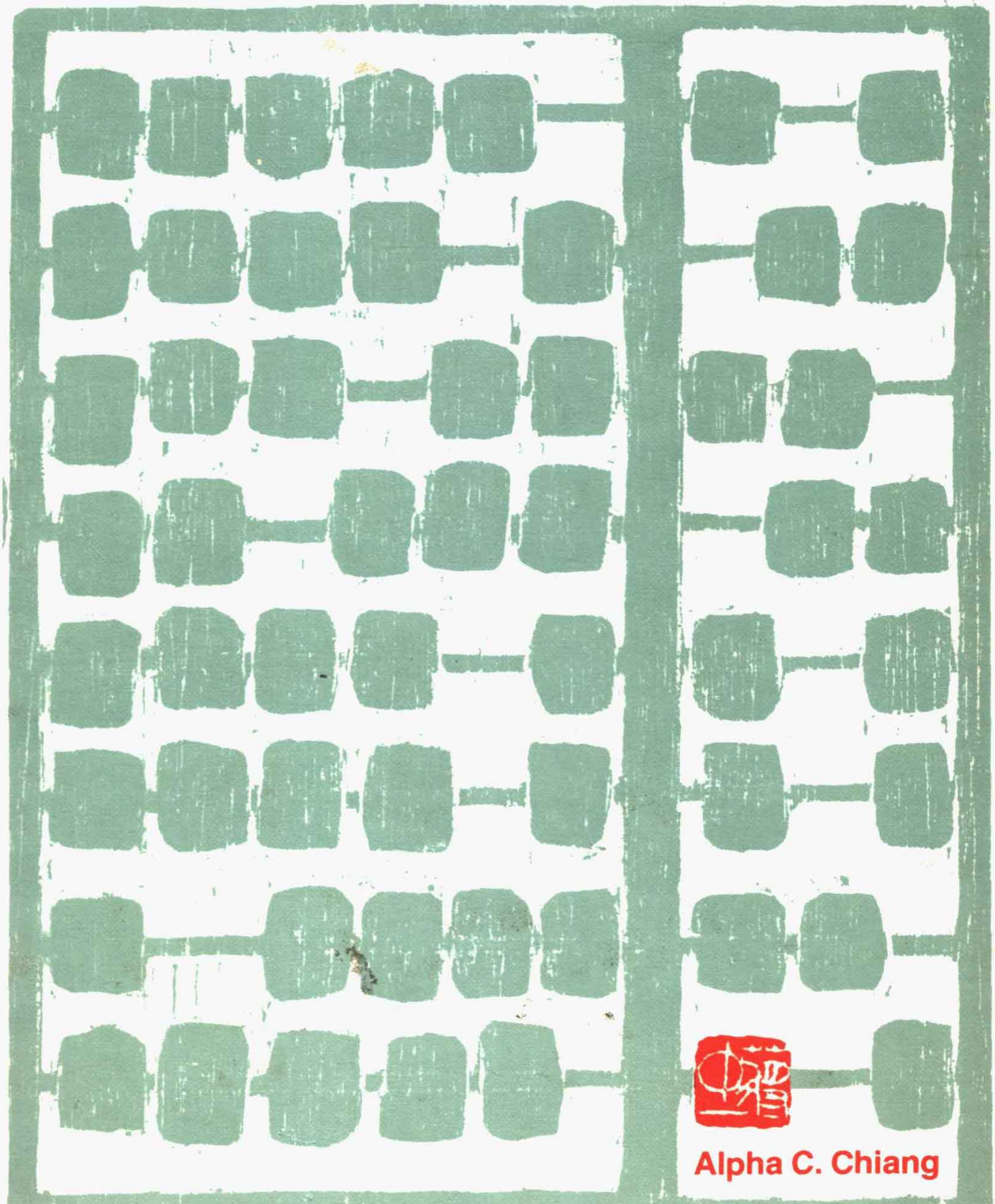
FUNDAMENTAL METHODS OF MATHEMATICAL ECONOMICS

Third Edition



Alpha C. Chiang

FUNDAMENTAL METHODS OF MATHEMATICAL ECONOMICS

Third Edition

Alpha C. Chiang

Professor of Economics
The University of Connecticut

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PREFACE

This book is written for those students of economics intent on learning the basic mathematical methods that have become indispensable for a proper understanding of the current economic literature. Unfortunately, studying mathematics is, for many, something akin to taking bitter-tasting medicine—necessary and inescapable, but extremely tortuous. Such an attitude, referred to as "math anxiety," has its roots—I believe—largely in the inauspicious manner in which mathematics is often presented to students. In the belief that conciseness means elegance, explanations offered are sometimes too brief for clarity, puzzling students, and giving them an undeserved sense of intellectual inadequacy. An overly formal style of presentation, when not accompanied by any intuitive illustrations, or demonstrations of "relevance," can impair motivation. An uneven progression in the level of the material can make certain mathematical topics appear more difficult than they actually are. And, finally, exercise problems that are excessively sophisticated may tend to shatter students' confidence, rather than stimulate thinking as intended.

