ELECTROMAGNETIC FIELDS

Theory and Applications

Volume I—Mapping of Fields

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To

CH. SONYA WEBER

whose unselfish and inspiring attitude

made this book possible

PREFACE

The subject of electromagnetic theory as formulated by James Clerk Maxwell has become classical, and it is hardly possible to add basically new material. Yet, the astounding developments in physics and electrical engineering have shown clearly that the utilization of electromagnetic phenomena has not reached the point of saturation. For this reason, a book giving a rather comprehensive survey of the methods of analysis and of results obtained with them should prove of value to the student and the teacher in advanced courses as well as to the professional engineer and the physicist in the research and development laboratory.

Now, it has become clear that the scope of electromagnetic theory and its applications to problems of interest to the engineer, the physicist, and the applied mathematician is much too great to be covered in one volume of practical size. Fortunately, the subject may be rather naturally divided into two fundamental branches: one dealing with static electric and magnetic fields and leading to methods of solving the family of potential equations in various forms; the other dealing with the dynamic interaction of electric and magnetic fields and leading to methods of solving the family of wave equations in various forms. This division has been followed here, and the first volume presents a survey of the methods of mapping the distribution of static electric and magnetic fields.

Many authors who have dealt with this subject have had a tendency to present a particular version or viewpoint, or to emphasize one particular method of analysis. Admirable as such treatises may be by themselves, they are less suitable for use in graduate courses where emphasis must lie upon guidance to a basic understanding of different ways of reasoning and of formulating ideas. Graduate study must concern itself primarily with basic concepts, of which there are always but a few; it should demonstrate the connection between them through generic principles and should lead to the critical understanding of their full implications. Only when this aim has been reached—illuminated by constructive applications—can one speak of mastery of the subject.

In order for the graduate teacher in engineering or applied science to achieve this aim it is imperative that the basic facts upon which theory rests and from which it receives support and confirmation be presented in broad strokes; and it demands a presentation not of mathematical detail of existence theorems, but of illustrative examples which demonstrate the variety of formulations and applications of the few principles, so frequently disguised under the names of specific "laws." Of course, as in any quantitative treatment, mathematics must be used as the most precise and most satisfying means of expression, and it is quite necessary to recognize, and conveniently refer to, the proofs of existence and of uniqueness of solutions which have been developed by pure mathematicians. The burden of this great debt to the mathematicians has been lightened only because of the tremendous stimulation of mathematical research through the incessant need for new solutions.

The recognition of the fundamental importance in electrical engineering of well-founded field concepts in all advanced development and design, as well as in research, has led to the requirement of a course in electromagnetic theory in nearly all major graduate schools. Where this course is given by the Department of Physics, mathematical theory may predominate, and where it is given by the Department of Electrical Engineering, design information may be emphasized. In order to combine the emphasis on the basic aspects common to all potential fields with a *comprehensive* treatment of the available *analytical and practical* methods of field plotting, this volume has been organized in a somewhat unconventional manner. Instead of the usual vertical division into electrostatics, magnetostatics, and electrokinetics, a horizontal division of the subject matter is used. Thus, all the physical relationships are established first, and the methods of actually obtaining static field distributions are demonstrated subsequently. This avoids considerable repetition and leads to a clearer understanding of the fact that methods of analysis are independent of the specific branch of application, and that nomenclature is frequently accidental and by no means the essence of knowledge. It is, of course, assumed that the reader possesses a general knowledge of the electromagnetic field as normally gained in a pertinent undergraduate course and that he is familiar with the principles of vector notation. To be sure, the field-mapping methods are generally formulated in specific coordinate systems as conditioned by the geometry of the fields studied; but the vector notation proves of definite advantage for the presentation of the basic relations in electric, magnetic, and other fields, as treated in the first three chapters.

