Mathematics Monograph Series **14**

# General Theory of Map Census

Yanpei Liu

（地图计数通论）

SCIENCE PRESS
Beijing

中国科学院科学出版基金资助出版

Mathematics Monograph Series 14

Yanpei Liu

# General Theory of Map Census

(地图计数通论)

SCIENCE PRESS
Beijing

Responsible Editor: Zhao Yanchao


Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, P. R. China

Printed in Beijing



ISBN 978-7-03-024435-2

# Preface

Since the first monograph titled *Enumerative Theory of Maps* appeared on the subject considered in 1999, many advances have been made by the author himself and those directed by him under such a theoretical foundation.

Because of that book with much attention to maps on surface of genus zero, this monograph is in principle concerned with maps on surfaces of genus not zero. Via main theoretical lines, this book is divided into four parts except Chapter 1 for preliminaries.

Part one contains Chapters 2 through 10. The theory is presented for maps on general surfaces of genus not necessary to be zero. For the theory on a surface of genus zero is comprehensively improved for investigating maps on all surfaces of genera not zero.

Part two consists of only Chapter 11. Relationships are established for building up a bridge between super maps and embeddings of a graph via their automorphism groups.

Part three consists of Chapters 12 and 13. A general theory for finding genus distribution of graph embeddings, handle polynomials and crosscap polynomials of super maps are constructed on the basis of the joint tree method which enables us to transform a problem in a high dimensional space into a problem on a polygon.

All other chapters, i.e., Chapters 14 through 17, as part four are concerned with several aspects of main extensions to distinct directions.

An appendix serves as atlas of super maps of typical graphs of small size on surfaces for the convenience of readers to check their understanding.

On this occasion, some of my former and present graduates such as Dr. Junliang Cai, Dr. Han Ren, Dr. Rongxia Hao, Dr. Linfan Mao, Dr. Zhaoxiang Li, Dr. Erling Wei, Dr. Liangxia Wan, Dr. Yichao Chen, Dr. Yan Xu, Dr. Wenzhong Liu, Dr. Zeling Shao, Dr. Yan Yang, Dr. Guanghua Dong et al should be particularly mentioned for their successful work in related topics.

Most new research results in this book such as Theorems 1.4.5, 1.6.3 and 1.6.4 , Chapter 4, §5.5, §6.5, §7.4, §8.5, §9.6, §10.5, §11.2~§11.4, §12.2~§12.4, §13.3~§13.4, §14.5, Chapter 15 etc are partially supported by the NNSF in China under Grant Numbers: 60373030 and 10871021.

