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Graph Connections

Relationships Between Graph Theory and other Areas of Mathematics

Edited by

LOWELL W. BEINEKE

and

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OXFORD SCIENCE PUBLICATIONS

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Preface

In a mathematical world of increasing specialization, it is important not to lose sight of how different fields are related to one another, and of how various ideas fit together. Those are the premises on which this book was conceived, as they apply to graph theory. On one level, graphs appear throughout mathematics (indeed, throughout life), since a graph is just a model of a relation. However, that ubiquity may obscure the deeper connections that have been developed between graph theory and other branches of mathematics. The purpose of this book is to present examples of these connections.

The connections are not all of the same kind. Some form a body of material that overlaps two fields, whereas others consist primarily of applications of one area to another. The applications may be to graph theory or from it. Whatever the form of the connections, light is shed on both areas, and that is in itself an excellent reason for examining the connections.

Although we do not claim to cover all of the areas of mathematics which have connections with graph theory, we believe that this collection contains most of the important ones and that there is sufficient diversity in these to illustrate their wide variety.

Wherever feasible, uniform notation and terminology are used throughout the book. Much of this, as well as some other relevant background material, is provided in an introductory chapter. Otherwise, the individual chapters are independent, except for an occasional cross-reference.

The origin of the book was a highly successful one-day conference sponsored by the British Combinatorial Committee and held at the Open University in Milton Keynes in 1994. Several of the chapters are based on talks given there; additional topics were added in order to present a wider range.

That conference was open to all, but was designed primarily for graduate students to learn about 'graph theory across the field of mathematics'. This concept has been carried over to the book; it is a resource for learning about how graph theory interacts with other branches of mathematics. As such, it can function as the basis of a graduate-level seminar, or can be used by individuals or groups interested in particular topics.

Acknowledgements

We are grateful to the authors of the various chapters for their willingness to share their expertise and for their cooperation with our efforts to make the book more than a collection of individual essays.

We are also indebted to the British Combinatorial Committee for supporting the conference that gave rise to the book. Further thanks go to the Department of Mathematical Sciences at Indiana University–Purdue University Fort Wayne, the Mathematical Institute at Oxford University, and the Faculty of Mathematics and Computing at the Open University.

Finally, we want to express our particular thanks to Nicky Kempton, Toni Cokayne, Steve Best and Alison Cadle at the Open University for preparing the manuscript, and to Elizabeth Johnston, Julia Tompson and Keith Mansfield at Oxford University Press for guiding it through the stages of publication.

Fort Wayne, Indiana, USA L.W.B.
Milton Keynes R.J.W.
July 1996

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1

Introduction

ROBIN J. WILSON

We present those definitions and theorems in graph theory that are assumed throughout this book. Further explanation of these terms, together with the proofs of stated results, can be found in the standard texts listed below, although not all of the terminology is standardized. Definitions and results not included here are introduced later, as needed.

1.1 Graphs

A *graph* G consists of a finite non-empty set $V(G)$ of elements called *vertices* and a finite set $E(G)$ of distinct unordered pairs of distinct elements of $V(G)$ called *edges* (see Fig. 1.1). We call $V(G)$ the *vertex set* of G and $E(G)$ the *edge set* of G; these are sometimes abbreviated to V and E, respectively. The number n of vertices of G is the *order* of G, and the number of edges of G is denoted by m. The edge $\{v, w\}$ (where v and w are vertices of G) is often denoted by vw.

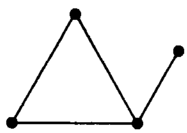


FIG. 1.1

If, in the definition of a graph, we remove the restriction that the edges are distinct, then we obtain a *multigraph* (see Fig. 1.2); two or more edges joining the same pair of vertices are *multiple edges*. If we also remove the restriction that the edges join distinct vertices, thus allowing the existence of *loops*, then the resulting object is a *general graph* (see Fig. 1.3). If loops and multiple edges are excluded, then we use the term *simple graph*.