Following the summary of the basic physical relations, the comparative physical quantities in six branches of physics and engineering are listed in table 9.1, which serves as the key for the translation of field solutions in any one branch into solutions of analogous problems in the other branches. Chapter 4 deals with the simple applications of the superposition principle, such as systems of point and line charges, line currents, and simple geometries of spatially distributed charges and currents. For more complicated geometries, it is frequently—though not always—simplest to map the field distributions experimentally; the experimental methods that have been used successfully are described in Chapter 5, including the analogies utilized in the electrolytic trough. As alternatives to the experimental procedure, graphical and numerical field-plotting methods are taken up in Chapter 6 with emphasis on the practical phases of actual applications; rather extensive treatments are given of the uses of electrical and magnetic images and of inversion—methods which are not always sufficiently emphasized. Next, the use of analytic functions for the solutions of two-dimensional field problems is shown in Chapter 7, and in particular the extremely powerful methods of "conjugate" functions and of conformal mapping, which are amply demonstrated. Finally, Chapter 8 gives the mathematical treatment of three-dimensional field problems, involving by necessity a thorough discussion of orthogonal coordinate systems that is supported by many illustrations which—it is hoped—will make it easier to visualize clearly the geometrical aspects.

In order to aid a teacher in organizing the material into feasible courses, several suggestions are offered in line with courses which have been taught by the author. For the first part of a course on Electromagnetic Theory dealing with static fields one might combine Chapters 1 and 2 and section 8 of Chapter 3 with selected examples from Chapters 4 and 6 and section 25 of Chapter 7. For a one-semester course in Applications of Functions of a Complex

Variable one might take the material of Chapter 7, sections 25 to 28. Again, for a course in Classical Boundary Value Problems dealing with the potential equation, one might combine section 9 with the two-dimensional applications in section 29 and Chapter 8 on three-dimensional applications. To satisfy individual requirements, still other combinations are possible.

Fortunately, it is no longer necessary to apologize for the use of the rationalized MKS system of units in a book dealing with electromagnetic theory and its applications. There might, however, be criticism of the fact that the engineering notation $\sqrt{-1} = j$ has been carried into the classical realm of analytic functions. This fact should not be construed as a serious offense, for notation is not the essence; rather, it should be taken for what it is, a choice necessitated by the severe conflict of $i = \sqrt{-1}$ with the symbol for the instantaneous value of current $i = I_m \sin \omega t$, which is internationally standardized and customarily defined as the imaginary part of $I_m \epsilon^{j\omega t}$, with the effective (root mean square) value $I = I_m/\sqrt{2}$, all of which will occur frequently in Volume II.

The original suggestion of a small volume on mapping of fields was made in 1935 by the late V. Karapetoff, Cornell University, as Chairman of a Sub-Committee on Monographs of the Committee on Electrical Insulation, National Research Council. A crude draft of the manuscript had the benefit of his criticisms as well as those of J. F. H. Douglas, Marquette University, and H. Poritzky, Schenectady. World War II interfered with the plans for this monograph. Furthermore, the various graduate courses given by the author at the Polytechnic Institute of Brooklyn slowly changed the original conception of the monograph to the rather different one of this volume. The contact with many graduate students has had a strong educational influence upon me, and I wish to acknowledge to them my deep appreciation. Certainly through their persistent questioning and their gratifying response, they have made teaching the delightful profession it is. I am greatly indebted also to a number of my colleagues, especially to Paul Mariotti, who assisted in the preparation of the drawings; to Professor William R. MacLean, who read parts of the manuscript and made constructive suggestions; and to Professor Charles A. Hachemeister, who read most of the proof and made many helpful comments. As the preface occupies a prominent place in the book, I am very happy and grateful that I could enlist for its composition the invaluable assistance of Professor Leo E. Saidla, head of the Department of English. Finally, I take great pleasure in acknowledging the encouragement and support which I received from President Harry S. Rogers in writing this book.

Ernst Weber

Brooklyn, New York
April, 1950

NOTES FOR THE READER

The symbols of field quantities are tabulated in Appendix 1.