Yanpei Liu
Beijing, P.R. China
March 2009

# Contents

**Preface**

**Chapter 1 Preliminaries** ······ **1**

§1.1 Maps ······ 1

§1.2 Polynomials on maps ······ 7

§1.3 Enufunctions ······ 16

§1.4 Polysum functions ······ 20

§1.5 The Lagrangian inversion ······ 24

§1.6 The shadow functional ······ 31

§1.7 Asymptotic estimation ······ 35

§1.8 Notes ······ 37

**Chapter 2 Outerplanar Maps** ······ **39**

§2.1 Plane trees ······ 39

§2.2 Wintersweets ······ 45

§2.3 Unicyclic maps ······ 53

§2.4 General outerplanar maps ······ 59

§2.5 Notes ······ 65

**Chapter 3 Triangulations** ······ **67**

§3.1 Outerplanar triangulations ······ 67

§3.2 Planar triangulations ······ 71

§3.3 Triangulations on the disc ······ 78

§3.4 Triangulations on the projective plane ······ 85

§3.5 Triangulations on the torus ······ 91

§3.6 Notes ······ 95

**Chapter 4 Quadrangulations** ······ **97**

§4.1 Outerplanar quadrangulations ······ 97

§4.2 Outerplanar quadrangulations on the disc ······ 101

§4.3 Hamiltonian quadrangulations on the sphere ······ 104

§4.4 Inner endless planar quadrangulations ······ 105

§4.5 Quadrangulations on the projective plane ······ 110

§4.6 Quadrangulations on the Klein bottle ······ 115

§4.7 Notes ······ 119

**Chapter 5 Eulerian Maps** ······ **120**

§5.1 Planar Eulerian maps ······ 120

§5.2 Tutte formula ········ 125
§5.3 Eulerian planar triangulations ········ 129
§5.4 Regular Eulerian planar maps ········ 134
§5.5 Eulerian maps on surfaces ········ 139
§5.6 Notes ········ 143
**Chapter 6 Nonseparable Maps ········ 145**
§6.1 Outerplanar nonseparable maps ········ 145
§6.2 Eulerian nonseparable maps ········ 151
§6.3 Planar nonseparable maps ········ 158
§6.4 Nonseparable maps on surfaces ········ 163
§6.5 Bridgeless maps on surfaces ········ 168
§6.6 Notes ········ 172
**Chapter 7 Simple Maps ········ 174**
§7.1 Loopless maps ········ 174
§7.2 General simple maps ········ 190
§7.3 Simple bipartite maps ········ 197
§7.4 Loopless maps on surfaces ········ 202
§7.5 Notes ········ 205
**Chapter 8 General Maps ········ 208**
§8.1 General planar maps ········ 208
§8.2 Planar *c*-nets ········ 213
§8.3 Convex polyhedra ········ 219
§8.4 Quadrangulations via *c*-nets ········ 224
§8.5 General maps on surfaces ········ 231
§8.6 Notes ········ 236
**Chapter 9 Chrosum Equations ········ 239**
§9.1 Tree equations ········ 239
§9.2 Outerplanar equations ········ 243
§9.3 General equations ········ 249
§9.4 Triangulation equations ········ 254
§9.5 Well definedness ········ 257
§9.6 Chrosums on surfaces ········ 261
§9.7 Notes ········ 266
**Chapter 10 Polysum Equations ········ 268**
§10.1 Polysums for bitrees ········ 268
§10.2 Outerplanar polysums ········ 273
§10.3 General polysums ········ 276
§10.4 Nonseparable polysums ········ 281

§10.5 Polysums on surfaces ··· 283
§10.6 Notes ··· 288
**Chapter 11 Maps via Embeddings ··· 290**
§11.1 Automorphism group of a graph ··· 290
§11.2 Embeddings of a graph ··· 293
§11.3 Super maps of a graph ··· 296
§11.4 Maps from embeddings ··· 301
§11.5 Notes ··· 304
**Chapter 12 Locally Oriented Maps ··· 305**
§12.1 Planar Hamiltonian maps ··· 305
§12.2 Biboundary inner rooted maps ··· 306
§12.3 Boundary maps ··· 310
§12.4 Cubic boundary maps ··· 314
§12.5 Notes ··· 316
**Chapter 13 Genus Polynomials of Graphs ··· 318**
§13.1 Joint tree model ··· 318
§13.2 Layer divisions ··· 320
§13.3 Graphs from smaller ··· 323
§13.4 Pan-bouquets ··· 327
§13.5 Notes ··· 333
**Chapter 14 From Rooted to Unrooted ··· 335**
§14.1 Symmetric relations ··· 335
§14.2 An application ··· 336
§14.3 Symmetric principles ··· 338
§14.4 General examples ··· 339
§14.5 From under graphs ··· 341
§14.6 Notes ··· 343
**Chapter 15 From Planar to Nonplanar ··· 344**
§15.1 Trees with boundary ··· 344
§15.2 Cutting along vertices ··· 346
§15.3 Cutting along faces ··· 349
§15.4 Maps with a plane base ··· 351
§15.5 Vertex partition ··· 353
§15.6 Notes ··· 357
**Chapter 16 Chromatic Solutions ··· 358**
§16.1 General solution ··· 358
§16.2 Cubic triangles ··· 364
§16.3 Invariants ··· 372

§16.4 Four color solutions ········ 379
§16.5 Notes ········ 384

**Chapter 17 Stochastic Behaviors** ········ **386**
§17.1 Asymptotics for outerplanar maps ········ 386
§17.2 The average on tree-rooted maps ········ 391
§17.3 Hamiltonian circuits per map ········ 394
§17.4 The asymmetry on maps ········ 398
§17.5 Asymptotics via equations ········ 407
§17.6 Notes ········ 411

**Appendix Atlas of Super Maps for Small Graphs** ········ **414**
Ax.1 Bouquets $B_m$, $4 \geqslant m \geqslant 1$ ········ 414
Ax.2 Link bundles $L_m$, $6 \geqslant m \geqslant 3$ ········ 419
Ax.3 Complete bipartite graphs $K_{m,n}, 4 \geqslant m, n \geqslant 3$ ········ 427
Ax.4 Wheels $W_n$, $5 \geqslant n \geqslant 4$ ········ 432
Ax.5 Triconnected cubic graphs of size in $[6, 15]$ ········ 434

