RANDOM SIGNALS ESTIMATION AND IDENTIFICATION

ANALYSIS AND APPLICATIONS

Nirode Mohanty

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To
Dr. Richard Ernest Bellman,
Professor of Mathematics, Electrical Engineering
and Medicine
University of Southern California

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Preface

The techniques used for the extraction of information from received or observed signals are applicable in many diverse areas such as radar, sonar, communications, geophysics, remote sensing, acoustics, meteorology, medical imaging systems, and electronics warfare. The received signal is usually disturbed by thermal, electrical, atmospheric, channel, or intentional interferences. The received signal cannot be predicted deterministically, so that statistical methods are needed to describe the signal. In general, therefore, any received signal is analyzed as a random signal or process.

The purpose of this book is to provide an elementary introduction to random signal analysis, estimation, filtering, and identification. The emphasis of the book is on the computational aspects as well as presentation of common analytical tools for systems involving random signals. The book covers random processes, stationary signals, spectral analysis, estimation, optimization, detection, spectrum estimation, prediction, filtering, and identification. The book is addressed to practicing engineers and scientists. It can be used as a text for courses in the areas of random processes, estimation theory, and system identification by undergraduates and graduate students in engineering and science with some background in probability and linear algebra.

Part of the book has been used by the author while teaching at State University of New York at Buffalo and California State University at Long Beach. Some of the algorithms presented in this book have been successfully applied to industrial projects.

Random signal processes are dealt with in Chapter 1, with emphasis on Gaussian, Brownian, Poisson, and Markov processes. Mean square calculus and renewal theory are briefly discussed.

Chapter 2 is devoted to stationary random processes along with spectral analysis, narrow-band processes, the Karhunen–Loeve expansion, entropy, zero crossing detectors, and nonlinear systems with random inputs.

A comprehensive account of estimation theory is given in Chapter 3. Maximum-likelihood estimation, mean square estimation, maximum a priori estimation, the Cramer–Rao bound, and interval estimation are covered. Optimum filters including Wiener filtering are discussed for white and colored noise. Elements of signal detection are included in this chapter.

Spectral estimation methods including the periodogram, autoregressive, maximum entropy, maximum likelihood, Pisarenko, and Prony methods are discussed in Chapter 4. Adaptive spectral density estimation and cross-spectral estimation are briefly described.

Chapter 5 deals with prediction, filtering, and identification. Kalman filtering, extended Kalman filtering, and recursive identification algorithms are given in this chapter.

Each chapter contains a set of worked-out problems, exercises, and bibliographic notes for further study.

Appendix 1 contains a brief survey of linear systems analysis, Z transforms, sampling theory, matrices, and orthogonal transforms. Elementary probability, random variables and distribution theory are treated in Appendix 2. Appendix 3 deals with the stochastic integral. Elements of Hilbert space are discussed in Appendix 4.

The book concludes with a detailed bibliography and reference list. Many of the publications listed therein are suitable for further study.

The author would like to thank the authors of the books and papers cited in the bibliography, which have greatly helped to write this book. He would also thank his friends and the management at The Aerospace Corporation for their encouragement and support. Readers are invited to send their comments and corrections which will be appreciated very much.

Los Angeles, California. Nirode Mohanty

Notations

Capital letters X, Y, Z etc. denote random variables or matrices. However, N is used also as a number of samples. Bold letters X, x, A, a etc. indicate vectors or matrices. x, y, z etc. stand for real numbers or dummy variables.

$P[X \leqslant x]$ = Probability of r.v. X is less than equal to x.

$\mathbf{X} = (X_1, X_2, \ldots, X_n)$
(n dimensional vector)

$F_X(x)$ = Probability distribution function of r.v. X, x is a real number
$\triangleq F(X)$

$f_X(x)$ = Probability density function of r.v. X, x is a real number
$\triangleq f(X)$

$f_{\mathbf{X}}(\mathbf{x}|s)$ = Conditional probability density function of random vector $\mathbf{X}$, $\mathbf{x}$ is a vector quantity given s
$\triangleq f(\mathbf{X}|s)$

$E(X)$ = Expected value of r.v. X, E is the expected operator

$\mathrm{Var}(X)$ = Variance of r.v. X.

