OPTIMAL CONTROL

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PREFACE

This book is intended for use in a second graduate course in modern control theory. A background in the state-variable representation of systems is assumed. Matrix manipulations are the basic mathematical vehicle, and for those whose memory needs refreshing, Appendix A provides a short review.

The book is also intended as a reference. Numerous tables make it easy to find the equations needed to implement optimal controllers for practical applications.

Our interactions with nature can be divided into two categories: observation and action. While observing, we process data from an essentially uncooperative universe to obtain knowledge. Based on this knowledge, we act to achieve our goals. This book treats the control of systems assuming perfect and complete knowledge. The dual problem of estimating the state of our surroundings is assumed to have been solved. A course in optimal estimation is required to conscientiously complete the picture begun in this text.

My intention was to present optimal control theory in a clear and direct fashion. This goal naturally obscures the more subtle points and unanswered questions that are scattered throughout the field of modern system theory. What appears here as a completed picture is in actuality a growing body of knowledge that can be interpreted from several points of view and that takes on different personalities as new research is completed.

I have tried to show with many examples that computer simulations of optimal controllers are easy to implement and are an essential part of gaining an intuitive feel for the equations. Students should be able to write simple programs as they progress through the book to convince themselves that they have confidence in the theory and understand its practical implications.

Relations to classical control theory have been pointed out, and a root-locus approach to steady-state controller design is included. A chapter on optimal control of polynomial systems is included to provide a background for further study in the field of adaptive control.

This book is dedicated to my teachers: J. B. Pearson, who gave me the initial excitement and passion for the field; E. W. Kamen, who tried to teach me persistence and attention to detail; B. L. Stevens, who forced me to consider applications to real situations; R. W. Newcomb, who gave me self-confidence; and A. H. Haddad, who showed me the big picture and the humor behind it all. It is also dedicated to my students, who forced me to take the work seriously and become a part of it.

Acknowledgments and grateful thanks are due to Pam Majors and Peggy Knight, without whose monumental efforts at typing and meeting deadlines this book would not be a reality, and to Mary Ann Tripp, who converted my incomprehensible hand-drawn sketches into the figures in their final forms.

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STATIC OPTIMIZATION

In this chapter we discuss optimization when time is not a parameter. The discussion is preparatory to dealing with time-varying systems in subsequent chapters. A reference that provides an excellent treatment of this material is Bryson and Ho (1975), and we shall sometimes follow their point of view.

Appendix A should be reviewed, particularly the section that discusses matrix calculus.

1.1 OPTIMIZATION WITHOUT CONSTRAINTS

A scalar *performance index* $L(u)$ is given that is a function of a *control* or *decision vector* $u \in R^m$. We want to select the value of u that results in a minimum value of $L(u)$.

To solve this optimization problem, write the Taylor series expansion for an increment in L as

$$dL = L_u^T du + \tfrac{1}{2} du^T L_{uu} du + O(3), \tag{1.1-1}$$

where $O(3)$ represents terms of order three. The gradient of L with respect to u is the column m vector

$$L_u \triangleq \frac{\partial L}{\partial u} \tag{1.1-2}$$

and the hessian matrix is

$$L_{uu} = \frac{\partial^2 L}{\partial u^2}. \tag{1.1-3}$$

L_{uu} is called the *curvature matrix*. For more discussion on these quantities, see Appendix A. Note that the gradient is defined throughout the book as a *column* vector, which is at variance with some authors, who define it as a row vector.

A *critical* or *stationary point* occurs when the increment dL is zero to first order for all increments du in the control. Hence

$$L_u = 0 \tag{1.1-4}$$

for a critical point.

Suppose that we are at a critical point, so $L_u = 0$ in (1.1-1). In order for the critical point to be a local minimum, we require that

$$dL = \tfrac{1}{2}\, du^{\mathrm{T}} L_{uu}\, du + O(3) \tag{1.1-5}$$

be positive for all increments du. This is guaranteed if the curvature matrix L_{uu} is positive definite,

$$L_{uu} > 0. \tag{1.1-6}$$

If L_{uu} is negative definite, the critical point is a local maximum; and if L_{uu} is indefinite, the critical point is a saddle point. If L_{uu} is semidefinite, then higher terms of the expansion (1.1-1) must be examined to determine the type of critical point.

Recall that L_{uu} is positive (negative) definite if all its eigenvalues are positive (negative), and indefinite if it has both positive and negative eigenvalues, all nonzero. It is semidefinite if it has some zero eigenvalues. Hence if $|L_{uu}| = 0$, the second-order term does not completely specify the type of critical point.