**Bibliography** ········ **451**
**Subject Index** ········ **471**
**Author Index** ········ **477**

# Chapter 1

# Preliminaries

Throughout, for the sake of brevity, we adopt the following logical conventions: disjunction, conjunction, negation, implication, equivalence, universal quantification and existential quantification denoted by the symbols: $\vee$, $\wedge$, $\neg$, $\Rightarrow$, $\Leftrightarrow$, $\forall$ and $\exists$, respectively.

In the context, $(i.j.k)$ (or $i.j.k$) refers to item $k$ of section $j$ in chapter $i$ for formulae (or theorems and the like).

A reference $[k]$ refers to item $k$ under the corresponding author(s) in the bibliography.

Fundamental concepts and notations not explained in this book can be found from Liu, Y.P.[63, 68, 81].

## §1.1 Maps

A *map*, denoted by $M$, is a mathematical concept which can, of course, be seen as a kind of abstraction from that appearing in geography, is defined to be a basic permutation $\mathcal{P}$ on a disjoint union $\mathcal{X}$ of quadricells with Axiom 1 and Axiom 2 bellow.

Let $X$ be a finite set, and $K$ the Klein group of four elements which are denoted by 1, $\alpha$, $\beta$, and $\alpha\beta$. For $x \in X$, the set $Kx=\{x, \alpha x, \beta x, \alpha\beta x\}$ is said to be a *quadricell.*

We may write $\mathcal{X} = \sum_{x\in X} Kx$. Naturally, both $\alpha$ and $\beta$ themselves are permutations on $\mathcal{X}$. A permutation $\mathcal{P}$ on $\mathcal{X}$ is said to be *basic* if for any $x \in \mathcal{X}$ there does not exist an integer $k$ such that $\mathcal{P}^k x = \alpha x$.

**Axiom 1** $\alpha\mathcal{P} = \mathcal{P}^{-1}\alpha$.

**Axiom 2** *The group $\Psi_J$ which is generated by $J = \{\alpha, \beta, \mathcal{P}\}$ is transitive on $\mathcal{X}$.*

Thus, we may write the map $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$. From Axiom 1, $\alpha$ and $\beta$ are asymmetric, i.e., $(\mathcal{X}_{\alpha,\beta}(X), \mathcal{P}) \neq (\mathcal{X}_{\beta,\alpha}(X), \mathcal{P})$ in general. Sometimes, $\alpha$ is called the *first operator* and $\beta$, the *second operator.* Generally, for a map $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$, it is not necessary that $(\mathcal{X}_{\beta,\alpha}(X), \mathcal{P})$ is also a map because it is not guaranteed to have Axiom 1 for $\beta$. In fact, Axiom 1 allows us to define the *vertices* of a map as the pairs of *conjugate* orbits of $\mathcal{P}$ on $\mathcal{X}$. We always write $\mathcal{P}$ as the product of the orbits (in cyclic order) obtained by choosing exactly one, which represents a vertex as well, in each *conjugate pair* determined by $x$ and $\alpha x$ for $x \in \mathcal{X}$.

For a map $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$ given, from the definition it is easy to check that $M^* = (\mathcal{X}_{\beta,\alpha}(X), \mathcal{P}\alpha\beta)$ is also a map with $\beta$ as the first operator and $\alpha$, the second. We call $M^*$ the *dual* (map) of $M$. From the duality, the *faces* of $M$ are defined to be the vertices of $M^*$. Moreover, an *edge* of $M$ is defined to be the quadricell $Kx = \{x, \alpha x, \beta x, \alpha\beta x\}$ for $x \in X$. An edge $\{x, \alpha x, \beta x, \alpha\beta x\}$ can be seen as a pair of *semiedges* $\{x, \alpha x\}$ and $\{\beta x, \alpha\beta x\}$, or $\{x, \beta x\}$ and $\{\alpha x, \alpha\beta x\}$ as well.

The graph whose vertices and edges are those of a map $M$ is said to be the *underlying graph* of $M$ and is denoted by $G(M)$. From Axiom 2, $G(M)$ has to be connected. Conversely, a map $M$ whose vertices and edges are those of a graph $G$ is said to be an *underlain map* of $G$ and denoted by $M(G)$. Of course, $M$ is an underlain map of $G(M)$. Although any map has a unique underlying graph, a graph is in general allowed to have many underlain maps.

