Nonlinear and Optimal Control Systems

Thomas L. Vincent
Walter J. Grantham

NONLINEAR AND OPTIMAL CONTROL SYSTEMS

THOMAS L. VINCENT
Department of Aerospace and Mechanical Engineering
University of Arizona

WALTER J. GRANTHAM
School of Mechanical and Materials Engineering
Washington State University

A Wiley-Interscience Publication
JOHN WILEY & SONS, INC.
New York • Chichester • Weinheim • Brisbane • Singapore • Toronto

This text is printed on acid-free paper.

Library of Congress Cataloging in Publication Data:

Vincent, Thomas L.
Nonlinear and optimal control systems / Thomas L. Vincent, Walter J. Grantham.
p. cm.
Includes index.
ISBN 0-471-04235-8 (cloth : alk. paper)
1. Feedback control systems. 2. Nonlinear systems. I. Grantham, Walter J. (Walter Jervis), 1944– . II. Title.
TJ216.V56 1997 96-37129

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

PREFACE

This text provides an introduction to the analysis techniques used in the design of nonlinear and optimal feedback control systems. The fundamental topics of stability, controllability, and optimality are developed and presented in a unique unified fashion that establishes strong connections between all three topics.

There are two important approaches to the design of feedback controllers for nonlinear systems: Lyapunov stability methods and optimal control theory. On the surface, these two approaches might not appear to have much in common. Lyapunov stability does not address optimality, while optimal control theory generally provides only open-loop control, which need not provide stability. In between these two approaches there is a large, and until now, relatively unexplored area of function minimizing feedback controllers. These controllers address optimality, but on an instantaneous basis, yielding closed-loop feedback controllers without having to solve an optimal control two-point boundary value problem. Instantaneous function minimization can be expected to miss global aspects of optimality that are provided in an optimal control solution. However, when function minimization is combined with Lyapunov stability methods, the resulting controllers do contain global information. These function minimizing Lyapunov controllers have proven to be highly effective and also robust with respect to uncertainties in system models, noisy inputs, and so on.

With the recent growth in interest in nonlinear systems, many engineering departments are beginning to add courses on nonlinear control systems. At the same time, optimal control theory has matured to the point where optimal controls courses can be modified to include other topics in nonlinear control systems design.

This text provides an integrated approach to both nonlinear and optimal control systems. The emphasis is on the fundamental topics of stability, controllability, and optimality, and on the corresponding geometry associated with these topics. In Chapter 1 we discuss background material on nonlinear dynamical systems and various phenomena that can occur only in nonlinear systems, such as limit cycles, chatter, and chaotic motion. In addition, we present numerical algorithms for solving systems of nonlinear differential equations, so that students can quickly begin simulating nonlinear systems, using these routines or commercial software, such as MATLAB. Chapter 2 presents an introduction to nonlinear control systems. In Chapter 3 we present basic results in nonlinear parameter optimization and parametric two-player games, results that are used in later chapters. Chapter 4 covers Lyapunov stability theory, and Chapter 5 applies these results to the design of Lyapunov optimizing feedback controllers. Chapter 6 is devoted to controllability concepts associated with nonlinear control systems. In particular, the controllability minimum principle is developed, which allows for the determination of the boundary of the set of points that are controllable to a specified target. This leads directly to a development of Pontryagin's minimum principle in Chapter 7. This principle is used to solve optimal control problems and for the design of optimal controllers in Chapter 8. The same geometric ideas introduced in Chapters 6 and 7 also lead to necessary conditions for two-player differential games, discussed in Chapter 9.

This text is designed for one-semester introductory senior- or graduate-level nonlinear and optimal control systems courses and does not presume any background on the student's part beyond differential equations. In particular, we do not assume that the student has taken a linear control systems course. Although our focus is on nonlinear systems, this does not preclude us from referring to linear systems or using a number of linear systems (with bounded controls, which make these systems nonlinear) as examples. Any linear control systems background that is required to understand any allusion to linear systems is presented directly in the text. Throughout the text we focus on a variety of example systems. Viewing such systems in terms of stability, controllability, and optimality provides the student with an opportunity to compare the same systems in different design contexts. In these examples we consider not only physical dynamical systems, but also biological systems, economic systems, and so on. Thus this text is suitable for students from a wide range of disciplines.

There are a variety of approaches to structuring separate one-semester nonlinear or optimal control systems courses using this text. Chapter 1 provides mathematical background on the nature of solutions to nonlinear dynamical systems and on some classical analysis techniques for such systems. Depending on the objectives of the course, this material could be covered either briefly or in detail. Chapters 1–6 provide the basis for a nonlinear controls course. We have found that it is also possible to cover at least some of the material in Chapter 7 as well, creating a nonlinear and optimal controls course with an emphasis on nonlinear controls and an introduction to optimal control systems. Chapters 2, 3, and 6–9 form the basis for an optimal controls course, with the emphasis on optimization, calculus of variations, optimal control, and two-player differential games.