$X(f) = \mathscr{F}(x(t))$
$\triangleq$ Fourier transform of $x(t)$

$X^*(f)$ = Complex conjugate of $X(f)$

$X(s) = \mathscr{L}(x(t))$
$\triangleq$ Laplace transform of $x(t)$

$X(z) \triangleq$ Z transform of $\{X_k\}$ or $\{X(k)\}$

A^{-1} = Inverse of matrix A

X' = Transpose of vector X or matrix X

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1
Random Signals

1.0. INTRODUCTION

Signals whose parameters are random variables are called random signals. Random signals are random or stochasic processes. We will extend the concept of random samples to the sampling of a random process. The characterization of a random process is given in terms of its time-dependent distribution and density functions. The characterization and classification is described in Section 1.1. Two important characteristics of a random process from the point of view of applications are its correlation and covariance functions (Section 1.2). Three important random processes, the Gaussian, Brownian, and Poisson processes, are discussed in Sections 1.3 and 1.4. Mean-square calculus for random processes is presented in Section 1.5. Markov processes and renewal processes are discussed in Sections 1.6 and 1.7. The chapter concludes with bibliographical notes (Section 1.8).

1.1. CHARACTERIZATION AND CLASSIFICATION

A signal is called random if its values or observed values are random variables. A signal of this type is also called a random function or process, or stochastic signal or process. A *random signal* $\{X_t, t \in T\}$ is a family of random variables with the indexed parameter t and defined on a common probability space $(\Omega, \mathscr{F}, P)$. The signal is denoted by $X(t, w) \triangleq X(t)$ or X_t. If the index set T is of the form $T = (0, \pm 1, \pm 2, \ldots)$ or $T = (0, 1, 2, \ldots)$, then the signal is called a *discrete parameter process*. When $T = \{t: -\infty < t < \infty\}$ or $T = \{t: t \geqslant 0\}$, the signal is called a *continuous parameter process*. Note that for a given w, $\{X(w): t \in T\}$ is a function defined on T and is called a *sample function* or realization of the process. For each t, X_t is a *random variable* (r.v.), a measurable function. Let $t_1 < t_2 < t_3 < \cdots < t_n$ be a subset of T; then $X(t_1) \triangleq X_1$, $X(t_2) \triangleq X_2$, $X(t_3) \triangleq X_3$, $X(t_n) \triangleq X_n$ are the samples or observable values of the process or signal. In many practical problems, particularly in the extraction of some signal parameters, it is desired to extract or estimate the entire signal from some number of finite samples. It is possible to characterize a random signal by the joint distribution function of every finite family of r.v.s $\{X_1, \ldots, X_n\}$. We define the joint distribution function as

$$F_{\mathbf{x}}(x_1, x_2, \ldots, x_n, t_1, \ldots, t_n) = P[X_1 \leqslant x_1, X_2 \leqslant x_2, \ldots, X_n \leqslant x_n] \quad (1.1\text{-}1)$$

where

$$t_1 < t_2 < t_3 < \cdots < t_n, \; \mathbf{X} = (X_1, X_2, \ldots, X_n)$$

Here $x_1, x_2, \ldots, x_n$ are observed values, whereas $X_1, X_2, \ldots, X_n$ are r.v.s. The joint distribution function must satisfy the following two conditions:

1. The *symmetry condition*:

$$F_{\mathbf{x}}(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = F_{\mathbf{x}}(x_{j_1}, \ldots, x_{j_n}; t_{j_1}, \ldots, t_{j_n}) \qquad (1.1\text{-}2)$$

where $j_1, j_2, \ldots, j_n$ is any permutation of indices $1, 2, \ldots, n$. For example, when $n = 3$

$$\begin{aligned} F_{\mathbf{x}}(x_1, x_2, x_3; t_1, t_2, t_3) &= F_{\mathbf{x}}(x_2, x_1, x_3; t_2, t_1, t_3) \\ &= F_{\mathbf{x}}(x_3, x_1, x_2; t_3, t_1, t_2) \\ &= \text{etc.} \end{aligned}$$

2. *The compatibility condition:*

$$F_{\mathbf{x}}(x_1, x_j; t_1, \ldots, t_j) = F_{\mathbf{x}}(x_1, x_2, x_3, \ldots, x_j, \infty, \ldots, \infty; t_1, \ldots, t_j, \ldots) \qquad (1.1\text{-}3)$$