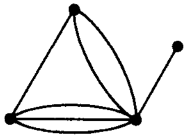


FIG. 1.2

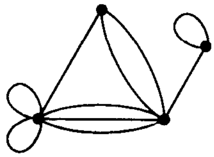


FIG. 1.3

There are many other variations on the concept of a graph. If one vertex is distinguished from the rest, then we have a *rooted graph*; the distinguished vertex is the *root*, indicated by a small square (see Fig. 1.4). A *labelled graph* of order n is a graph whose vertices have been assigned the numbers $1, 2, \ldots, n$ so that no two vertices are assigned the same number (see Fig. 1.5). A *signed graph* is a graph to each edge of which is assigned either + or − (see Fig. 1.6).

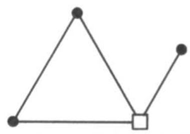


FIG. 1.4

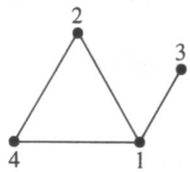


FIG. 1.5

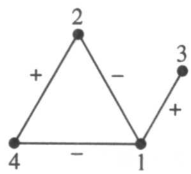
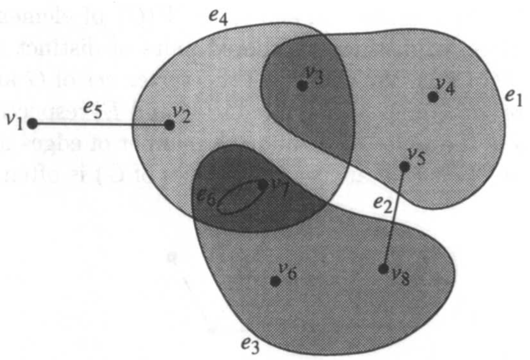


FIG. 1.6

A hypergraph is like a graph, except that the edges consist of any subset of vertices; more formally, a *hypergraph* H consists of a finite non-empty set $V(H)$ of elements called *vertices* and a finite set $E(H)$ of distinct sets of distinct elements of $V(H)$ called *hyperedges* (see Fig. 1.7).



$e_1 = v_3v_4v_5,\ e_2 = v_5v_8,\ e_3 = v_6v_7v_8,\ e_4 = v_2v_3v_7,\ e_5 = v_1v_2,\ e_6 = v_7$

FIG. 1.7

We also define *infinite graphs*, in which we no longer insist that $V(G)$ and $E(G)$ be finite; a *countable graph* is one in which $V(G)$ and $E(G)$ are finite or countably infinite, and a *locally finite graph* is one in which the number of edges incident with each vertex is finite.

Finally, we consider directed graphs, in which each edge is assigned a direction. More formally, a *digraph* D consists of a finite non-empty set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of distinct *ordered* pairs of distinct elements of $V(D)$ called *arcs* (see Fig. 1.8). The arc (v, w) (where v and w are vertices of D) is often denoted by vw. A *simple digraph* is a digraph with no loops vv or multiple arcs. If D is a digraph, then the *underlying graph* of D is the graph or multigraph obtained from D by replacing each arc by an undirected edge joining the same pair of vertices.

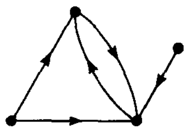


FIG. 1.8

1.2 Adjacency and incidence

If $e = vw$ is an edge of a graph G, then e *joins* the vertices v and w, and these vertices are *adjacent*; in this case, we say that e is *incident* with v and w, and that w is a *neighbour* of v. The *neighbourhood* $N(v)$ of v is the set of all vertices of G adjacent to v. Two edges of G incident with the same vertex are *adjacent edges*.

Two graphs G and H are *isomorphic* (written $G \cong H$) if there is a one-to-one correspondence between their vertex sets that preserves the adjacency of vertices. An *automorphism* of G is a one-to-one mapping ϕ of $V(G)$ onto itself with the property that $\phi(v)$ and $\phi(w)$ are adjacent if and only if v and w are. The automorphisms of G form a group $\Gamma(G)$ under composition, called the *automorphism group* of G; $\Gamma(G)$ is *transitive* if it contains automorphisms mapping each vertex of G to every other vertex, and *edge-transitive* if it contains automorphisms mapping each edge of G to every other edge.

