Geometrical methods of mathematical physics

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CAMBRIDGE UNIVERSITY PRESS

Cambridge

London New York New Rochelle Melbourne Sydney

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
32 East 57th Street, New York, NY 10022, USA
296 Beaconsfield Parade, Middle Park, Melbourne 3206, Australia

First published 1980

Printed and bound in Great Britain
at The Pitman Press, Bath

British Library Cataloguing in Publication Data

Schutz, Bernard F
Geometrical methods of mathematical physics.
1. Geometry, Differential
2. Mathematical physics
I. Title
516'.36'002453   QC20.7.D52   80-40211

ISBN 0-521-23271-6
ISBN 0-521-29887-3 Pbk

PREFACE

Why study geometry?

This book aims to introduce the beginning or working physicist to a wide range of analytic tools which have their origin in differential geometry and which have recently found increasing use in theoretical physics. It is not uncommon today for a physicist's mathematical education to ignore all but the simplest geometrical ideas, despite the fact that young physicists are encouraged to develop mental 'pictures' and 'intuition' appropriate to physical phenomena. This curious neglect of 'pictures' of one's mathematical tools may be seen as the outcome of a gradual evolution over many centuries. Geometry was certainly extremely important to ancient and medieval natural philosophers; it was in geometrical terms that Ptolemy, Copernicus, Kepler, and Galileo all expressed their thinking. But when Descartes introduced coordinates into Euclidean geometry, he showed that the study of geometry could be regarded as an application of algrebra. Since then, the importance of the study of geometry in the education of scientists has steadily declined, so that at present a university undergraduate physicist or applied mathematician is not likely to encounter much geometry at all.

One reason for this suggests itself immediately: the relatively simple geometry of the three-dimensional Euclidean world that the nineteenth-century physicist believed he lived in can be mastered quickly, while learning the great diversity of analytic techniques that must be used to solve the differential equations of physics makes very heavy demands on the student's time. Another reason must surely be that these analytic techniques were developed at least partly in response to the profound realization by physicists that the laws of nature could be expressed as differential equations, and this led most mathematical physicists genuinely to neglect geometry until relatively recently.

However, two developments in this century have markedly altered the balance between geometry and analysis in the twentieth-century physicist's outloook. The first is the development of the theory of relativity, according to which the Euclidean three-space of the nineteenth-century physicist is only an approximation to the correct description of the physical world. The second development, which is only beginning to have an impact, is the realization by twentieth-century [illegible] by Cartan, that the relation between geometry and analysis [illegible]: on the one hand analysis may be the foundation of the study [illegible], but on the other hand the study of geometry leads naturally to the [illegible] of certain analytic tools (such as the Lie derivative and the exterior [illegible]) and certain concepts (such as the manifold, the fiber bundle, and the identification of vectors with derivatives) that have great power in applications of analysis. In the modern view, geometry remains subsidiary to analysis. For example, the basic concept of differential geometry, the differentiable manifold, is defined in terms of real numbers and differentiable functions. But this is no disadvantage: it means that concepts from analysis can be expressed geometrically, and this has considerable heuristic power.

Because it has developed this intimate connection between geometrical and analytic ideas, modern differential geometry has become more and more important in theoretical physics, where it has led to a greater simplicity in the mathematics and a more fundamental understanding of the physics. This revolution has affected not only special and general relativity, the two theories whose content is most obviously geometrical, but other fields where the geometry involved is not always that of physical space but rather of a more abstract space of variables: electromagnetism, thermodynamics, Hamiltonian theory, fluid dynamics, and elementary particle physics.

Aims of this book

In this book I want to introduce the reader to some of the more important notions of twentieth-century differential geometry, trying always to use that geometrical or 'pictorial' way of thinking that is usually so helpful in developing a physicist's intuition. The book attempts to teach mathematics, not physics. I have tried to include a wide range of applications of this mathematics to branches of physics which are familiar to most advanced undergraduates. I hope these examples will do more than illustrate the mathematics: the new mathematical formulation of familiar ideas will, if I have been successful, give the reader a deeper understanding of the physics.