In fact, any underlain map of a graph (connected of course) is an embedding on a surface. This enables us to denote a map $M$ by $(G, F)$ such that $G = (V, E) = G(M)$ where $V$, $E$ and $F$ are the vertex, edge and face sets respectively. Only one vertex without edge is always defined to be a map, which is called the *trivial map*, or the *vertex map*. If a map has a single edge, then it is called an *edge map* denoted by $L$. If an edge map has a loop, then it is called a *loop map*; otherwise, the *link map*. Apparently, only two possible loop maps exist. They are $L_1 = (\mathcal{X}, (x, \alpha\beta x))$ and $L_2 = (\mathcal{X}, (x, \beta x))$. The unique link map is $L_0 = (\mathcal{X}, (x)(\alpha\beta x))$.

Let $\nu$, $\varepsilon$ and $\phi$ be the numbers of vertices, edges and faces of a map $M$ respectively. The number

$$\mathrm{Eul}(M) = \nu - \varepsilon + \phi \tag{1.1.1}$$

is said to be the *Euler characteristic* of $M$.

Further, if a map $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$ satisfies the following Axiom 3, then it is said to be *nonorientable*; otherwise, *orientable*.

**Axiom 3** *The group $\Psi_L$ generated by $L = \{\alpha\beta, \mathcal{P}\}$ is transitive on $\mathcal{X}_{\alpha,\beta}(X)$.*

Because it can be shown that if the group $\Psi_L$ is not transitive on $\mathcal{X}_{\alpha,\beta}(X)$ then it has exactly two orbits one of which is conjugate of the other on $\mathcal{X}_{\alpha,\beta}(X)$, a map $(\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$ is orientable if, and only if, the group $\Psi_L$ has two orbits on $\mathcal{X}_{\alpha,\beta}(X)$.

Let $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$ be a map and $e_x = \{x, \alpha x, \beta x, \alpha\beta x\}$ be the edge incident with $x \in \mathcal{X}_{\alpha,\beta}(X)$. For convenience, we always write $\mathcal{X}$ instead of $\mathcal{X}_{\alpha,\beta}(X)$ without specific indication and see that

$$\mathcal{X} = X + \alpha X + \beta X + \alpha\beta X, \tag{1.1.2}$$

where $\gamma X = \{\gamma x | \forall x \in X\}$ for $\gamma = \alpha,\ \beta$, or $\alpha\beta$. Moreover, an edge $e_x$ for $x \in X$ is simply denoted by $e$.

Now, we introduce two kinds of operations for an edge $e$ on a map $M$. By the *deletion* of e on $M$, denoted by $M - e$, is meant that

$$M - e = (\mathcal{X} - e, \mathcal{P}\langle e\rangle), \tag{1.1.3}$$

where $\mathcal{P}\langle e\rangle$ is the restriction of $\mathcal{P}$ on $\mathcal{X} - e$. The other, called *contraction* of e on $M$ and denoted by $M \bullet e$, is

$$M \bullet e = (\mathcal{X} - e, \mathcal{P}[e]), \tag{1.1.4}$$

where $\mathcal{P}[e]$ is obtained by composing the two vertices $u$ and $v$ incident to $e$ as

$$\{AB, \alpha B^{-1}A^{-1}\}$$

when

$$u = \{xA, \alpha x\alpha A^{-1}\} \text{ and } v = \{\alpha\beta xB, \beta x\alpha B^{-1}\}$$

while all other vertices are in agreement with those for $\mathcal{P}$.

**Theorem 1.1.1** *Any map $M$ has* $\mathrm{Eul}(M) \leqslant 2$.

**Proof** Because the deletion of an edge on the common boundary of two faces in a map $M$ reduces one in the face number of $M$, we can always find a map $M'$, $\nu(M') = \nu(M)$, such that $M'$ has only one face and $\mathrm{Eul}(M') = \mathrm{Eul}(M)$ by a series of the operations. From the connectedness, $\nu(M') \leqslant \varepsilon(M') + 1$. Therefore,