For each vertex v in a graph G, the number of edges incident with v is the *degree* of v, denoted by $\deg(v)$. The maximum degree in G is denoted by Δ. A vertex of degree 0 is an *isolated vertex*, and a vertex of degree 1 is an *end-vertex*. The *degree list* of G is the set of degrees of the vertices of G, often arranged in non-decreasing order; for example, the degree list of the graph in Fig. 1.1 is $(1, 2, 2, 3)$. If all of the vertices of G have the same degree k, then G is *regular of degree k* or *k-regular*. A 3-regular graph is a *cubic graph*.

Analogous concepts can be defined for digraphs. If $e = vw$ is an arc of a digraph D, then v and w are *adjacent*, and e is *incident from* v and *incident to* w. If v is a vertex of a digraph D, then its *out-degree* $\text{outdeg}(v)$ is the number of arcs in D of the form vw, and its *in-degree* $\text{indeg}(v)$ is the number of arcs in D of the form wv.

An *independent* (or *stable*) *set of vertices* in a graph G is a set of vertices of G no two of which are adjacent, and the size of a largest such set is the *independence* (or *stability*) *number* of G. Similarly, an *independent set of edges* or *matching* is a set of edges of G no two of which are adjacent, and the size of a largest such set is the *edge-independence number* of G. An independent set of edges that includes every vertex of G is a *1-factor* or *perfect matching* in G.

Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{e_1, e_2, \ldots, e_m\}$. The *adjacency matrix* of G is the $n \times n$ matrix $\mathbf{A}(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if not,} \end{cases}$$

and the *incidence matrix* of G is the $n \times m$ matrix $\mathbf{B}(G) = (b_{ij})$, where

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is incident with } e_j, \\ 0, & \text{if not.} \end{cases}$$

Note that the eigenvalues of $\mathbf{A}(G)$ are independent of the way in which the vertices are labelled. We refer to them as the *eigenvalues* of G, and to the characteristic polynomial of $\mathbf{A}(G)$ as the *characteristic polynomial* of G.

1.3 Paths and cycles

A sequence of edges of the form $v_0v_1, v_1v_2, \ldots, v_{r-1}v_r$ (sometimes abbreviated to $v_0v_1 \ldots v_r$) is a *walk of length* r between v_0 and v_r. If these edges are all distinct, then the walk is a *trail*, and if the vertices $v_0, v_1, \ldots, v_r$ are also distinct, then the walk is a *path* (or *open path*). Two paths are *edge-disjoint* if they share no common edges, and are *vertex-disjoint* if they share no common vertices, although one frequently relaxes this condition to allow the end-vertices of the paths to coincide. A walk or trail is *closed* if $v_0 = v_r$, and for $r > 0$ a closed walk in which the vertices $v_0, v_1, \ldots, v_{r-1}$ are all distinct is a *cycle*.

A cycle of length 3 is a *triangle*. The length of a shortest cycle in a graph G is the *girth* of G. If v and w are vertices in G, then the length $d(v, w)$ of any shortest path from v to w is the *distance* between v and w. The largest distance between two vertices in G is the *diameter* of G.

These definitions extend to directed graphs and infinite graphs. Thus, a *trail* in a digraph is a sequence of distinct arcs of the form $v_0v_1, v_1v_2, \ldots, v_{r-1}v_r$, a *path* is such a sequence in which the vertices are all distinct, and for $r > 0$ a *cycle* is a sequence of arcs of the form $v_0v_1, v_1v_2, \ldots, v_{r-1}v_0$, where $v_0, v_1, \ldots, v_{r-1}$ are distinct. In an infinite graph, a *two-way infinite path* is a sequence of distinct edges of the form

$$\ldots, v_{-r}v_{-r+1}, \ldots, v_{-1}v_0, v_0v_1, \ldots, v_rv_{r+1}, \ldots .$$

A graph G is *connected* if there is a path joining each pair of vertices of G; a graph that is not connected is called *disconnected*. Every disconnected graph can be split into maximal connected subgraphs, called *components*. There are analogous definitions for digraphs; a digraph D is *strongly connected* if there is a (directed) path in D joining each pair of vertices in each direction, and *connected* if the underlying graph is connected.