Example 1.1-1: Quadratic Surfaces

Let $u \in R^2$ and

$$L(u) = \tfrac{1}{2}\, u^{\mathrm{T}} \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} u + [s_1 \quad s_2]\, u \tag{1}$$

$$\triangleq \tfrac{1}{2} u^{\mathrm{T}} Q u + S^{\mathrm{T}} u. \tag{2}$$

The critical point is given by

$$L_u = Qu + S = 0 \tag{3}$$

or

$$u^* = -Q^{-1} S, \tag{4}$$

where u^* denotes the optimizing control.

The type of critical point is determined by examining the hessian

$$L_{uu} = Q. \tag{5}$$

The point u^* is a minimum if $L_{uu} > 0$, or (Appendix A) $q_{11} > 0$, $q_{11}q_{22} - q_{12}^2 > 0$. It is a maximum if $L_{uu} < 0$, or $q_{11} < 0$, $q_{11}q_{22} - q_{12}^2 > 0$. If $|Q| < 0$, then u^* is a saddle point. If $|Q| = 0$, then u^* is a *singular point* and we cannot determine whether it is a minimum or a maximum from L_{uu}.

By substituting (4) into (2) we find the extremal value of the performance index to be

$$L^* \triangleq L(u^*) = \tfrac{1}{2}S^TQ^{-1}QQ^{-1}S - S^TQ^{-1}S$$

$$= -\tfrac{1}{2}S^TQ^{-1}S. \tag{6}$$

Let

$$L = \tfrac{1}{2}u^T\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}u + \begin{bmatrix}0 & 1\end{bmatrix}u. \tag{7}$$

Then

$$u^* = -\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}1\\ -1\end{bmatrix} \tag{8}$$

is a minimum, since $L_{uu} > 0$. Using (6), we see that the minimum value of L is $L^* = -\tfrac{1}{2}$.

The contours of the $L(u)$ in (7) are drawn in Fig. 1.1-1, where $u = [u_1\ u_2]^T$. The arrows represent the gradient

$$L_u = Qu + S = \begin{bmatrix}u_1 + u_2\\ u_1 + 2u_2 + 1\end{bmatrix}. \tag{9}$$

Note that the gradient is always perpendicular to the contours and pointing in the direction of increasing $L(u)$. ■

We shall use an asterisk to denote optimal values of u and L when we want to be explicit. Usually, however, the asterisk will be omitted.

Example 1.1-2: Optimization by Scalar Manipulations

We have discussed optimization in terms of vectors and the gradient. As an alternative approach, we could deal entirely in terms of scalar quantities.

To demonstrate, let

$$L(u_1, u_2) = \tfrac{1}{2}u_1^2 + u_1u_2 + u_2^2 + u_2, \tag{1}$$

where u_1 and u_2 are scalars. A critical point occurs where the derivatives of L with

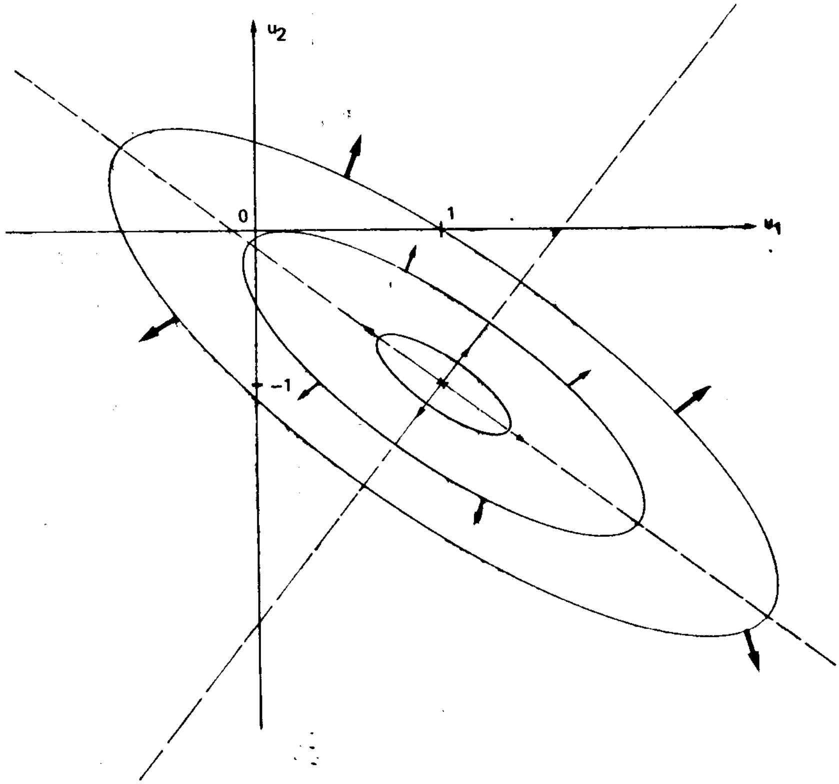

FIGURE 1.1-1 Contours and the gradient vector.