$$\mathrm{Eul}(M) = \nu(M') - \varepsilon(M') + 1 \leqslant 2.$$

The theorem is proved. □

Two more operations which are often used have to be explained. Suppose $v = (AB)$ is a vertex of a map $M$. Let $\mathcal{P}'$ be obtained by substituting $(Ax)$ and $(\alpha\beta xB)$ for $(AB)$ in $\mathcal{P}$ where $x$ is incident with the new edge. Then, the map $M' = (\mathcal{X} + Kx, \mathcal{P}')$ is said to be obtained by *splitting* the vertex $v$ on $M$. If $v = (xy)$ is a vertex in a map $M$, then the map

$$M' = (\mathcal{X} - Kx - Ky + Kz, \mathcal{P}'), \quad z = x = y,$$

where $\mathcal{P}'$ is the resultant one of deleting $(xy)$ from $\mathcal{P}$, is said to be obtained by *missing* the vertex $v$ on $M$. The inverse of deletion of an edge is called the *addition* of an edge and the inverse of missing a vertex, *subdividing* an edge.

Of course, the inverse of contraction of an edge is splitting a vertex as defined above. It is easily seen that the Euler characteristic is unchanged under the contraction of an edge, missing a vertex and their inverses: splitting a vertex, subdividing an edge.

However, the invariance of the Euler characteristic under edge deletion and its inverse, the edge addition, is only for an edge on the common boundary of two faces, or say, under *standard* deletion and addition.

Because any map can be transformed into another which has only one face such that the Euler characteristic is unchanged by virtue of what appears in the proof of Theorem 1.1.1, we are allowed to consider one face maps for the sake of finding the simplest one with a given Euler characteristic. For brevity, a map is represented by its faces with the convention: $x^{-1} = \alpha\beta x$ and hence $(\alpha x)^{-1} = \beta x$. On the whole, we are allowed to realize $x = \alpha x$ and hence $\beta x = \alpha\beta x$.

For orientable maps we have the following two properties: Orien.1 and Orien.2 which can be derived from the operations mentioned above.

**Orien.1** If a one face map $M = (Rxx^{-1}Q)$, $R, Q \neq \varnothing$, then

$$\mathrm{Eul}(M) = \mathrm{Eul}(RQ).$$

**Orien.2** If a one face map $M = (PxQyRx^{-1}Sy^{-1}T)$, then

$$\mathrm{Eul}(M) = \mathrm{Eul}(PSRQTxyx^{-1}y^{-1}).$$

For nonorientable maps, we have the following two properties: Norien.1 and Norien.2 which can be derived from the operations as well.

**Norien.1** If a one face map $M = (PxQxR)$, then

$$\mathrm{Eul}(M) = \mathrm{Eul}(PQ^{-1}Rxx).$$

**Norien.2** If a one face map $M = (Axxyzy^{-1}z^{-1})$, then

$$\mathrm{Eul}(M) = \mathrm{Eul}(Ax_1x_1x_2x_2x_3x_3).$$

**Theorem 1.1.2** *If a map $M$ is orientable, then we have* $\mathrm{Eul}(M) = 0 \pmod 2$. *Moreover, $M$ is on the surface of genus $p$ (orientable), $p \geqslant 0$, if, and only if,*

$$\mathrm{Eul}(M) = 2 - 2p,$$

*where* $\mathrm{Eul}(M)$ *is the Euler characteristic of $M$ defined by* (1.1.1).

**Proof** From the orientability, each edge $e = Kx$ is only allowed to have $\{x, \alpha\beta x\} = \{x, x^{-1}\}$ (or $\{\alpha x, \beta x\}$ as well) in one of the two orbits of the group $\Psi_L$ on $\mathcal{X}$. By using the properties: Orien.1 and Orien.2 as far as possible, we may finally find that $\mathrm{Eul}(M)$ is equal to either $\mathrm{Eul}(O_0)$, $O_0 = (xx^{-1})$, or

$$\mathrm{Eul}(O_p), \quad O_p = \left(\prod_{i=1}^{p} x_i y_i x_i^{-1} y_i^{-1}\right)$$

for an integer $p \geqslant 1$. By counting the numbers of vertices, edges and faces in $O_0$ and $O_p$, the first statement of the theorem can be obtained. The second statement is a conclusion of the characterization of orientable surfaces. $\square$