For $n = 3$,

$$F_{\mathbf{x}}(x_1, x_2, \infty; t_1, t_2, t_3) = F_{\mathbf{x}}(x_1, x_2; t_1, t_2)$$

If the joint distribution $F_x(x_1, x_2 \ldots; t_1, t_2 \ldots)$ is differentiable, then it can be represented as

$$F_{\mathbf{x}}(x_1, \ldots, x_n; \mathbf{t}) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \int_{-\infty}^{x_n} f_{\mathbf{x}}(y_1, \ldots, y_n; \mathbf{t})\, dy_1 \ldots dy_n \qquad (1.1\text{-}4)$$

$$(n = \text{fold}) \quad \mathbf{t} = (t_1, t_2, \ldots, t_n) \in T$$

where, $f_{\mathbf{x}}(x_1, \ldots, x_n; \mathbf{t})$ is called the joint probability density function (p.d.f.) of the signal X_t and it has the properties

1. $$f_x(x_1, \ldots, x_n; \mathbf{t}) \geq 0 \tag{1.1-5}$$

2. $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\mathbf{x}}(x_1, \ldots, x_n; \mathbf{t})\, dx_1 \ldots dx_n = 1$$

$(n = \text{fold})$

for all $\mathbf{t} = (t_1, \ldots, t_n) \in T$. The notation for the joint distribution function and joint probability density function uses the vector $\mathbf{t}$ to emphasize that both functions vary with $\mathbf{t} = (t_1, \ldots, t_n)$, the ordered set of indices. The random signal can also be described by its joint characteristic function. The *joint characteristic function* is defined by

$$\phi_{\mathbf{x}}(v_1, \ldots, v_n; t_1, \ldots, t_n) = E\left\{\exp\left[i \sum_{j=1}^{n} x_{t_j} v_j\right]\right\} \tag{1.1-6}$$

where E is the expected operator.† The characteristic function depends also on $t_1, t_2, \ldots, t_n$. The joint probability density function and characteristic function form a Fourier transform pair. For a continuous random signal having a joint probability density function, we can define the characteristic function as

$$\begin{aligned}\phi_{\mathbf{x}}&(v_1, v_2, \ldots, v_n; t_1, \ldots, t_n)\\ &= E[\exp(iv_1 x_1 + iv_2 x_2 + \cdots + iv_n x_n)]\\ &\triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\exp\left(i \sum_{j=1}^{n} x_{t_j} v_j\right)\right] f_x(x_1, \ldots, x_n, t_1, \ldots, t_n)\, dx_1 \ldots dx_n\end{aligned}$$

$(n = \text{fold})$ (1.1-7)

For a discrete-type stochastic process, we define the characteristic function as

---

† $EX = \sum_{i=1}^{n} x_i P(X = x_i)$, if $\{x_i\}$ are discrete random variables

$= \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x)\, dx$, if $\{x_i\}$ are continuous random variables

$= \int x\, dF_{\mathbf{x}}(x)$, $F_{\mathbf{x}}(x)$ is the distribution function of r.v. X.

$$\phi_{\mathbf{x}}(v_1,\ldots,v_n;t_1,\ldots,t_n)$$
$$= \sum_{x_1}\cdots\sum_{x_j}\left[\exp\left(i\sum_{j=1}^{n} x_{t_j}v_j\right)\right]P[X_{t_1}=x_1, X_{t_2}=x_2,\ldots,X_{t_n}=x_n] \tag{1.1-8}$$

The general form for the characteristic function for a random signal is given by

$$\phi_{\mathbf{x}}(v_1,\ldots,v_n;t_1,\ldots,t_n)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[i(v_1x_1+v_2x_2+\cdots+v_nx_n)]\,dF_{\mathbf{x}}(x_1,\ldots,x_n;t_1,t_2,\ldots,t_n)$$
$$(n = \text{fold}) \tag{1.1-9}$$

where $F_{\mathbf{x}}(x_1,\ldots,x_n;t_1,\ldots,t_n)$ is the joint distribution function.

EXAMPLE 1.1-1. BASEBAND SIGNAL. Consider the baseband signal of the form:

$$X(t) = \begin{cases} 1q(t) & \text{with probability } p \\ -1q(t) & \text{with probability } 1-p. \end{cases} \tag{1.1-10}$$

where $q(t) = [u(t-(n-1)T) - u(t-nT)]$ for some integer n and paremeter T.* This process is known as *Bernoulli Process.* Find the density function.