respect to *all arguments* are equal to zero:

$$\frac{\partial L}{\partial u_1} = u_1 + u_2 = 0, \tag{2a}$$

$$\frac{\partial L}{\partial u_2} = u_1 + 2u_2 + 1 = 0. \tag{2b}$$

Solving these simultaneous equations yields

$$u_1 = 1, \qquad u_2 = -1, \tag{3}$$

so a critical point is $(1, -1)$.

Note that (1) is an expanded version of (7) in Example 1.1-1, so we have just derived the same answer by another means.

Vector notation simplifies the bookkeeping involved in dealing with multidimensional quantities, and for that reason it is very attractive for our purposes. ■

1.2 OPTIMIZATION WITH EQUALITY CONSTRAINTS

Now let the scalar performance index be $L(x,u)$, a function of the control vector $u \in R^m$ and an *auxiliary* (state) *vector* $x \in R^n$. The problem is to select u to minimize $L(x,u)$ and simultaneously satisfy the *constraint equation*

$$f(x,u) = 0. \tag{1.2-1}$$

The auxiliary vector x is determined for a given u by the relation (1.2-1), so that f is a set of n scalar equations, $f \in R^n$.

To find necessary and sufficient conditions for a local minimum also satisfying $f(x,u) = 0$, we shall proceed exactly as we did in the previous section, first expanding dL in a Taylor series and then examining the first- and second-order terms. Let us first gain some insight into the problem, however, by considering it from three points of view (Bryson and Ho 1975, Athans and Falb 1966).

Lagrange Multipliers and the Hamiltonian

At a stationary point, dL is equal to zero to first order for all increments du when df is zero. Thus we require that

$$dL = L_u^{\mathrm{T}}\,du + L_x^{\mathrm{T}}\,dx = 0 \tag{1.2-2}$$

and

$$df = f_u\,du + f_x\,dx = 0. \tag{1.2-3}$$

Since (1.2-1) determines x for a given u, the increment dx is determined by (1.2-3) for a given control increment du. Thus, the Jacobian matrix f_x is nonsingular and

$$dx = -f_x^{-1} f_u\,du. \tag{1.2-4}$$

Substituting this into (1.2-2) yields

$$dL = \left(L_u^{\mathrm{T}} - L_x^{\mathrm{T}} f_x^{-1} f_u\right) du. \tag{1.2-5}$$

The derivative of L with respect to u holding f constant is therefore given by

$$\left.\frac{\partial L}{\partial u}\right|_{df=0} = \left(L_u^{\mathrm{T}} - L_x^{\mathrm{T}} f_x^{-1} f_u\right)^{\mathrm{T}} = L_u - f_u^{\mathrm{T}} f_x^{-\mathrm{T}} L_x. \tag{1.2-6}$$

where $f_x^{-\mathrm{T}}$ means $(f_x^{-1})^{\mathrm{T}}$. Note that

$$\left.\frac{\partial L}{\partial u}\right|_{dx=0} = L_u. \tag{1.2-7}$$

In order that dL equal zero to first order for arbitrary increments du when $df = 0$, we must have

$$L_u - f_u^{\mathrm{T}} f_x^{-\mathrm{T}} L_x = 0. \tag{1.2-8}$$

This is a necessary condition for a minimum. Before we derive a sufficient condition, let us develop some more insight and a very valuable tool by examining two more ways to obtain (1.2-8).