1.4 New graphs from old

A *subgraph* of a graph $G = (V(G), E(G))$ is a graph $H = (V(H), E(H))$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then H is a *spanning subgraph* of G. If W is any set of vertices in G, then the *subgraph induced by W* is the subgraph of G obtained by joining those pairs of vertices in W that are joined in G. An *induced subgraph of G* is a subgraph that is induced by some subset W of $V(G)$. Similar definitions can be given for digraphs and multigraphs.

If e is an edge of G, then the *edge-deleted subgraph* $G - e$ or $G \backslash e$ is the graph obtained from G by removing the edge e; more generally, $G - \{e_1, \ldots, e_k\}$ is the graph obtained from G by removing the edges $e_1, \ldots, e_k$. Similarly, if v is a vertex of G, then the *vertex-deleted subgraph* $G - v$ is the graph obtained from G by removing the vertex v together with all its incident edges; more generally, $G - \{v_1, \ldots, v_k\}$ is the graph obtained from G by removing the vertices $v_1, \ldots, v_k$ and all edges incident with any of them. These concepts are illustrated in Fig. 1.9.

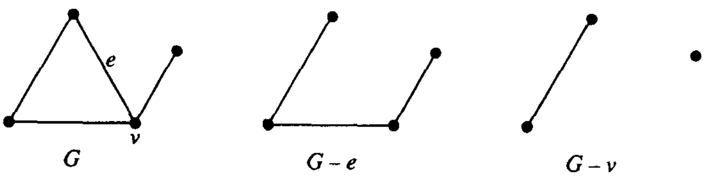


FIG. 1.9

We can also obtain a new graph from G by removing the edge $e = vw$ and identifying v and w so that the resulting vertex is incident to all edges (other than e) that were originally incident with v or w; this is called *contracting the edge e* (see Fig. 1.10), and the resulting graph is denoted by G/e. If the graph H can be obtained from G by a succession of edge contractions such as this, then G is *contractible* to H. A *minor* of G is any graph obtained from G by a succession of edge-deletions and edge-contractions.

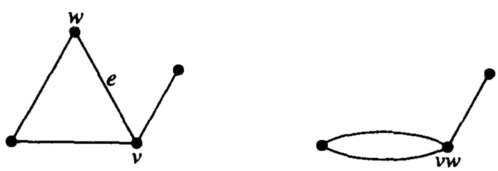


FIG. 1.10

If $e = vw$ is an edge of G, then we obtain a new graph by replacing e by two new edges vz and zw, where z is a new vertex; this is called *subdividing the edge* (see Fig. 1.11). Two graphs that can be obtained from the same graph by subdividing its edges are *homeomorphic*.

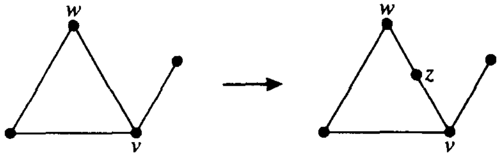


FIG. 1.11

If G and G' are graphs with the same vertex set, then their *intersection* $G \cap G'$ is the graph with edge set $E(G) \cap E(G')$, and their *union* $G \cup G'$ is the graph with edge set $E(G) \cup E(G')$. If G and G' are disjoint graphs, then their *disjoint union* $G \cup G'$ is the graph with vertex set $V(G) \cup V(G')$ and edge set $E(G) \cup E(G')$; the disjoint union of k copies of G is written kG. The *join* $G * G'$ is obtained from the disjoint union of G and G' by adding an edge between each vertex of G and each vertex of G'. The *Cartesian product* $G \times G'$ is the graph with vertex set $V(G) \times V(G')$ in which the vertex (v, w) is adjacent to the vertex (v', w') whenever $v = v'$ and w is adjacent to w', or $w = w'$ and v is adjacent to v'.

The *complement* $\overline{G}$ of G is the graph with the same vertex set as G, but where two vertices are adjacent whenever they are *not* adjacent in G; a graph is *self-complementary* if it is isomorphic to its complement. The *line graph* $L(G)$ of G is the graph whose vertices correspond to the edges of G, and where two vertices are joined whenever the corresponding edges of G are adjacent.