I will discuss the background I have assumed of the reader in more detail below, but here it may be helpful to give a brief list of some of the 'familiar' ideas which are seen in a new light in this book: vectors, tensors, inner products, special relativity, spherical harmonics and the rotation group (and angular-momentum operators), conservation laws, volumes, theory of integration, curl and cross-product, determinants of matrices, partial differential equations and their integrability conditions, Gauss' and Stokes' integral theorems of vector calculus, thermodynamics of simple systems, Caratheodory's theorem (and the second law of thermodynamics), Hamiltonian systems in phase space, Maxwell's equations, fluid dynamics (including the laws governing the conservation of circulation), vector calculus in curvilinear coordinate systems, and the quantum theory of a charged scalar field. Besides these more or less familiar subjects, there are a few others which are not usually taught at undergraduate level but which most readers would certainly have heard of: the theory of Lie groups and symmetry, open and closed cosmologies, Riemannian geometry, and gauge theories of physics. That all of these subjects can be studied by the methods of differential geometry is an indication of the importance differential geometry is likely to have in theoretical physics in the future.

I believe it is important for the reader to develop a pictorial way of thinking and a feeling for the 'naturalness' of certain geometrical tools in certain situations. To this end I emphasize repeatedly the idea that tensors are geometrical objects, defined independently of any coordinate system. The rôle played by components and coordinate transformations is submerged into a secondary position: whenever possible I write equations without indices, to emphasize the coordinate-independence of the operations. I have made no attempt to present the material in a strictly rigorous or axiomatic way, and I have had to ignore many aspects of our subject which a mathematician would regard as fundamental. I do, of course, give proofs of all but a handful of the most important results (references for the exceptions are provided), but I have tried wherever possible to make the main geometrical ideas in the proof stand out clearly from the background of manipulation. I want to show the beauty, elegance, and naturalness of the mathematics with the minimum of obscuration.

How to use this book

The first chapter contains a review of the sort of elementary mathematics assumed of the reader plus a short introduction to some concepts, particularly in topology, which undergraduates may not be familiar with. The next chapters are the core of the book: they introduce tensors, Lie derivatives, and differential forms. Scattered through these chapters are some applications, but most of the physical applications are left for systematic treatment in chapter 5. The final chapter, on Riemannian geometry, is more advanced and makes contact with areas of particle physics and general relativity in which differential geometry is an everyday tool.

The material in this book should be suitable for a one-term course, provided the lecturer exercises some selection in the most difficult areas. It should also be possible to teach the most important points as a unit of, say, ten lectures in an advanced course on mathematical methods. I have taught such a unit to graduate students, concentrating mainly on §§ 2.1–2.3, 2.5–2.8, 2.12–2.14, 2.16, 2.17, 2.19–2.28, 3.1–3.13, 4.1–4.6, 4.8, 4.14–4.18, 4.20–4.23, 4.25, 4.26, 5.1, 5.2, 5.4–5.7, and 5.15–5.18. I hope lecturers will experiment with their own choices of material, especially because there are many people for whom geometrical reasoning is easier and more natural than purely analytic reasoning, and for them an early exposure to geometrical ideas can only be helpful. As a general guide to selecting material, section headings within chapters are printed in two different styles. Fundamental material is marked by **boldface** headings, while more advanced or supplementary topics are marked by ***boldface italics***. All of the last chapter falls into this category. The same convention of type-face distinguishes those exercises which are central to the development of the mathematics from those which are peripheral.

The exercises form an integral part of the book. They are inserted in the middle of the text, and they are designed to be worked when they are first encountered. Usually the text after an exercise will assume that the reader has worked and understood the exercise. The reader who does not have the time to work an exercise should nevertheless read it and try to understand its result. Hints and some solutions will be found at the end of the book.

Background assumed of the reader

Most of this book should be understandable to an advanced undergraduate or beginning graduate student in theoretical physics or applied mathematics. It presupposes reasonable facility with vector calculus, calculus of many variables, matrix algebra (including eigenvectors and determinants), and a little operator theory of the sort one learns in elementary quantum mechanics. The physical applications are drawn from a variety of fields, and not everyone will feel at home with them all. It should be possible to skip many sections on physics without undue loss of continuity, but it would probably be unrealistic to attempt this book without some familiarity with classical mechanics, special relativity, and electromagnetism. The bibliography at the end of chapter 1 lists some books which provide suitable background.

I want to acknowledge my debt to the many people, both colleagues and teachers, who have helped me to appreciate the beauty of differential geometry and understand its usefulness in physics. I am especially indebted to Kip Thorne, Rafael Sorkin, John Friedman, and Frank Estabrook. I also want to thank the first two and many patient students at University College, Cardiff, for their comments on earlier versions of this book. Two of my students, Neil Comins and Brian Wade, deserve special mention for their careful and constructive suggestions. It is also a pleasure to thank Suzanne Ball, Jane Owen, and Margaret Wilkinson for their fast and accurate typing of the manuscript through all its revisions. Finally, I thank my wife for her patience and encouragement, particularly during the last few hectic months.