Write (1.2-2) and (1.2-3) as

$$\begin{bmatrix} dL \\ df \end{bmatrix} = \begin{bmatrix} L_x^{\mathrm{T}} & L_u^{\mathrm{T}} \\ f_x & f_u \end{bmatrix} \begin{bmatrix} dx \\ du \end{bmatrix} = 0. \tag{1.2-9}$$

This set of linear equations defines a stationary point, and it must have a solution $[dx^{\mathrm{T}}\ du^{\mathrm{T}}]^{\mathrm{T}}$. The only way this can occur is if the $(n+1) \times (n+m)$ coefficient matrix has rank less than $n+1$. That is, its rows must be linearly dependent so there exists an n vector λ such that

$$[1 \quad \lambda^{\mathrm{T}}] \begin{bmatrix} L_x^{\mathrm{T}} & L_u^{\mathrm{T}} \\ f_x & f_u \end{bmatrix} = 0. \tag{1.2-10}$$

Then

$$L_x^{\mathrm{T}} + \lambda^{\mathrm{T}} f_x = 0, \tag{1.2-11}$$

$$L_u^{\mathrm{T}} + \lambda^{\mathrm{T}} f_u = 0. \tag{1.2-12}$$

Solving (1.2-11) for λ gives

$$\lambda^{\mathrm{T}} = -L_x^{\mathrm{T}} f_x^{-1}, \tag{1.2-13}$$

and substituting in (1.2-12) again yields the condition (1.2-8) for a stationary point.

It is worth noting that the left-hand side of (1.2-8) is the transpose of the Schur complement of L_u^{T} in the coefficient matrix of (1.2-9) (Appendix A).

The vector $\lambda \in R^n$ is called a *Lagrange multiplier*, and it will turn out to be an extremely useful tool for us. To give it some additional meaning now, let $du = 0$ in (1.2-2), (1.2-3) and eliminate dx to get

$$dL = L_x^{\mathrm{T}} f_x^{-1}\, df. \tag{1.2-14}$$

Therefore

$$\left.\frac{\partial L}{\partial f}\right|_{du=0} = \left(L_x^{\mathrm{T}} f_x^{-1}\right)^{\mathrm{T}} = -\lambda, \tag{1.2-15}$$

so that $-\lambda$ is the partial of L with respect to the constraint holding the control u constant. It shows the effect on the performance index of holding the control constant when the constraints are changed.

As a third method of obtaining (1.2-8), let us develop the approach we shall use for our analysis in subsequent chapters. Adjoint the constraints to the performance index to define the *Hamiltonian* function

$$H(x, u, \lambda) = L(x, u) + \lambda^T f(x, u), \tag{1.2-16}$$

where $\lambda \in R^n$ is an as yet undetermined Lagrange multiplier. To choose x, u, and λ to yield a stationary point, proceed as follows.

Increments in H depend on increments in x, u, and λ according to

$$dH = H_x^T\, dx + H_u^T\, du + H_\lambda^T\, d\lambda. \tag{1.2-17}$$

Note that

$$H_\lambda = \frac{\partial H}{\partial \lambda} = f(x, u), \tag{1.2-18}$$

so suppose we choose some value of u and demand that

$$H_\lambda = 0. \tag{1.2-19}$$

Then x is determined for the given u by $f(x, u) = 0$, which is the constraint relation. In this situation the Hamiltonian equals the performance index:

$$H|_{f=0} = L. \tag{1.2-20}$$

Recall that if $f = 0$, then dx is given in terms of du by (1.2-4). We should rather not take into account this coupling between du and dx, so it is convenient to choose λ so that

$$H_x = 0. \tag{1.2-21}$$

Then, by (1.2-17), increments dx do not contribute to dH. Note that this yields a value for λ given by

$$\frac{\partial H}{\partial x} = L_x + f_x^T \lambda = 0 \tag{1.2-22}$$

or (1.2-13).

If (1.2-19) and (1.2-21), hold, then

$$dL = dH = H_u^T\, du, \tag{1.2-23}$$

since $H = L$ in this situation. To achieve a stationary point, we must therefore finally impose the *stationarity condition*

$$H_u = 0. \tag{1.2-24}$$

In summary, necessary conditions for a minimum point of $L(x, u)$ that also satisfies the constraint $f(x, u) = 0$ are

$$\frac{\partial H}{\partial \lambda} = f = 0, \tag{1.2-25a}$$

$$\frac{\partial H}{\partial x} = L_x + f_x^{\mathrm{T}}\lambda = 0, \tag{1.2-25b}$$

$$\frac{\partial H}{\partial u} = L_u + f_u^{\mathrm{T}}\lambda = 0, \tag{1.2-25c}$$

with $H(x, u, \lambda)$ defined by (1.2-16). The way we shall often use them, these three equations serve to determine x, λ, and u in that respective order. Compare the last two of these equations to (1.2-11) and (1.2-12).

