Interpolation of Operators

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Interpolation of Operators

To Margaret and Carla

Preface

Three classical interpolation theorems form the foundation of the modern theory of interpolation of operators. They are the M. Riesz convexity theorem (1926), G.O. Thorin's complex version of Riesz' theorem (1939), and the J. Marcinkiewicz interpolation theorem (1939). The ideas of Thorin and Marcinkiewicz were reworked some twenty years later into an abstract theory of interpolation of operators on Banach spaces and more general topological spaces. Thorin's technique has given rise to what is now known as the complex method of interpolation, and Marcinkiewicz' to the real method. Both have found widespread application, have extensive literatures attached to them, and remain very much alive as subjects of current research.

This is a book about the real method of interpolation. Our goal has been to motivate and develop the entire theory from its classical origins, that is, through the theory of spaces of measurable functions. Although the influence of Riesz, Thorin, and Marcinkiewicz is everywhere evident, the work of G. H. Hardy, J. E. Littlewood, and G. Pólya on rearrangements of functions also plays a seminal role. It is through the Hardy–Littlewood–Pólya relation that spaces of measurable functions and interpolation of operators come together, in a simple blend which has the capacity for great generalization. Interpolation between L^1 and L^∞ is thus the prototype for interpolation between more general pairs of Banach spaces. This theme airs constantly throughout the book.

The theory and applications of interpolation are as diverse as language itself. Our goal is not a dictionary, or an encyclopedia, but instead a brief biography of interpolation, with a beginning and an end, and (like interpolation itself) some substance in between.

The book should be accessible to anyone familiar with the fundamentals of real analysis, measure theory, and functional analysis. The standard advanced undergraduate or beginning graduate courses in these disciplines should suffice. The exposition is essentially self-contained.

We wish to thank Dr. Carl Riehm and McMaster University, without whose support this project would not have begun, and the University of South Carolina, which provides such a stimulating and exciting environment in which to work. We are particularly indebted to Dr. Sherman Riemenschneider and Mr. Natarajan Sivakumar of the University of Alberta, who suggested many improvements and caught several errors. We thank them for their generosity and their humor, which will someday surely be repaid. Our thanks are due also to Department of Mathematics staff members Jane Squires, Jessie Smith, Sue Darlington, and Dana Ward for their invaluable assistance in preparing the manuscript. It is a pleasure to acknowledge the efficiency and professionalism of our editors, Bill Sribney and Pascha Gerlinger of Harcourt Brace Jovanovich. Above all, we thank our families for their patience and support during the preparation of this work.

Columbia, South Carolina

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1 Banach Function Spaces

Although the Lebesgue spaces L^p $(1 \leq p \leq \infty)$ play a primary role in many areas of mathematical analysis, there are other classes of Banach spaces of measurable functions that are also of interest. The larger classes of Orlicz spaccs and Lorentz spaces, for example, are of intrinsic importance. There is a considerable literature dealing with each of these classes. In this chapter, however, we shall concentrate not on the differences between such classes but instead on their similarities. This common ground provides the foundation for the abstract theory of *Banach function spaces.*

Banach function spaces are Banach spaces of measurable functions in which the norm is related to the underlying measure in an appropriate way. This allows for a fruitful interplay between functional-analytic and measure-theoretic techniques. The theory is further enriched by thc presence of a natural order structure on the function elements themselves, and so may be subsumed in a more general treatment of *Banach lattices*, or *Riesz spaces*, as they are sometimes called. For our purposes, however, this more general point of view will be neither necessary nor desirable.

The Banach function space axioms are displayed in Section 1, where some elementary properties are derived from them. The concept of the *associate space* is introduced in Section 2 and this sets the scene for the discussion of duality, reflexivity, and separability in Sections 3, 4, and 5.

The program follows the same lines as any standard development of the L^p-spaces. In fact, the reader may find it instructive to keep the L^p-spaces in mind as a model for the entire theory of Banach function spaces.

The reader may also find it useful to reflect on the motivation for the particular choice of axioms. The literature shelters more than one axiomatic system under the general umbrella of Banach function spaces. Some use weaker versions of the Fatou property (property (P3) in Definition 1.1), while others rely on a different class of distinguished "bounded" sets in the underlying measure space (properties (P4), (P5)). For example, in a totally σ-finite measure space (R, μ), one could select once and for all an increasing sequence $(R_n)_{n=1}^{\infty}$ of measurable subsets of finite measure whose union is all of R. A measurable subset of R might then be declared "bounded" if it is contained in some set R_n. The approach we have adopted is simpler: the "bounded" sets are just the sets of finite measure. The resulting theory is less technical but also less general at this initial stage. There is, however, no real loss of generality when we specialize to the rearrangement-invariant spaces in the next chapter.

1. BANACH FUNCTION SPACES

Let (R, μ) be a measure space, in the sense described above. Let $\mathcal{M}^+$ be the cone of μ-measurable functions on R whose values lie in $[0, \infty]$. The characteristic function of a μ-measurable subset E of R will be denoted by χ_E.

Definition 1.1. A mapping $\rho : \mathcal{M}^+ \to [0, \infty]$ is called a *Banach function norm* (or simply a *function norm*) if, for all f, g, f_n, $(n = 1, 2, 3, \ldots)$, in $\mathcal{M}^+$, for all constants $a \geq 0$, and for all μ-measurable subsets E of R, the following properties hold:

(P1) $\rho(f) = 0 \;\Leftrightarrow\; f = 0$ μ-a.e.; $\quad \rho(af) = a\rho(f)$;
$\rho(f + g) \leq \rho(f) + \rho(g)$

(P2) $0 \leq g \leq f$ μ-a.e. $\;\Rightarrow\; \rho(g) \leq \rho(f)$

(P3) $0 \leq f_n \uparrow f$ μ-a.e. $\;\Rightarrow\; \rho(f_n) \uparrow \rho(f)$

(P4) $\mu(E) < \infty \;\Rightarrow\; \rho(\chi_E) < \infty$

(P5) $\mu(E) < \infty \;\Rightarrow\; \int_E f \, d\mu \leq C_E \rho(f)$

for some constant C_E, $0 < C_E < \infty$, depending on E and ρ but independent of f.