Cardiff, 30 June 1979 Bernard Schutz

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1 SOME BASIC MATHEMATICS

This chapter reviews the elementary mathematics upon which the geometrical development of later chapters relies. Most of it should be familiar to most readers, but we begin with two topics, topology and mappings, which many readers may find unfamiliar. The principal reason for including them is to enable us to define precisely what is meant by a manifold, which we do early in chapter 2. Readers to whom topology is unfamiliar may wish to skip the first two sections initially and refer back to them only after chapter 2 has given them sufficient motivation.

1.1 The space R^n and its topology

The space R^n is the usual n-dimensional space of vector algebra: a point in R^n is a sequence of n real numbers $(x_1, x_2, \ldots, x_n)$, also called an *n-tuple* of real numbers. Intuitively we have the idea that this is a *continuous* space, that there are points of R^n arbitrarily close to any given point, that a line joining any two points can be subdivided into arbitrarily many pieces that also join points of R^n. These notions are in contrast to properties we would ascribe to, say, a lattice, such as the set of all n-tuples of integers $(i_1, i_2, \ldots, i_n)$. The concept of continuity in R^n is made precise in the study of its *topology*. The word 'topology' has two distinct meanings in mathematics. The one we are discussing now may be called *local topology*. The other is *global topology*, which is the study of large-scale features of the space, such as those which distinguish the sphere from the torus. We shall have something to say about global topology later, particularly in the chapter on differential forms. But first we must take a brief look at local topology.

The fundamental concept is that of a neighborhood of a point in R^n, which we can define after introducing a *distance function* between any two points $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ of R^n:

$$d(\mathbf{x}, \mathbf{y}) = [(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2]^{1/2}. \qquad (1.1)$$

A neighborhood of radius r of the point $\mathbf{x}$ in R^n is the set of points $N_r(\mathbf{x})$ whose distance from $\mathbf{x}$ is less than r. For R^2 this is illustrated in figure 1.1. The continuity of the space can now be more precisely defined by considering very small neighborhoods. A set of points of R^n is *discrete* if each point has a neighborhood which contains no other points of the set. Clearly R^n itself is not discrete. A set of points S of R^n is said to be *open* if every point **x** in S has a neighborhood entirely contained in S. Clearly, discrete sets are not open, and from now on we will have no use for discrete sets. A simple example of an open set in R^1 (also known simply as R) is all points x for which $a < x < b$ for two real numbers a and b. An important thing to understand is that the set of points for which $a \leqslant x < b$ is *not* open, because the point $x = a$ does not have a neighborhood entirely contained in the set: some points of *any* neighborhood of $x = a$ must be less than a and therefore outside the set. This is illustrated in figure 1.2. This is, of course, a very general property: any reasonable 'chunk' of R^n will be open if we do not include the boundary of the chunk in the set.

Fig. 1.1. The distance function $d(\mathbf{x}, \mathbf{y})$ defines a neighborhood in R^2 which is the interior of the disc bounded by the circle of radius r. The circle itself is not part of this neighborhood.

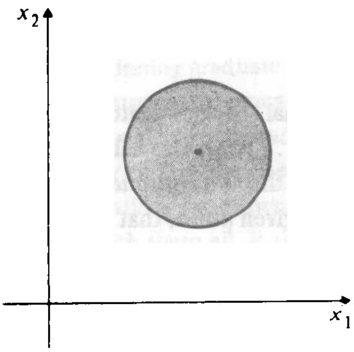
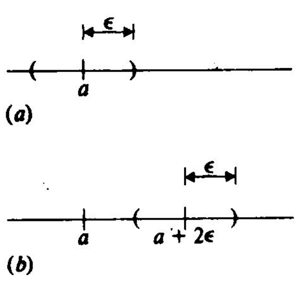


Fig. 1.2. (a) Any neighborhood of the point $x = a$ must include points to the left of a, while (b) any point to the right of a has a neighborhood entirely to the right of a.



The idea that a line joining any two points of R^n can be infinitely subdivided can be made more precise by saying that any two points of R^n have neighborhoods which do not intersect. (They will also have some neighborhoods which *do* intersect, but if we choose small enough neighborhoods we can make them disjoint.) This is called the *Hausdorff property* of R^n. It is possible to construct non-Hausdorff spaces, but for our purposes they are artificial and we shall ignore them.