In most applications we are not interested in the value of λ, but we must find its value since it is an intermediate variable that allows us to determine the quantities of interest, u, x, and the minimum value of L.

The usefulness of the Lagrange-multiplier approach can be summarized as follows. In reality dx and du are not independent increments, because of (1.2-4). By introducing an undetermined multiplier λ, however, we obtain an extra degree of freedom, and λ can be selected to make dx and du behave *as if* they were independent increments. Setting independently to zero the partials of H with respect to *all arguments* as in (1.2-25) therefore yields a stationary point. (Compare this with Example 1.1-2.) By introducing Lagrange multipliers, we have thus been able to replace the problem of minimizing $L(x, u)$ subject to the constraint $f(x, u) = 0$ with the problem of *minimizing the Hamiltonian* $H(x, u, \lambda)$ *without constraints*.

Conditions (1.2-25) determine a stationary point. We are now ready to derive a test that guarantees that this point is a minimum. We shall proceed as we did in Section 1.1.

Write Taylor series expansions for increments in L and f as

$$dL = \begin{bmatrix} L_x^{\mathrm{T}} & L_u^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} dx \\ du \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} dx^{\mathrm{T}} & du^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} L_{xx} & L_{xu} \\ L_{ux} & L_{uu} \end{bmatrix}\begin{bmatrix} dx \\ du \end{bmatrix} + O(3), \tag{1.2-26}$$

$$df = \begin{bmatrix} f_x & f_u \end{bmatrix}\begin{bmatrix} dx \\ du \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} dx^{\mathrm{T}} & du^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} f_{xx} & f_{xu} \\ f_{ux} & f_{uu} \end{bmatrix}\begin{bmatrix} dx \\ du \end{bmatrix} + O(3), \tag{1.2-27}$$

where

$$f_{xu} \triangleq \frac{\partial^2 f}{\partial u\, \partial x}$$

and so on. (What are the dimensions of f_{xu}?)

To introduce the Hamiltonian, use these equations to see that

$$[1 \quad \lambda^{\mathrm{T}}]\begin{bmatrix} dL \\ df \end{bmatrix} = \begin{bmatrix} H_x^{\mathrm{T}} & H_u^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} dx \\ du \end{bmatrix} + \tfrac{1}{2}[dx^{\mathrm{T}} \quad du^{\mathrm{T}}]\begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix}\begin{bmatrix} dx \\ du \end{bmatrix} + O(3). \tag{1.2-28}$$

Now, for a stationary point we require $f = 0$, and also that dL be zero to first order for all increments dx, du. Since f is held equal to zero, df is also zero, and so these conditions require $H_x = 0$ and $H_u = 0$ exactly as in (1.2-25).

To find sufficient conditions for a minimum, let us examine the second-order term. First, it is necessary to include in (1.2-28) the dependence of dx on du. Hence, let us suppose we are at a critical point so that $H_x = 0$, $H_u = 0$, and $df = 0$. Then by (1.2-27)

$$dx = -f_x^{-1} f_u\, du + O(2). \tag{1.2-29}$$

Substituting this relation into (1.2-28) yields

$$dL = \tfrac{1}{2}\, du^{\mathrm{T}}\begin{bmatrix} -f_u^{\mathrm{T}} f_x^{-\mathrm{T}} & I \end{bmatrix}\begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix}\begin{bmatrix} -f_x^{-1} f_u \\ I \end{bmatrix} du + O(2). \tag{1.2-30}$$

To ensure a minimum, dL in (1.2-30) should be positive for all increments du. This is guaranteed if the *curvature matrix with constraint f equal to zero*

$$L_{uu}^f \triangleq L_{uu}|_f = \begin{bmatrix} -f_u^{\mathrm{T}} f_x^{-\mathrm{T}} & I \end{bmatrix}\begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix}\begin{bmatrix} -f_x^{-1} f_u \\ I \end{bmatrix}$$

$$= H_{uu} - f_u^{\mathrm{T}} f_x^{-\mathrm{T}} H_{xu} - H_{ux} f_x^{-1} f_u + f_u^{\mathrm{T}} f_x^{-\mathrm{T}} H_{xx} f_x^{-1} f_u \tag{1.2-31}$$

is positive definite. Note that if the constraint $f(x, u)$ is identically zero for all x and u, then (1.2-31) reduces to L_{uu} in (1.1-6).

If (1.2-31) is negative definite (indefinite), then the stationary point is a constrained maximum (saddle point).