot{x} = f(t, x) = a(t)x$$

where

$$a(t) = \begin{cases} 0 & \text{if } t \in \mathbb{Q} \\ 1 & \text{if } t \in \mathbb{R} \setminus \mathbb{Q} \end{cases} .$$

For each initial pair $(0, x_0)$ with $x_0 \neq 0$, the set of classical solutions is empty, but there is a Carathéodory solution of the form $x = e^t x_0$. ■

Peano's and Carathéodory's Theorems only guarantee in general the existence of local solutions. A typical additional assumption is *local Lipschitz continuity* with respect to x:

($\mathbf{A}_4$) for each point $(\bar{t}, \bar{x}) \in [0, +\infty) \times \mathbb{R}^n$ there exist $\delta > 0$ and a positive function $l(t) : [0, +\infty) \to \mathbb{R}$ such that $l(t)$ is locally integrable and

$$||f(t, x') - f(t, x'')|| \leq l(t)||x' - x''||$$

for each t, x' and x'' such that $|t - \bar{t}| \leq \delta$, $||x' - \bar{x}|| \leq \delta$ and $||x'' - \bar{x}|| \leq \delta$.

Under the assumptions ($\mathbf{A}_1$), ($\mathbf{A}_2$), ($\mathbf{A}_3$) and ($\mathbf{A}_4$), it is possible to prove local uniqueness and continuity of solutions with respect to the initial data. In particular, the following holds.

(C) let $(\bar{t}, \bar{x}) \in [0, +\infty) \times \mathbb{R}^n$ be fixed, and assume that $x(t; \bar{t}, \bar{x})$ is defined on some closed interval $[\alpha, \beta]$ (with $\alpha \le \bar{t} \le \beta$). Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each pair (τ, ξ) with

$$|\tau - \bar{t}| < \delta \ , \qquad ||\xi - \bar{x}|| < \delta$$

the solution $x(t; \tau, \xi)$ is defined for $\alpha \le t \le \beta$ and

$$\|x(t; \tau, \xi) - x(t; \bar{t}, \bar{x})\| < \varepsilon$$

for each $t \in [\alpha, \beta]$.

These and other results about ordinary differential equations can be found in many usual textbooks (see for instance [125], [70], [60]).

1.2 Differential inclusions

In this section we illustrate how differential inclusions arise in the mathematical theory of control systems. Moreover, we recall the main existence results needed in the following chapters. In particular, we show that the existence of Filippov solutions for discontinuous differential equations can be actually deduced from an existence theorem for differential inclusions.

1.2.1 The upper semi-continuous case

As already mentioned in the Introduction, for certain applications of control theory we need to resort to differential equations whose right hand side is discontinuous not only with respect to t, but also with respect to the state variable x. Indeed, even if the system is modeled by smooth vector fields, discontinuities may be inevitably introduced when closed loop solutions of certain problems are required.

Note that if the right hand side of (1.1) is not continuous with respect to x, then the usual notions of solution (classical or Carathéodory) do not apply. The more common way to overcome the difficulty is to replace (1.1) by a differential inclusion of the form

$$\dot{x} \in F(t, x) \ . \tag{1.2}$$

A *solution* of (1.2) is any function $x(t)$ defined on some interval $I \subseteq [0, +\infty)$ which is absolutely continuous on each compact subinterval of I and such that