If G is a connected graph, and if the graph $G - e$ is disconnected for some edge e, then e is a *bridge* (or *cut-edge* or *isthmus*) of G. More generally, a *cutset* (or *edge-cut*) in G is a set of edges whose removal disconnects G. A graph G is *k-edge-connected* if every two vertices v and w are connected by at least k edge-disjoint paths, and the *edge-connectivity* $\lambda(G)$ of G is the largest value of k for which G is k-edge-connected.

If G is a connected graph, and if the graph $G - v$ is disconnected for some vertex v, then v is a *cut-vertex* of G. More generally, a *separating set of vertices* in G is a set of vertices whose removal disconnects G. A graph G with at least $k + 1$ vertices is *k-connected* if every two vertices v and w are connected by at least k paths that are pairwise disjoint except for the vertices v and w; a 2-connected graph is a *block* or a *non-separable graph.* The *connectivity* $\kappa(G)$ of G is the largest value of k for which G is k-connected; note that $\kappa(G) \leq \lambda(G)$.

The most important result relating these concepts is *Menger's Theorem*; it takes several forms, among which are the following.

Theorem 1.1 (Menger's Theorem) *Let G be a connected graph with at least $k + 1$ vertices. Then*

(a) G is k-connected if and only if G cannot be disconnected by the removal of $k - 1$ or fewer vertices;

(b) G is k-edge-connected if and only if G cannot be disconnected by the removal of $k - 1$ or fewer edges.

1.5 Examples of graphs

A graph in which every two vertices are adjacent is a *complete graph*; the complete graph with n vertices and $n(n-1)/2$ edges is denoted by K_n. The *cycle graph* C_n of order n consists of the vertices and edges of an n-gon. The *wheel* W_n is the graph $C_{n-1} * K_1$, and the path graph P_n is obtained by removing an edge from C_n. The *null graph* N_n of order n is the graph with n vertices and no edges. The graphs K_5, C_5, W_5, P_5 and N_5 are shown in Fig. 1.12. It is also occasionally useful to introduce the *empty graph* (not really a graph at all), consisting of no vertices or edges.

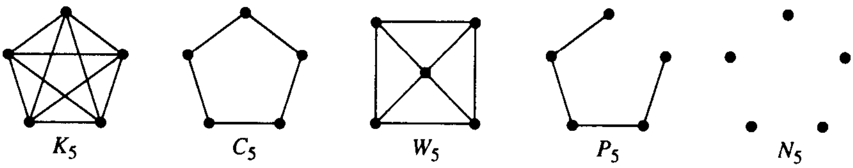


FIG. 1.12

A *clique* in a graph G is a complete subgraph of G, and a *maximum clique* is a clique of maximum order in G. The *clique number* $\omega(G)$ of G is the order of a maximum clique. A *tournament* is an 'oriented complete graph'—that is, a digraph in which every two vertices are joined by exactly one arc.

A *bipartite graph* is a graph whose vertex set can be partitioned into two sets so that each edge joins a vertex of the first set and a vertex of the second set. A *complete bipartite graph* is a bipartite graph in which each vertex in the first set is adjacent to every vertex in the second set; if the two sets contain r and s vertices, then the complete bipartite graph is denoted by $K_{r,s}$. A *complete k-partite graph* is obtained by partitioning the vertex set into k sets, and joining two vertices whenever they lie in different sets; if all of these sets have size r, then the resulting graph is the complement of rK_k, and is denoted by $K_{r,\ldots,r}$ or $K_{k(r)}$. The graphs $K_{3,3}$ and $K_{3,3,3}$ are shown in Fig. 1.13.

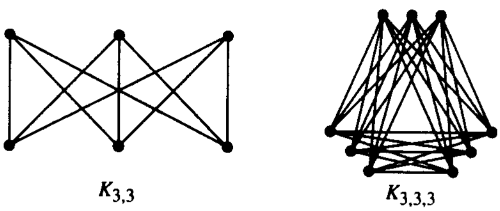


FIG. 1.13

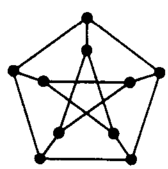


FIG. 1.14

The *Petersen graph* is the graph shown in Fig. 1.14; it is the complement of the line graph of K_5. The *Platonic graphs* are the graphs corresponding to the vertices and edges of the five regular solids—the tetrahedron, cube, octahedron,