**Theorem 1.1.3** *For a nonorientable map $M$, $M$ is on the surface of genus $q$ (nonorientable) if, and only if, $M$ has*

$$\mathrm{Eul}(M) = 2 - q,$$

*where $q \geqslant 1$.*

**Proof** From the nonorientability, there always is $x$ in $\mathcal{X}$ such that both $x$ and $\alpha x$ appear in the face of a one face map. Or in our words here, $x$ appears twice. By using the properties Norien.1 and Norien.2 as far as possible, we may finally find that

$$\mathrm{Eul}(M) = \mathrm{Eul}(N_q), \quad N_q = \left(\prod_{i=1}^{q} x_i x_i\right)$$

for an integer $q \geqslant 1$. Hence, from counting the numbers of vertices, edges, and faces in $N_q$, by virtue of the characterization of nonorientable surfaces the theorem is soon obtained. $\square$

All $O_p$, $p \geqslant 0$ and $N_q$, $q \geqslant 1$, are called *standard maps* on the corresponding surface. If $\mathrm{Eul}(M) = 2$, i.e., $p(M) = 0$, then $M$ is said to be *planar*. The cases of $p(M) = 1$, $q(M) = 1$ and 2, which are often encountered, show that $M$ is on the *torus*, the *projective plane* and the *Klein bottle* respectively.

For two maps $M_1 = (\mathcal{X}_{\alpha,\beta}(X_1), \mathcal{P}_1)$ and $M_2 = (\mathcal{X}_{\alpha,\beta}(X_2), \mathcal{P}_2)$, if there exists a bijection

$$\tau : \mathcal{X}_{\alpha,\beta}(X_1) \longrightarrow \mathcal{X}_{\alpha,\beta}(X_2)$$

such that the diagrams (1.1.5) as shown below are commutative for $\gamma_1 = \gamma_2 = \alpha$, for $\gamma_1 = \gamma_2 = \beta$ and for $\gamma_1 = \mathcal{P}_1$ and $\gamma_2 = \mathcal{P}_2$, then we say $M_1$ and $M_2$ are *isomorphic* while $\tau$ is called an *isomorphism* between them.

$$\begin{array}{ccc}
\mathcal{X}_{\alpha,\beta}(X_1) & \xrightarrow{\quad\tau\quad} & \mathcal{X}_{\alpha,\beta}(X_2) \\
\gamma_1 \big\downarrow & & \big\downarrow \gamma_2 \\
\mathcal{X}_{\alpha,\beta}(X_1) & \xrightarrow[\quad\tau\quad]{} & \mathcal{X}_{\alpha,\beta}(X_2)
\end{array} \qquad (1.1.5)$$

An isomorphism of a map $M$ to itself is called an *automorphism* of $M$. All automorphisms of a map $M$ form a group which is called the *automorphism group* of $M$ and denoted by $\mathrm{Aut}(M)$. The order of $\mathrm{Aut}(M)$ is written as $\mathrm{aut}(M) = |\mathrm{Aut}(M)|$.

If a map $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$ has an element, the *root* denoted by $r = r(M)$, in $\mathcal{X}_{\alpha,\beta}(X)$ marked beforehand, then $M$ is called a *rooted map* and the marked edge, the *rooted* edge of $M$, which is usually denoted by $a = e_r(M)$. And likewise, the *rooted vertex* and the *rooted face*. Two rooted maps are said to be *isomorphic* if there is an isomorphism between them such that their roots are in correspondence.

**Theorem 1.1.4** *For any rooted map, its automorphism group is the trivial group.*

**Proof** Let $\tau$ be an automorphism of a map $M$ with $r$ being the root. Because $\tau(r) = r$, from (1.1.5) we see that

$$\tau(\alpha r) = \alpha r, \quad \tau(\beta r) = \beta r \quad \text{and } \tau(\mathcal{P}r) = \mathcal{P}r.$$

Thus, for any $\psi \in \Psi_J$, the group generated by $J = \{\alpha, \beta, \mathcal{P}\}$, we have $\tau(\psi r) = \psi r$. From Axiom 2, the theorem follows. □

Based on this theorem, we may find

**Theorem 1.1.5** *Let $\nu_i$ and $\phi_i$ be the respective number of vertices and faces of valency $i$, $i \geqslant 1$, on a map $M$. Then,*

$$\text{aut}(M) \;\Big|\; (2i\nu_i,\ 2j\phi_j \mid \forall i,\ i \geqslant 1,\ \forall j,\ j \geqslant 1), \tag{1.1.6}$$