Notice that we have used the distance function $d(\mathbf{x}, \mathbf{y})$ to define neighborhoods and thereby open sets. We say that $d(\mathbf{x}, \mathbf{y})$ has *induced a topology* on R^n. By this we mean that it has enabled us to define open sets of R^n which have the properties:

(Ti) if O_1 and O_2 are open, so is their intersection, $O_1 \cap O_2$; and

(Tii) the union of any collection (possibly infinite in number) of open sets is open.

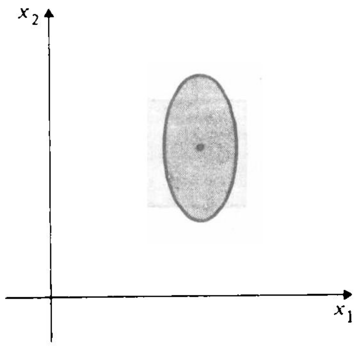
In order to make (Ti) apply to all open sets of R^n, we *define* the empty set (or null set) to be open, and in order to make (Tii) work we likewise define R^n itself to be open. (In more advanced treatments one defines a *topological space* to be a collection of points with a definition of open sets satisfying (Ti) and (Tii). In this sense the distance function $d(\mathbf{x}, \mathbf{y})$ has enabled us to make R^n into a topological space.)

At this point we must ask whether the induced topology depends very much on the precise form of $d(\mathbf{x}, \mathbf{y})$. Suppose, for example, that we use a different distance function

$$d'(\mathbf{x}, \mathbf{y}) = [4(x_1 - y_1)^2 + 0.1(x_2 - y_2)^2 + \ldots + (x_n - y_n)^2]^{1/2}. \quad (1.2)$$

This also defines neighborhoods and open sets, as shown in figure 1.3 for R^2.

Fig. 1.3. The distance function $d'(\mathbf{x}, \mathbf{y}) = [4(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2}$ defines a neighborhood in R^2 which is the interior of the disc bounded by the ellipse $4(x_1 - y_1)^2 + (x_2 - y_2)^2 = r^2$. As in figure 1.1, the ellipse itself is not in the neighborhood.



The key point is that any set which is open according to $d'(\mathbf{x}, \mathbf{y})$ is also open according to $d(\mathbf{x}, \mathbf{y})$, and vice versa. The proof of this is not hard, and it rests on the fact that any given d-type neighborhood of $\mathbf{x}$ contains a d'-type neighborhood entirely within it, and vice versa. That is, given a d-type neighborhood of radius ϵ about $\mathbf{x}$, one can choose a number δ so small that a d'-type neighborhood of $\mathbf{x}$ of radius δ is entirely within the original (see figure 1.4). So we can conclude that if a set is open as defined by $d(\mathbf{x}, \mathbf{y})$ it is also open as defined by $d'(\mathbf{x}, \mathbf{y})$, and vice versa. We therefore say that both d and d' induce the same topology on R^n. The reader may wish to show that the distance functions

Fig. 1.4. In R^2 a d-neighborhood of radius ϵ (bounded by the circle) entirely contains a d'-neighborhood of radius δ (bounded by the ellipse defined in figure 1.3) if $\delta < \epsilon$. If $\delta > 2\epsilon$ the inclusion is reversed.

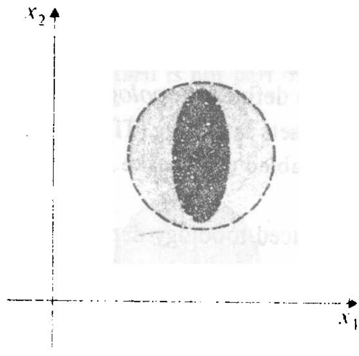
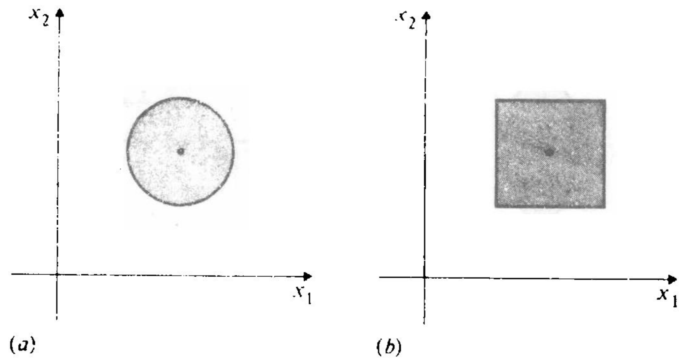


Fig. 1.5. (a) In R^2 the distance function d'' has circular neighborhoods smaller for a given radius r, than those of d. (b) The neighborhoods of d''' are bounded by squares of side $2r$.