To transform the relations from the rationalized MKS unit system to other unit systems, consult Appendix 2.

A brief review of vector analysis is given in Appendix 3.

Equations are numbered consecutively in each section; references to equations in different sections carry the section number, thus (5·4) means equation (4) in section 5.

The Bibliography in Appendix 4 lists only books to which several references are made in the text; such references, e.g. Attwood,[A2] p. 243, give the page and the author, the superscript indicating number 2 of section A of the Bibliography.

CONTENTS

1. THE ELECTROSTATIC FIELD

1. FUNDAMENTAL RELATIONS IN THE ELECTROSTATIC FIELD

From primitive observations, electrostatics divides all materials into only two groups, conductors and insulators. The first group is endowed with infinite mobility of electric charges such that any redistribution occurs in an unobservably short time. The second group has zero mobility of electric charges; any redistribution occurs in an uninterestingly long time. Admittedly, this is a radical division, but it leads to a much simpler theory of the electrostatic field than would be possible otherwise. In addition, the results are of direct practical value, and deviations in specific cases can readily be indicated.

The basic quantitative relationship of electrostatics is Coulomb's law of force action between two charges Q_1 and Q_2,

$$F_e = \frac{1}{4\pi\varepsilon} \cdot \frac{Q_1 Q_2}{r^2} \tag{1}$$

The charges are assumed to be confined to very small regions (point charges) so that the distance r can be identified as the distance between centers, and ε is the absolute dielectric constant of the homogeneous infinitely extended medium in which the force F_e is measured; one usually expresses $\varepsilon = \varepsilon_v \varepsilon_r$, where ε_v is the absolute dielectric constant of free space (vacuum) (see Appendix 2 for unit relations). The relative dielectric constant ε_r is the numeric value generally found in the tables of material constants. Throughout this volume, only *isotropic* dielectric media will be considered, so that ε is always assumed to be independent of direction.

The study of electrostatics, then, is primarily concerned with the equilibrium distribution of charges on the various conductors comprising a particular system, under the influence of this Coulomb force. If the charge Q_2 is very small, so that it causes a negligible and only local distortion of the field of charge Q_1, it can be used as a *probe* for the exploration of the force field of charge Q_1. From (1), the limit value for vanishing Q_2

$$\lim_{Q_2 \to 0} \frac{F_e}{Q_2} = E_1 = \frac{1}{4\pi\varepsilon} \frac{Q_1}{r^2} \tag{2}$$

is then interpreted as the electric intensity or *field strength* of charge Q_1. In the case of a single positive point charge, the field strength E has radial, outward direction, in vector notation (see Appendix 3 for a brief review of vector analysis)

$$\mathbf{E}_1 = \frac{1}{4\pi\varepsilon} \frac{Q_1}{r^3} \mathbf{r} \tag{3}$$

where $\mathbf{r}/r$ serves to indicate the radial direction. In the case of any general distribution of a total charge Q, one can subdivide it into small elements Q_α, consider each to be a point charge, and by use of the principle of superposition obtain the resultant field vector $\mathbf{E}$ at any point P

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \sum_{\alpha=1}^{n} \frac{Q_\alpha}{r_\alpha{}^3} \mathbf{r}_\alpha \tag{4}$$

where the $\mathbf{r}_\alpha$ are the radius vectors from the charges Q_α to the point P.

If one places a very small charge $\bar{Q}$ into the electric field of any number of charges Q_α, and if one is permitted to disregard the effect of $\bar{Q}$ upon the charge distribution of the Q_α, then such a small charge is again called a probe charge, since it can well serve to probe or explore the electric field of the charge assembly by means of the force action upon it, which is given by (2) as, $\bar{Q}\mathbf{E}$. Left free to move, at very low speed, this probe charge will trace the direction of the vector $\mathbf{E}$ in space and the path described is called a *field line* or also line of force; it has the vector $\mathbf{E}$ everywhere as tangent. Defining the path element as $d\mathbf{s}$, its components dx, dy, dz must be proportional to those of $\mathbf{E}$, so that

$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z} \tag{5}$$

which is the differential equation of the field lines. Since for any point charge the field lines diverge radially for positive sign and converge radially for negative sign, there can be no closed field lines.