*where $(2i\nu_i,\ 2j\phi_j \mid \forall i,\ i \geqslant 1,\ \forall j,\ j \geqslant 1)$ is the greatest common divisor of all the numbers in the parentheses.*

**Proof** From (1.1.5), an automorphism $\tau$ on $M$ has to have the property that for $x \in \mathcal{X}$ which is incident to a vertex of valency $i$, $i \geqslant 1$, and with a face of valency $j$, $j \geqslant 1$, $\tau(x)$ has to be incident to a vertex of valency $i$ and with a face of valency $j$ as well. We may classify the elements which are incident to a vertex of valency $i$ in $\mathcal{X}$ by the rule:

$$x \sim_{\text{Aut}} y \Longleftrightarrow \exists \tau \in \text{Aut}(M),\ x = \tau y.$$

And then, it is seen that all the classes obtained in this way have the same cardinality which is the order of the automorphism group $\text{Aut}(M)$ from Theorem 1.1.4. Since the number of the elements incident to a vertex of valency $i$ is $2i\nu_i$, we have $\text{aut}(M)|2i\nu_i$. Similarly, we may also find $\text{aut}(M)|2j\phi_j$. From the arbitrariness of the choice of $i$, $i \geqslant 1$, and $j$, $j \geqslant 1$, the theorem is obtained. □

From Theorem 1.1.5, one can soon find

$$\text{aut}(M) \leqslant (2i\nu_i,\ 2j\phi_j \mid \forall i,\ i \geqslant 1,\ \forall j,\ j \geqslant 1). \tag{1.1.7}$$

Because of the relation

$$4\varepsilon = 2\sum_{i=1}^{\nu} i\nu_i = 2\sum_{j=1}^{\phi} j\phi_j,$$

and then

$$(2i\nu_i,\ 2j\phi_j \mid \forall i,\ i \geqslant 1,\ \forall j,\ j \geqslant 1) \Big| \ 4\varepsilon, \tag{1.1.8}$$

Theorem 1.1.5 implies that

$$\mathrm{aut}(M) \Big| \ 4\varepsilon \text{ and hence } \mathrm{aut}(M) \leqslant 4\varepsilon. \tag{1.1.9}$$

Theorem 1.1.4 allows us to enumerate rooted maps without considering the symmetry. Meanwhile, Theorem 1.1.5 would have more advantages for enumerating general maps with the consideration of the symmetry from those obtained in the corresponding asymmetric case because an efficient algorithm for finding the automorphism group of a map can be designed on the basis.

## §1.2 Polynomials on maps

From the two operations defined by (1.1.3) and (1.1.4), we are allowed to introduce a number of polynomials on maps. Here, only those related to the two kinds of polynomials, the chromatic polynomials and the dichromatic polynomials, are discussed. For generality, each edge $e$ is assigned to have a weight: $w(e) = 0$ or 1 in other words, to be binary on a map $M$.

For a map $M$, not necessary to be planar, let $V$ and $E$ be the sets of vertices and edges of $M$ respectively.

Suppose each edge $e \in E$ is with a weight $w(e)$ whose value is assigned to be binary. We may introduce a map function $\Phi(M)$ by the following recursion: for $e \in E$,

$$\Phi(M) = \begin{cases} A(e)\Phi(M-e) + B(e)\Phi(M \bullet e), \\ \qquad \text{if } e \in E \text{ is neither a loop} \\ \qquad\qquad \text{nor an isthmus;} \\ (X+Yz)^{\bar{w}(e)}(Xz+Y)^{w(e)}\Phi(M-e), \\ \qquad \text{if } e \text{ is a loop;} \\ (X+Yz)^{w(e)}(Xz+Y)^{\bar{w}(e)}\Phi(M \bullet e), \\ \qquad \text{if } e \text{ is an isthmus,} \end{cases} \tag{1.2.1}$$

where

$$\begin{cases} A(e) = \bar{w}(e)X + w(e)Y; \\ B(e) = w(e)X + \bar{w}(e)Y \end{cases} \tag{1.2.2}$$

with the two conditions below being satisfied.

**Cond1** If $M$ is the vertex map, then

$$\Phi(M) = 1. \quad (1.2.3)$$

**Cond2** If $M = M_1+M_2$, or in other words, $M$ is the union of $M_1$ and $M_2$ provided without common vertex of $M_1$ and $M_2$, then

$$\Phi(M) = z\Phi(M_1)\Phi(M_2). \quad (1.2.4)$$

Let $k_i = k_i(M)$ and $l_i = l_i(M)$ be the numbers of isthmuses and loops of weight $i$, $i = 0, 1$, respectively.