With the above in mind, I have made a serious effort to minimize anxiety-causing features. To the extent possible, patient rather than cryptic explanations are offered. The style is deliberately informal and "user friendly," to borrow a phrase from the computer industry. As a matter of routine, I try to anticipate, and answer, questions that are likely to arise in the students' minds as they read. To underscore the relevance of mathematics to economics, I let the analytical needs of economists motivate the study of the related mathematical techniques, and then illustrate the latter with appropriate economic models immediately afterward. Also, the mathematical tool kit is built up on a carefully graduated schedule, with the elementary tools serving as stepping stones to the more sophisticated tools discussed later. Wherever appropriate, graphic illustrations give visual reinforcement to the algebraic results. And I have designed the exercise problems more as drills to help solidify grasp and bolster confidence, than as exacting challenges that might unwittingly frustrate and intimidate the novice.

In this book, the following major types of economic analysis are covered: statics (equilibrium analysis), comparative statics, optimization problems (as a special type of statics), dynamics, and mathematical programming (as a modern development of optimization). To tackle these, the following mathematical methods are introduced in due course: matrix algebra, differential and integral calculus, differential equations, difference equations, and convex sets. Because of the substantial number of illustrative economic models—both macro and micro—appearing here, this book should be useful also to those who are already mathematically trained but who still need a guide to usher them from the realm of mathematics into the land of economics. For the same reason, the book should serve not only as a text for a course on mathematical methods but also as supplementary reading in such courses as macroeconomic theory, microeconomic theory, and economic growth and development.

While retaining the principal objectives, style, and organization of the previous editions, the present edition nevertheless contains several significant changes. Chapters 11 and 12, on the classical approach to optimization, have been extensively rewritten. I have adopted the differential (as against derivative) forms of the first- and second-order conditions as a unifying theme throughout those two chapters, although the derivative forms of the conditions are still used as operational criteria in problem-solving. More importantly, I have added a full-fledged discussion of concavity and convexity in relation to the second-order conditions of free-extremum problems (Sec. 11.5), and a parallel discussion of quasiconcavity and quasiconvexity in the context of optimization with equality constraints (Sec. 12.4). This arrangement not only facilitates the understanding of second-order conditions, but also makes possible an earlier introduction to absolute extrema, usually not emphasized in the classical approach. These discussions are summarized in two new diagrams (Figs. 11.5 and 12.6).

Another major change is in the treatment of simultaneous differential equations and difference equations. Now placed in a new chapter (Chap. 18), this topic has been expanded to include an introduction to the technique of two-variable phase diagrams (Sec. 18.5), and the linearization and local-stability analysis of nonlinear differential-equation systems (Sec. 18.6).

There are various other refinements: I have added a brief discussion of homothetic function (Sec. 12.7); simplified the explanation of continuity and differentiability (Sec. 6.7); given more emphasis to the role of second-order *necessary* conditions (Secs. 9.4, 11.1 and 12.3); clarified the interpretation of Taylor expansion as an approximation to an arbitrary function (Sec. 9.5); and streamlined the presentation on exact differential equations (Sec. 14.4). As for economic illustrations, I have included models on the interaction of inflation and unemployment, which involve the use of such concepts as expectations-augmented Philips relation and adaptive expectations, as applications of differential equations and difference equations (Secs. 15.5, 17.3 and 18.4). In response to popular request, answers to selected exercise problems are now given at the end of the book. For easier reference, I have also appended a list of mathematical symbols. To accommodate all the new material, I have reluctantly deleted the chapter on game theory.

I am indebted to many people in the writing of this book. The immense debt to the mathematicians and economists whose ideas underlie this volume goes without saying. For the previous two editions, I had the benefit of comments and suggestions from (in alphabetical order): Nancy S. Barrett, Thomas Birnberg, E. J. R. Booth, Roberta Grower Carey, Emily Chiang, Lloyd R. Cohen, Harald Dickson, John C. H. Fei, Roger N. Folsom, Jack Hirchleifer, James C. Hsiao, Ki-Jun Jeong, Marc Nerlove, J. Frank Sharp, Dennis Starleaf, and Chiou-Nan Yeh. For the present edition, the following persons (in alphabetical order) have kindly offered me valuable suggestions: E. J. R. Booth, Charles E. Butler, Gary Cornell, Warren L. Fisher, Dennis R. Heffley, George Kondor, William F. Lott, Paul B. Manchester, Peter Morgan, Allan G. Sleeman and, last but not least, Henry Y. Wan, Jr. To all of them, I give my hearty thanks. Since I have not accepted all their suggestions, however, the responsibility for the final product is mine alone. In particular, I have again decided not to include a few chapters on dynamic optimization—a topic that requires a separate volume for an adequate treatment. Finally, I wish to express my deep appreciation to Gail Gavert of McGraw-Hill Book Company for her patient cooperation and skillful handling of a complex manuscript.