Carrying a small charge Q_2 over any finite path $\overline{P_1P_2}$ within

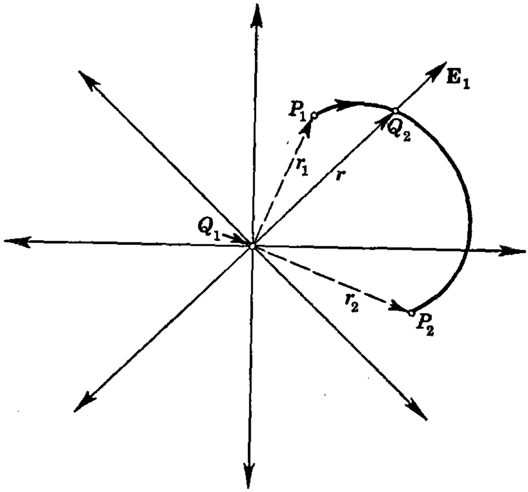

FIG. 1·1 Electrostatic Field of a Single Point Charge.

the field of a single point charge located at the origin as in Fig. 1·1 requires the work

$$W = \int_{P_1}^{P_2} \mathbf{F}_e \cdot d\mathbf{s} = Q_2 \int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{s} \tag{6}$$

However, $\mathbf{E}$ has only radial direction, so that $\mathbf{E} \cdot d\mathbf{s} = E\, dr$ and hence with the use of (2)

$$W = Q_2 Q_1 \frac{1}{4\pi\varepsilon} \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

The work is thus independent of the path; it depends only on the end points, and is therefore zero for a closed path. One can immediately generalize this fact because of (4) and characterize the electrostatic field as a *conservative* field. This means also, as seen from (6), that the line integral of the vector $\mathbf{E}$ vanishes for every closed path; all field lines emanate from, and terminate on, charges.

On the other hand, a line integral is independent of the path if the integrand represents a complete differential. This requires that the components of **E** can be identified as the derivatives of a single, scalar function Φ, the *electrostatic potential.* Vectorially,

$$\mathbf{E} = -\text{grad}\ \Phi = -\nabla\Phi \tag{7}$$

and for the single point charge the potential function becomes at once from the above

$$\Phi = \frac{Q}{4\pi\varepsilon} \cdot \frac{1}{r} \tag{8}$$

where r is the distance from the charge center. Since (7) defines only the derivatives of Φ, any arbitrary constant could be added in (8). For any number of point charges in a single medium ε, superposition again holds and one has

$$\Phi = \frac{1}{4\pi\varepsilon} \sum_{\alpha=1}^{n} \frac{Q_\alpha}{r_\alpha} \tag{9}$$

subject to some arbitrary constant. Obviously, the scalar summation involved in (9) is more convenient than the vector sum required in (4). The surfaces obtained for constant values of potential are called *equipotential surfaces* and are equally as characteristic for the field structure as the field lines; in fact, they form with the latter an orthogonal system of surfaces and lines. The objective of *field mapping* is precisely the evaluation of this orthogonal field geometry in quantitative terms.

Returning to the concepts of conductors and insulators in the ideal sense, it must be clear at once that conductors can have charges only on the surface and must have constant potential throughout their interior; any potential variation would cause a field vector and, therefore, a force action until a surface charge distribution is established which maintains constant potential. Conversely, any charge in the interior of the conductor would be a source of field lines which could be maintained only by a potential difference. Any conductor surface is, therefore, an equipotential surface, and the field lines terminate perpendicularly to it.