However, by design, this text is structured to allow for combined nonlinear and optimal controls courses. For such courses there are at least two approaches. For an emphasis on nonlinear controls one could skip the material on differential games, covering (perhaps only the first half of) Chapter 1 briefly, but covering Chapters 2–7 in detail. For an emphasis on optimal controls one could skip the second half of Chapter 4, dealing with the details of constructing Lyapunov functions and estimating stability regions. The remaining chapters can then be covered in detail, with Chapter 1 (or the first half only) being covered briefly.

Each chapter contains several examples and a variety of exercises to aid the student. In addition, several sections contain numerical algorithms. In particular, the algorithms that we present for nonlinear simulation and nonlinear parameter optimization form the basis for the current state-of-the-art algorithms in these areas. We have found these and other algorithms (such as the routines in MATLAB) to be useful, and we hope that the reader will also benefit from them. Finally, it is hoped that this text will stimulate the reader to further study in nonlinear and optimal control system design and to the application of these methods to practical problems of interest.

THOMAS L. VINCENT
WALTER J. GRANTHAM

Tucson, Arizona
Pullman, Washington
June 1997

CONTENTS

1

NONLINEAR DYNAMICAL SYSTEMS

1.1 FUNDAMENTALS

System Models

In this text we consider nonlinear dynamical control systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \tag{1.1-1}$$

where $\mathbf{x} \in R^{n_x}$ is the n_x-dimensional **state** vector, $\mathbf{u} \in \mathcal{U} \subseteq R^{n_u}$ is the n_u-dimensional **control** vector input (to be chosen), and $(\dot{\ })$ denotes differentiation with respect to time t. The set $\mathcal{U}$ is a **control constraint set**, reflecting the fact that control inputs typically have some constraints imposed upon them, such as bounded thrust on a rocket or bounded turning rate for a vehicle. The vector function $\mathbf{f}(\cdot)$ and its partial derivatives with respect to $\mathbf{x}$ and $\mathbf{u}$ are assumed to be continuously differentiable functions of $\mathbf{x}$ and $\mathbf{u}$, although the control input function $\mathbf{u}(\cdot)$ may be discontinuous, subject to certain restrictions that we discuss shortly. In general we are concerned with developing a **feedback control** law $\mathbf{u}(\mathbf{x})$ to make the system (1.1-1) behave in some specified fashion. We assume, throughout, that the state information needed for the control law $\mathbf{u}(\mathbf{x})$ can be measured precisely.

The dot notation used above to indicate the time derivative is not to be confused with the notation $(\cdot)$, which is used in place of a function argument, as in $\mathbf{f}(\cdot)$, when we wish to refer to the functional relationship itself rather than its value.

Target Sets There may be several design objectives involved in the choice of a control algorithm, but common among them is a specified target set $\mathcal{X}$ in state space. For example, an objective might be to achieve a certain target and maintain it, as in a rendezvous with an orbiting satellite, or simply to hit a target, as in an aircraft-missile intercept problem. In all cases the underlying basic controllability objective is one of finding a feedback control function $\mathbf{u}(\mathbf{x})$ that will transfer the state to a specified **target** set $\mathcal{X} \subset R^{n_x}$, usually defined by a system of n_g equalities

$$\mathcal{X} = \{\mathbf{x} | \mathbf{g}(\mathbf{x}) = 0\}, \tag{1.1-2}$$

with $\mathbf{g}(\mathbf{x}) = [g_1(\mathbf{x}) \cdots g_{n_g}(\mathbf{x})]^{\mathrm{T}}$ assumed to be continuous and continuously differentiable. Furthemore we assume that the gradient vectors $\partial g_i(\mathbf{x})/\partial \mathbf{x}, i = 1, \ldots n_g$, are linearly independent on $\mathcal{X}$.

Control Constraints In many applications the elements of the control vector $\mathbf{u}$ may have various constraints imposed upon them, which may limit our ability to transfer the state to the target set. For example, the magnitude of the thrust vector, for a variable thrust rocket, would vary from zero to some upper limit corresponding to the maximum thrust available. Also, the direction of the thrust vector relative to the rocket would generally be limited by the gimbals on which the exhaust nozzles are mounted.