Solution. The two consecutive values $X(t_1) = x_1$ and $X(t_2) = x_2$ can be any one of the pairs $(-1,-1)$, $(-1,1)$, $(1,-1)$, $(1,1)$. The ensemble of possible signals for two samples is shown in Figure 1.1-1.

The joint probability distribution in sample space of $(-1,-1)$, $(-1,1)$, $(1,-1)$, $(1,+1)$ is given by

$$P[X_1 = -1, X_2 = 1] = p^2$$
$$P[X_1 = 1, X_2 = -1] = p(1-p)$$

* $u(t)$ is a unit step function, i.e.,

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0, & t < 0 \end{cases}$$

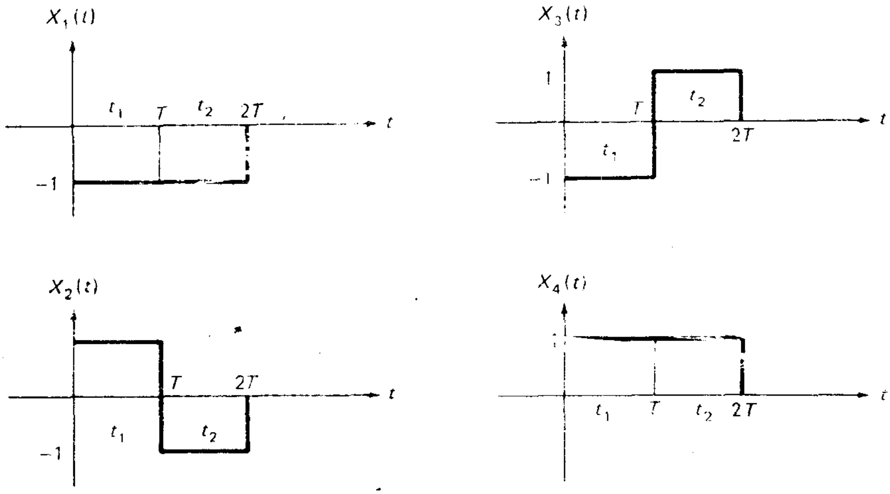

Fig. 1.1-1. Ensemble of baseband signal. $X_1(t)$, $X_2(t)$, $X_3(t)$, and $X_4(t)$ are the four sample functions of $X(t)$.

$$P[X_1 = -1, X_2 = 1] = (1 - p)p$$
$$P[X_1 = -1, X_2 = -1] = (1 - p)^2 \qquad (1.1\text{-}11)$$

The joint density of $\mathbf{X} = (x_1, x_2)$ is given by**

$$f_{\mathbf{x}}(x_1, x_2, t_1 t_2) = p^2\delta(x_1 - 1)\delta(x_2 - 1) + p(1 - p)\delta(x_1 + 1)\delta(x_2 - 1)$$
$$+ (1 - p)p\delta(x_1 - 1)\delta(x_2 + 1) + (1 - p)^2\delta(x_1 + 1)\delta(x_2 + 1) \qquad (1.1\text{-}12)$$

It can be verified that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\mathbf{x}}(x_1, x_2)\, dx_1\, dx_2 = p^2 + p(1 - p) + p(1 - p) + (1 - p)^2 = 1.$$

EXAMPLE 1.1-2. DIGITAL MODULATION. Consider a signal of the form

$$X(t) = A\cos(\omega_c t + \pi/2)$$

** $\delta(x)$ is the Dirac delta function defined as

$$\delta(x) = \begin{cases} \dfrac{1}{A}, & -\dfrac{A}{2} \leqslant x \leqslant \dfrac{A}{2} \\ 0, & \text{otherwise,} \end{cases}$$

When $A \to 0$. For any continuous function $X(t)$, we have $\int_{-\infty}^{\infty} X(t)\delta(t - t_0)\,dt = X(t_0)$.