Among the simplest examples of Banach function norms are those associated with the Lebesgue spaces L^p $(1 \le p \le \infty)$. Let

$$\rho_p(f) = \begin{cases} \left\{ \int_R f^p \, d\mu \right\}^{1/p}, & (1 \le p < \infty), \\ \operatorname*{ess\,sup}_R f, & (p = \infty), \end{cases} \qquad f \in \mathcal{M}^+. \tag{1.1}$$

Theorem 1.2. *The Lebesgue functionals ρ_p, $(1 \le p \le \infty)$, are function norms.*

Proof. The triangle inequality for ρ_p is the classical Minkowski inequality. The remaining parts of (P1) are obvious, as are (P2) and (P4). Property (P3) follows from the monotone convergence theorem, and (P5) from Hölder's inequality: if $1 < p < \infty$ and $1/p + 1/p' = 1$, then

$$\int_E f \, d\mu = \int_R f\chi_E \, d\mu \le \left(\int_R f^p \, d\mu \right)^{1/p} \left(\int_R \chi_E^{p'} \, d\mu \right)^{1/p'} = C_E \rho(f),$$

with $C_E = \mu(E)^{1/p'}$. The cases $p = 1$ and $p = \infty$ are easier so their proofs are omitted. ■

Let $\mathcal{M}$ denote the collection of all extended scalar-valued (real or complex) μ-measurable functions on R and $\mathcal{M}_0$ the class of functions in $\mathcal{M}$ that are finite μ-a.e. As usual, any two functions coinciding μ-a.e. will be identified. The natural vector space operations are well defined on $\mathcal{M}_0$ (although not on all of $\mathcal{M}$), and when $\mathcal{M}_0$ is given the topology of convergence in measure on sets of finite measure it becomes a metrizable topological vector space (cf. Exercise 1).

Definition 1.3. Let ρ be a function norm. The collection $X = X(\rho)$ of all functions f in $\mathcal{M}$ for which $\rho(|f|) < \infty$ is called a *Banach function space*. For each $f \in X$, define

$$\|f\|_X = \rho(|f|). \tag{1.2}$$

Theorem 1.4. *Let ρ be a function norm and let $X = X(\rho)$ and $\|\cdot\|_X$ be as in Definition 1.3. Then under the natural vector space operations, $(X, \|\cdot\|_X)$ is a normed linear space for which the inclusions*

$$S \subset X \hookrightarrow \mathcal{M}_0 \tag{1.3}$$

hold, where S is the set of μ-simple functions on R. In particular, if $f_n \to f$ in X, then $f_n \to f$ in measure on sets of finite measure, and hence some subsequence converges pointwise μ-a.e. to f.

Proof. It follows from Definition 1.3 and property (P5) of Definition 1.1 that every function in X is locally integrable and hence finite μ-a.e. (because μ is σ-finite). The set X therefore inherits the vector space operations from $\mathcal{M}_0$ and then there is no difficulty in using (P1) and (1.2) to verify that $(X, \|\cdot\|_X)$ is a normed linear space. Property (P4) shows that X contains the characteristic function of every set of finite measure and hence, by linearity, every μ-simple function. This establishes the set-theoretic inclusions in (1.3).

It remains to show that the inclusion map from X to $\mathcal{M}_0$ is continuous. Since both spaces are metrizable it will suffice to show that every sequence convergent in X is convergent also in $\mathcal{M}_0$ (to the same limit, of course). But if $f_n \to f$ in X, then (1.2) shows that $\rho(|f - f_n|) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ and let E be any subset of R having finite measure. By property (P5),

$$\mu\{x \in E : |f(x) - f_n(x)| > \varepsilon\} \le \int_E \frac{1}{\varepsilon} |f - f_n| \, d\mu$$

$$\le \frac{1}{\varepsilon} C_E \rho(|f - f_n|),$$

which converges to 0 as $n \to \infty$ since C_E is independent of n. This shows that $f_n \to f$ in measure on every set of finite measure or, what is the same thing, $f_n \to f$ in $\mathcal{M}_0$. A standard result in measure theory [Ro, p.92] now provides the desired pointwise a.e.-convergent subsequence. ■

The Banach function spaces arising from the functionals ρ_p in (1.1) are of course the familiar Lebesgue spaces $L^p = L^p(R, \mu)$:

$$\|f\|_{L^p} = \begin{cases} \left(\int_R |f|^p \, d\mu \right)^{1/p}, & (1 \le p < \infty) \\ \operatorname*{ess\,sup}_R |f|, & (p = \infty). \end{cases} \tag{1.4}$$

The next result shows that one of the cornerstones of the L^p-theory, namely Fatou's lemma, has a natural analogue in every Banach function space. Note that the Fatou property (P3) plays a central role here.

Lemma 1.5. *Let $X = X(\rho)$ be a Banach function space and suppose $f_n \in X$, $(n = 1, 2, \ldots)$.*

(i) *If $0 \le f_n \uparrow f$ μ-a.e., then either $f \notin X$ and $\|f_n\|_X \uparrow \infty$, or $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$.*