We consider the case in which the control vector is required to satisfy a given set of constraints

$$\mathbf{u} \in \mathcal{U}, \tag{1.1-3}$$

where $\mathcal{U} \subseteq R^{n_u}$ is a specified **control constraint set**. If no control constraints are imposed, then $\mathcal{U} = R^{n_u}$. In the general case we allow for the possibility that the control constraints may depend on the current state, with the control constraint set being defined by a system of n_h inequalities

$$\mathcal{U} = \{\mathbf{u} | \mathbf{h}(\mathbf{x}, \mathbf{u}) \geq 0\}. \tag{1.1-4}$$

Most often, however, we restrict the discussion to state-independent control constraints of the form

$$\mathcal{U} = \{\mathbf{u} | \mathbf{h}(\mathbf{u}) \geq 0\}, \tag{1.1-5}$$

with $\mathbf{h}(\mathbf{u}) = [h_1(\mathbf{u}) \cdots h_{n_h}(\mathbf{u})]^{\mathrm{T}}$ assumed to be continuous and continuously differentiable. We also require that the gradient vectors $\partial h_i(\mathbf{u})/\partial \mathbf{u}$ satisfy a **regularity** assumption, which is satisfied if they are linearly independent. This requirement is discussed in detail in Section 3.2, where we develop first-order necessary conditions governing solutions to nonlinear constrained parameter optimization problems.

The state vector $\mathbf{x}$ might also have constraints imposed upon it, such as a minimum and maximum allowed speed for an aircraft. The case where a portion of a trajectory runs along a state constraint boundary is considered in Section 7.7. However, for the most part, we do not consider state constraints explicitly and, instead, assume that trajectories lie inside the region defined by the state constraints.

Example 1.1-1: An Introductory Optimal Control Problem Consider a drag race in which the objective is to go in minimum time from a standing start to a full stop at the other end of the drag strip. Consider just one car, of mass m, and let $y(t)$ denote the car's position along the drag strip. Under an applied force $F(t)$, which we get to pick, the equation of motion may be written as

$$m\ddot{y} = F(t).$$

The control input $F(t)$ can be either positive (acceleration) or negative (braking) and is bounded by the constraints

$$F_{\min} \leq F \leq F_{\max}.$$

We can convert this problem to state-space form by letting $x_1 = y$ and $x_2 = \dot{y}$. With $u = F/m$ the equations of motion become

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u. \end{aligned}$$

The optimal control problem is to find $u(t)$ to drive the system in minimum time from the initial state $\mathbf{x}(0) = (0, 0)$, corresponding to $y(0) = \dot{y}(0) = 0$, to a target final state $\mathbf{x}(t_f) = (L, 0)$, corresponding to a full stop $y(t_f) = L$, $\dot{y}(t_f) = 0$ at the far end of the drag strip. The performance measure to be minimized is the travel time

$$t_f = \int_0^{t_f} dt$$

and the control constraints are of the form

$$u_{\min} \leq u(t) \leq u_{\max}.$$

Solution techniques for optimal control problems are presented in Chapter 7. For this example, however, it is intuitively clear that the optimal control is maximum acceleration followed by maximum braking, that is,

$$u(t) = \begin{cases} u_{\max} & \text{if } t < t_s \\ u_{\min} & \text{if } t > t_s \end{cases}$$

with the switching time t_s, that is, the onset of braking, being chosen to bring the vehicle to rest precisely at the end of the drag strip. Note that the switching time will depend on L. A complete solution to this problem involves finding the optimal control as a function of the current state $u(\mathbf{x})$, rather than as a function of time $u(t)$. ■

Control System Concepts

For a given specified control law $\mathbf{u}(\mathbf{x})$ and a given initial point $\mathbf{x}$ in state space, we may think of the solution $\mathbf{x}(t)$ of (1.1-1) as a function that transfers the initial state at time zero to some other state at time t. Our study of nonlinear control systems focuses on the stability, controllability, and optimization properties of this solution. In particular, we are concerned with the relationship between a specified control law and the region of the state space that can be transferred to a designated target set under that control law. In this context:

Stability Is concerned with determining those initial states that, under a specified feedback control law, will be transferred to and maintained at the specified target set.

Controllability Is concerned with determining those initial states where a feedback control exists that will transfer the state to a specified target.

Optimality Has to do with finding the "best" feedback controller, based on some specified performance measure, which will transfer any controllable state to a specified target.

Note that stability implies controllability; however, controllability need not imply stability since controllability does not require that the system be maintained at the target. The remainder of this chapter is devoted to a discussion of the nature of solutions to nonlinear dynamical systems with the control input function $\mathbf{u}(\cdot)$ already specified so that the right-hand side of (1.1-1) is simply $\mathbf{f}(\mathbf{x})$ or $\mathbf{f}(\mathbf{x}, t)$. Chapter 2 removes the assumption that $\mathbf{u}(\cdot)$ has been specified and specifically deals with some of the aspects of nonlinear control systems. Chapter 3 introduces nonlinear optimization techniques needed for the control design methods introduced in the chapters that follow. Stability is discussed in Chapters 4 and 5. Controllability is considered in Chapters 6 and optimality in Chapters 7 and 8. Finally, in Chapter 9, the fundamental concepts are