SUGGESTIONS FOR THE USE OF THIS BOOK

Because of the gradual building-up of the mathematical tool kit in the organization of this book, the ideal way of study is to follow closely its specific sequence of presentation. The one important exception is that it makes no difference whether Part Five (Dynamic Analysis) or Part Six (Mathematical Programming) is read first.

Some other alterations in the sequence of reading are also possible: Upon completing the study of matrix algebra (Chap. 5), you can proceed directly to linear programming (Chaps. 19 and 20) without difficulty. Similarly, after finishing the study of optimization with equality constraints (Chap. 12), it is feasible to go directly to nonlinear programming (Chap. 21) with or without the background of linear programming.

In case comparative statics is not an area of primary concern to you, you may skip the comparative-static analysis of general-function models (Chap. 8), and jump from Chap. 7 to Chap. 9. In that case, however, it would become necessary also to omit Sec. 11.7 and the comparative-static portion of Sec. 12.5.

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PART ONE

INTRODUCTION

CHAPTER

ONE

THE NATURE OF MATHEMATICAL ECONOMICS

Mathematical economics is not a distinct branch of economics in the sense that public finance or international trade is. Rather, it is an *approach* to economic analysis, in which the economist makes use of mathematical symbols in the statement of the problem and also draws upon known mathematical theorems to aid in reasoning. As far as the specific subject matter of analysis goes, it can be micro- or macroeconomic theory, public finance, urban economics, or what not.

Using the term *mathematical economics* in the broadest possible sense, one may very well say that every elementary textbook of economics today exemplifies mathematical economics insofar as geometrical methods are frequently utilized to derive theoretical results. Conventionally, however, mathematical economics is reserved to describe cases employing mathematical techniques beyond simple geometry, such as matrix algebra, differential and integral calculus, differential equations, difference equations, etc. It is the purpose of this book to introduce the reader to the most fundamental aspects of these mathematical methods—those encountered daily in the current economic literature.

1.1 MATHEMATICAL VERSUS NONMATHEMATICAL ECONOMICS

Since mathematical economics is merely an approach to economic analysis, it should not and does not differ from the *non*mathematical approach to economic analysis in any fundamental way. The purpose of any theoretical analysis, regardless of the approach, is always to derive a set of conclusions or theorems from a given set of assumptions or postulates via a process of reasoning. The major difference between "mathematical economics" and "literary economics" lies principally in the fact that, in the former, the assumptions and conclusions are stated in mathematical symbols rather than words and in equations rather than sentences; moreover, in place of literary logic, use is made of mathematical theorems—of which there exists an abundance to draw upon—in the reasoning process. Inasmuch as symbols and words are really equivalents (witness the fact that symbols are usually defined in words), it matters little which is chosen over the other. But it is perhaps beyond dispute that symbols are more convenient to use in deductive reasoning, and certainly are more conducive to conciseness and preciseness of statement.

The choice between literary logic and mathematical logic, again, is a matter of little import, but mathematics has the advantage of forcing analysts to make their assumptions explicit at every stage of reasoning. This is because mathematical theorems are usually stated in the "if-then" form, so that in order to tap the "then" (result) part of the theorem for their use, they must first make sure that the "if" (condition) part does conform to the explicit assumptions adopted.

Granting these points, though, one may still ask why it is necessary to go beyond geometric methods. The answer is that while geometric analysis has the important advantage of being visual, it also suffers from a serious dimensional limitation. In the usual graphical discussion of indifference curves, for instance, the standard assumption is that only *two* commodities are available to the consumer. Such a simplifying assumption is not willingly adopted but is forced upon us because the task of drawing a three-dimensional graph is exceedingly difficult and the construction of a four- (or higher) dimensional graph is actually a physical impossibility. To deal with the more general case of 3, 4, or n goods, we must instead resort to the more flexible tool of equations. This reason alone should provide sufficient motivation for the study of mathematical methods beyond geometry.

In short, we see that the mathematical approach has claim to the following advantages: (1) The "language" used is more concise and precise; (2) there exists a wealth of mathematical theorems at our service; (3) in forcing us to state explicitly all our assumptions as a prerequisite to the use of the mathematical theorems, it keeps us from the pitfall of an unintentional adoption of unwanted implicit assumptions; and (4) it allows us to treat the general n-variable case.