An insulator or dielectric, on the other hand, will normally not carry any charges at all; it will serve primarily to separate charged conductors. In certain instances, space charges produced by thermionic or other emission, by glow discharges, or by arcs can exist within insulators. Assume again a single point charge Q in a homogeneous dielectric; then (2) will give the field strength as depending on the dielectric constant ε. However, the quantity $\varepsilon E = D$ is independent of the dielectric and appears as density of the charge were it distributed uniformly over the surface of a sphere of radius r. It is designated as a vector called *dielectric flux density* (or electric displacement),

$$\mathbf{D} = \varepsilon\mathbf{E} \tag{10}$$

for homogeneous dielectrics for which ε is a constant. Again generalizing for many point charges, the integral of $\mathbf{D}\cdot\mathbf{n}$ over any closed surface S gives then the sum of all charges contained within this surface (Gauss's dielectric flux theorem),

$$\iint_S \mathbf{D}\cdot\mathbf{n}\, dS = \Sigma Q_\alpha \tag{11}$$

no matter what their distribution. For a continuous space charge distribution of finite volume density ρ, the right-hand side of (11) is better written as the integral over the volume τ bounded by the closed surface S. Transforming also the left-hand surface integral, one has then

$$\iiint (\operatorname{div} \mathbf{D})\, d\tau = \iiint \nabla\cdot\mathbf{D}\, d\tau = \iiint \rho\, d\tau$$

Applying this relation to very small dimensions one concludes that

$$\operatorname{div} \mathbf{D} = \nabla\cdot\mathbf{D} = \rho \tag{12}$$

or any space charge is a source or sink of the vector $\mathbf{D}$ independent of the dielectric medium.

In isotropic dielectrics, with no space charge, div $\mathbf{D} = 0$ and the vectors $\mathbf{E}$ and $\mathbf{D}$ have the same direction according to (10), so that the field lines of the vector $\mathbf{E}$ can also be interpreted as *dielectric flux lines*, being tangential to the vector $\mathbf{D}$ at every point. Since the total dielectric flux coming from a charge Q is numerically equal to the charge, one can conceive of a chosen number of *flux lines* to represent the charge value. In the case of several charges, the flux lines will then quantitatively represent the dielectric flux distribution. For conductors of arbitrary shape in a uniform dielectric, **D** is normal to the surface and its value is identical with the surface density of charge,

$$D \equiv D_n = \sigma \tag{13}$$

This follows from (11) since no electric field can exist within the conductor. The flux lines bounding a finite surface element δS which carries a charge $\sigma\,\delta S = \delta Q$ form a *flux tube* which will lead to an element $\delta S'$ on another conductor where it delimits a charge $(-\delta Q) = \sigma'\,\delta S'$. These flux tubes are a valuable aid in the visualization of the field geometry if no space charge is present (see Fig. 3·1).

2· ANALYTICAL THEORY OF THE ELECTROSTATIC FIELD

On the basis of section 1, the general problem of electrostatics can be formulated as the evaluation of the field distribution in dielectrics and of the surface charge distribution on conductors subject to certain known potential or field strength values designated as boundary conditions. Actually, potential values as such are arbitrary, as pointed out in section 1; only potential differences can be measured, so that, to any solution of the electrostatic potential function, an arbitrary constant could be added. Usually, one chooses some reference conductor such as ground to be of zero potential in order to simplify numerical computations.

As already indicated, solution of electrostatic field problems usually becomes more convenient with the use of the scalar electrostatic potential. As defined in (1·7), the electric field strength **E** can be expressed as the negative gradient of the potential. Introducing this into relation (1·10) and then substituting into (1·12), one has

$$\nabla\cdot\mathbf{D} = -\nabla\cdot(\varepsilon\nabla\Phi) = \rho \tag{1}$$

or also [see Appendix 3, (21)]

$$\nabla\Phi\cdot\nabla\varepsilon + \varepsilon\nabla^2\Phi = -\rho \tag{1a}$$

This represents the most general differential equation for an inhomogeneous isotropic dielectric, wherein the variation of ε must be known. Though this general case has little practical value, it readily permits specialization for several important cases.