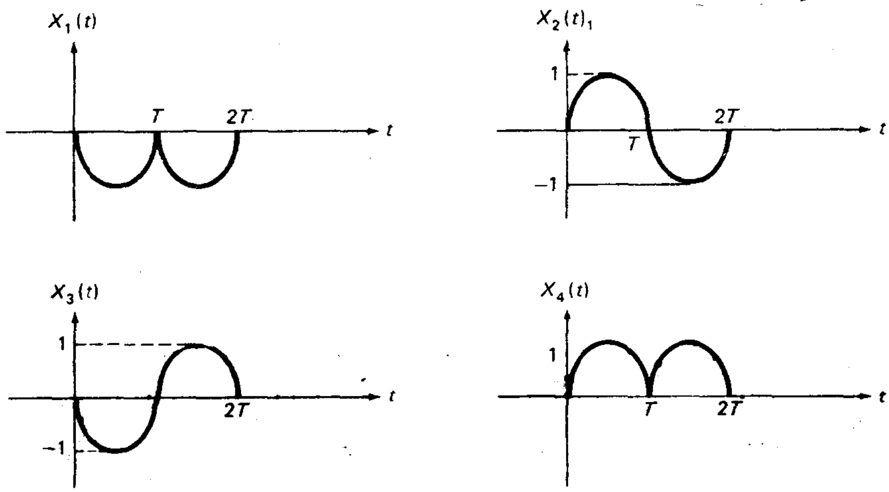

Fig. 1.1-2. **Sample functions of ASK signals for two consecutive periods.**

where

$$0 \leqslant t \leqslant T, \quad f_c = \frac{1}{2T}, \quad \omega_c = 2\pi f_c \tag{1.1-13}$$

where A is a random amplitude and is given by

$$A = \begin{cases} 1q(t) & \text{with probability } p \\ -1q(t) & \text{with probability } 1-p. \end{cases}$$

where $q(t) = [u(t-(n-1)T) - u(t-nT)]$ for integer n and parameter T, and $u(t)$ is a unit step function. Derive the joint p.d.f.

Solution. This signal is called the *Amplitude Shift Keying* (*ASK*). The sample functions are shown in Figure 1.1-2 for two consecutive periods. The joint p.d.f. is same as that given in Example 1.1-2.

EXAMPLE 1.1-3. Consider a signal which is the sum of two sinusoidal signals, i.e.,

$$X(t) = A\cos\omega_1 t + B\cos\omega_2 t,$$
$$\omega_i = 2\pi f_i, \quad i = 1, 2 \tag{1.1-14}$$

where A and B are independent Gaussian random variables with p.d.f.s

$$f_A(a) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left[-a^2/2\sigma_1^2\right]$$

$$f_B(b) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left[-b^2/2\sigma_2^2\right]$$

Find the p.d.f. of $X(t)$.

Solution. Since $X(t)$ is a linear combination of two Gaussian r.v.s, $X(t)$ is also Gaussian. Note that $E(A) = E(B) = 0$.

$$\begin{aligned}
\mathrm{Var}(A) &= \sigma_1^2, \quad \mathrm{Var}(B) = \sigma_2^2 \\
E[X(t)] &= E[A\cos(\omega_1 t) + B\cos(\omega_2 t)] \\
&= E[A]\cos(\omega_1 t) + E[B]\cos(\omega_2 t) \\
&= 0\cos(\omega_1 t) + 0\cos(\omega_2 t) \\
&= 0 \\
E[X^2(t)] &= E[A^2\cos^2(\omega_1 t) + B^2\cos^2(\omega_2 t) \\
&\quad + 2AB\cos(\omega_1 t)\cdot\cos(\omega_2 t)] \\
&= E[A^2]\cos(\omega_1 t) + E[B^2]\cos^2(\omega_2 t) \\
&\quad + 2E[A]\cdot E[B]\cos(\omega_1 t)\cos(\omega_2 t) \\
&= \sigma_1^2\cos^2(\omega_1 t) + \sigma_2^2\cos^2(\omega_2 t) \\
&\quad + 0\cdot 0\cdot 2\cos(\omega_1 t)\cdot\cos(\omega_2 t)
\end{aligned}$$

Therefore the p.d.f. of $X(t)$ is

$$\begin{aligned}
f_x(x,t) &= \frac{1}{\sqrt{2\pi(\sigma_1^2\cos^2(\omega_1 t) + \sigma_2^2\cos^2(\omega_2 t))}} \\
&\quad \times \exp\left[-\frac{x^2}{2(\sigma_1^2\cos^2(\omega_1 t) + \sigma_2^2\cos^2(\omega_2 t))}\right]
\end{aligned}$$

Note that the second moment is a function of t.

The different dependence relationships among X_{t_i} characterize the various random signals. Some of the important categories are listed below.