Against these advantages, one sometimes hears the criticism that a mathematically derived theory is inevitably *unrealistic*. However, this criticism is not valid. In fact, the epithet "unrealistic" cannot even be used in criticizing economic theory in general, whether or not the approach is mathematical. Theory is by its very nature an abstraction from the real world. It is a device for singling out only the most essential factors and relationships so that we can study the crux of the problem at hand, free from the many complications that do exist in the actual world. Thus the statement "theory lacks realism" is merely a truism that cannot be accepted as a valid criticism of theory. It then follows logically that it is quite meaningless to pick out any one approach to theory as "unrealistic." For example, the theory of firm under pure competition is unrealistic, as is the theory of firm under imperfect competition, but whether these theories are derived mathematically or not is irrelevant and immaterial.

In sum, we might liken the mathematical approach to a "mode of transportation" that can take us from a set of postulates (point of departure) to a set of conclusions (destination) at a good speed. Common sense would tell us that, if you intend to go to a place 2 miles away, you will very likely prefer driving to walking, unless you have time to kill or want to exercise your legs. Similarly, as a theorist who wishes to get to your conclusions more rapidly, you will find it convenient to "drive" the vehicle of mathematical techniques appropriate for your particular purpose. You will, of course, have to take "driving lessons" first; but since the skill thus acquired tends to be of service for a long, long while, the time and effort required would normally be well spent indeed.

For a serious "driver"—to continue with the metaphor—some solid lessons in mathematics are imperative. It is obviously impossible to introduce all the mathematical tools used by economists in a single volume. Instead, we shall concentrate on only those that are mathematically the most fundamental and economically the most relevant. Even so, if you work through this book conscientiously, you should at least become proficient enough to comprehend most of the professional articles you will come across in such periodicals as the *American Economic Review*, *Quarterly Journal of Economics*, *Journal of Political Economy*, *Review of Economics and Statistics*, and *Economic Journal*. Those of you who, through this exposure, develop a serious interest in mathematical economics can then proceed to a more rigorous and advanced study of mathematics.

1.2 MATHEMATICAL ECONOMICS VERSUS ECONOMETRICS

The term "mathematical economics" is sometimes confused with a related term, "econometrics." As the "metric" part of the latter term implies, econometrics is concerned mainly with the measurement of economic data. Hence it deals with the study of *empirical* observations using statistical methods of estimation and hypothesis testing. Mathematical economics, on the other hand, refers to the application of mathematics to the purely *theoretical* aspects of economic analysis, with little or no concern about such statistical problems as the errors of measurement of the variables under study.

In the present volume, we shall confine ourselves to mathematical economics. That is, we shall concentrate on the application of mathematics to deductive reasoning rather than inductive study, and as a result we shall be dealing primarily with theoretical rather than empirical material. This is, of course, solely a matter of choice of the scope of discussion, and it is by no means implied that econometrics is less important.

Indeed, empirical studies and theoretical analyses are often complementary and mutually reinforcing. On the one hand, theories must be tested against empirical data for validity before they can be applied with confidence. On the other, statistical work needs economic theory as a guide, in order to determine the most relevant and fruitful direction of research. A classic illustration of the complementary nature of theoretical and empirical studies is found in the study of the aggregate consumption function. The theoretical work of Keynes on the consumption function led to the statistical estimation of the propensity to consume, but the statistical findings of Kuznets and Goldsmith regarding the relative long-run constancy of the propensity to consume (in contradiction to what might be expected from the Keynesian theory), in turn, stimulated the refinement of aggregate consumption theory by Duesenberry, Friedman, and others.*

In one sense, however, mathematical economics may be considered as the more basic of the two: for, to have a meaningful statistical and econometric study, a good theoretical framework—preferably in a mathematical formulation—is indispensable. Hence the subject matter of the present volume should be useful not only for those interested in theoretical economics, but also for those seeking a foundation for the pursuit of econometric studies.

* John M. Keynes, *The General Theory of Employment, Interest and Money*, Harcourt, Brace and Company, Inc., New York, 1936, Book III; Simon Kuznets, *National Income: A Summary of Findings*, National Bureau of Economic Research, 1946, p. 53; Raymond Goldsmith, *A Study of Saving in the United States*, vol. I, Princeton University Press, Princeton, N.J., 1955, chap. 3; James S. Duesenberry, *Income, Saving, and the Theory of Consumer Behavior*, Harvard University Press, Cambridge, Mass., 1949; Milton Friedman, *A Theory of the Consumption Function*, National Bureau of Economic Research, Princeton University Press, Princeton, N.J., 1957.