**Theorem 1.2.1** *If the underlying graph of a map $M$ has each of its edges either an isthmus or a loop, then we have*

$$\Phi(M) = (X + Yz)^{k_1+l_0}(Xz + Y)^{k_0+l_1}. \quad (1.2.5)$$

**Proof** When $M$ has only one loop or isthmus, it is easy to check that (1.2.5) holds. In general, if $e$ is a loop, then from (1.2.1), we should have

$$\Phi(M) = (X + Yz)^{\bar{w}(e)}(Xz + Y)^{w(e)}\Phi(M - e)$$

and hence (1.2.5) by the hypothesis of induction; otherwise, we may also obtain (1.2.5) in the similar way for $e$ as an isthmus instead of a loop. □

In fact, we may further have

**Theorem 1.2.2** *For any map $M$, the function*

$$\Phi(M) = \Phi(M; X, Y, z)$$

*is always a polynomial of $X$, $Y$, and $z$ as undeterminate.*

**Proof** Because of (1.2.1)–(1.2.4), the function $\Phi(M)$ of any map $M$ can always be expressed into a form in linear combination of the functions of those with all the edges either loop or isthmus such that the coefficients are all polynomials of $X$, $Y$ and $z$.

Hence, from Theorem 1.2.1, the theorem follows. □

If two planar maps $M_1$ and $M_2$ are mutual dual, then it can be shown that the functions $\Phi(M_1)$ and $\Phi(M_2)$ have the same type.

**Theorem 1.2.3** *For a planar map $M$, let $M^*$ be its dual. Then, we have*

$$\Phi(M; X, Y, z) = \Phi(M^*; Y, X, z), \quad (1.2.6)$$

*where $\Phi$ is defined by* (1.2.1).

**Proof** Because of the duality between the two operations: deletion and contraction in the pair of $M$ and $M^*$, the theorem can be soon derived. □

If two maps are isomorphic such that corresponding edges are of the same weight, then they are seen to be equivalent. By a *combinatorial invariant* of maps, we shall mean such a function of maps that has the same value when two maps are in the same equivalent class.

**Theorem 1.2.4** *The function $\Phi$ is a combinatorial invariant on maps.*

**Proof** First, it is easily checked that if $M$ has at most one edge which is neither a loop nor an isthmus, the theorem holds.

We are then allowed to suppose $e_1$ and $e_2$ are two edges, each of which is neither a loop nor an isthmus.

Further, we are also allowed to assume that none of $e_1$ and $e_2$ is a multiedge from (1.2.1)–(1.2.4) without loss of generality. Since

$$\begin{aligned}
&\Big(A(e_1)\Phi(M-e_1)+B(e_1)\Phi(M\bullet e_1)\Big)\\
&\quad-\Big(A(e_2)\Phi(M-e_2)+B(e_2)\Phi(M\bullet e_2)\Big)\\
&=A(e_1)B(e_2)\Phi\Big((M-e_1)\bullet e_2\Big)\\
&\quad+B(e_1)A(e_2)\Phi\Big((M\bullet e_1)-e_2\Big)\\
&\quad-A(e_2)B(e_1)\Phi\Big((M-e_2)\bullet e_1\Big)\\
&\quad-B(e_2)A(e_1)\Phi\Big((M\bullet e_2)-e_1\Big)\\
&=0,
\end{aligned}$$

by considering that

$$\begin{cases}
\Phi((M-e_1)\bullet e_2) = \Phi((M\bullet e_2)-e_1);\\
\Phi((M\bullet e_1)-e_2) = \Phi((M-e_2)\bullet e_1),
\end{cases}$$

the theorem can be obtained recursively. □

If maps are restricted to planar maps, then it is known that the polynomial $\Phi$ is a kind of generalizations of the Jones polynomial and the bracket polynomial on knots in topology from Liu, Y.P.[58–59].

For simplicity, we would prefer to concentrate on the polynomials of maps with constant weights which are always assumed to be 1 such that the coefficients in the recursion for an edge which is neither loop nor isthmus in (1.2.1) are absolute constants which are assumed to be 1 or $-1$.