$$d''(\mathbf{x},\mathbf{y}) = \exp\,[d(\mathbf{x},\mathbf{y})] - 1, \qquad (1.3)$$

$$d'''(\mathbf{x},\mathbf{y}) = \text{maximum}\,(|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|) \qquad (1.4)$$

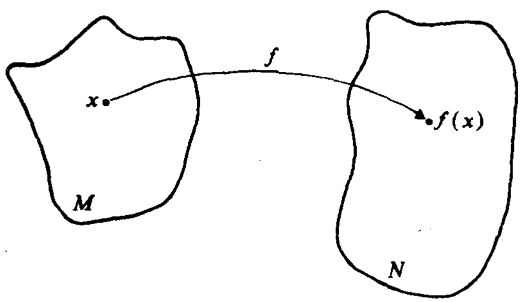
also induce the same topology. Their neighborhoods in R^2 are illustrated in figure 1.5. So although we began with the usual Euclidean distance function $d(\mathbf{x}, \mathbf{y})$, the topology we have defined is not very dependent on the form of d. This is called the 'natural' topology of R^n. Topology is a more 'primitive' concept than distance. We do not need to know the actual distance between points, since many different distance definitions will do. What we need is only a notion that the distance between points can be made arbitrarily small and that no two distinct points have zero distance between them.

Our definition of a neighborhood was tied to a particular distance function, but because the topology of a manifold is more general than any particular distance function the word 'neighborhood' is often used in a different sense. We will often find it convenient to let a neighborhood of a point $\mathbf{x}$ be any set containing an open set containing $\mathbf{x}$. It should always be clear from the context which sense of 'neighborhood' is intended.

1.2 Mappings

The concept of a mapping, simple though it is, will be so useful later that it is well to spend some time discussing it. A map f from a space M to a space N is a rule which associates with an element x of M a unique element y of N. It is useful to keep in one's mind a general picture of a map, such as figure 1.6. The simplest example of a map is an ordinary real-valued function on R. The function f associates a point x in R with a point $f(x)$ also in R. (This illustrates the fact that the spaces M and N need not be distinct.) Such a map is shown in the usual way in figure 1.7. Notice that the map gives a unique $f(x)$ for every x, but not necessarily a unique x for every $f(x)$. In the figure, both x_0 and

Fig. 1.6. A pictorial representation of the mapping f: $M \to N$ showing $x \mapsto f(x)$.



x_1 map into the same value. Such a map is called *many-to-one*. More generally, if f maps M to N then for any set S in M the elements in N mapped from points of S form a set T called the *image* of S under f, denoted by $f(S)$. Conversely, the set S is called the *inverse image* of T, denoted by $f^{-1}(T)$. If the map is many-to-one then the inverse image of a single point of N is not a single point of M, so there is no *map* f^{-1} from N to M, since every map must have a unique image. So in general the symbol $f^{-1}(T)$ must be read as a single symbol: it is not the image of T under a map f^{-1} but simply a set called $f^{-1}(T)$. On the other hand, if every point in $f(S)$ has a unique inverse image point in S, then f is said to be *one-to-one* (abbreviated 1–1) and there does exist another 1–1 map f^{-1}, called the *inverse* of f, which maps the image of M to M. These concepts, if not the words used to describe them, are familiar from elementary calculus. The function $f(x) = \sin x$ is many-to-one, since $f(x) = f(x + 2n\pi) = f((2n + 1)\pi - x)$ for any integer n. Therefore, a true inverse function does not exist. The usual inverse function, arcsin y or $\sin^{-1} y$, is obtained by restricting the original sine function to the 'principal' values, $-\pi/2 < x \leqslant \pi/2$, on which it is indeed 1–1 and invertible.

Another example of a 1–1 map is a geographical map of part of the Earth's surface: this maps a point of the Earth's surface to a point of a piece of paper. Yet another map is a rotation of a sphere about some diameter: this maps a point of the sphere to another one a fixed angular distance away as measured about the axis of rotation.

We shall now introduce some standard notation and terminology regarding maps. The statement that f maps M to N is abbreviated $f: M \to N$. The statement that f maps a particular element x of M to y of N has its own special notation, $f: x \mapsto y$. If the name of a map is f, the image of a point x is $f(x)$. When the map is a real-valued function of, say, n variables (so $f: R^n \to R$), it is conventional among physicists to use the symbol $f(x)$ to denote both the value of f on x and the function itself. When there is no chance of confusion we will follow that convention. If we have two maps, f and g, $f: M \to N$ and $g: N \to P$, then there is a

Fig. 1.7. A many-to-one map (function) of R to R.