THE FAST FOURIER TRANSFORM

E. ORAN BRIGHAM

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E–Systems, Inc.

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PREFACE

The Fourier transform has long been a principle analytical tool in such diverse fields as linear systems, optics, probability theory, quantum physics, antennas, and signal analysis. A similar statement is not true for the discrete Fourier transform. Even with the tremendous computing speeds available with modern computers, the discrete Fourier transform found relatively few applications because of the exorbitant amount of computation time required. However, with the development of the fast Fourier transform (an algorithm that efficiently computes the discrete Fourier transform), many facets of scientific analysis have been completely revolutionized.

As with any new development that brings about significant technological change, there is the problem of communicating the essential basics of the fast Fourier transform (FFT). A unified presentation which relates this technique to one's formal education and practical experience is dictated. The central aim of this book is to provide the student and the practicing professional a readable and meaningful treatment of the FFT and its basic application.

The book communicates with the reader not by the introduction of the topics but rather in the manner by which the topics are presented. Every major concept is developed by a three stage sequential process. First, the concept is introduced by an intuitive development which is usually pictorial in nature. Second, a non-sophisticated (but theoretically sound) mathematical treatment is developed to support the intuitive arguments. The third stage consists of practical examples designed to review and expand the concept being discussed. It is felt that this three step procedure gives *meaning* as well as mathematical substance to the basic properties of the FFT.

The book should serve equally well to senior or first year graduate students and to the practicing scientific professional. As a text, the material covered can be easily introduced into course curriculums including linear systems, transform theory, systems analysis, signal processing, simulation, communication theory, optics, and numerical analysis. To the practicing engineer the book offers a readable introduction to the FFT as well as providing a unified reference. All major developments and computing procedures are tabled for ease of reference.

Apart from an introductory chapter which introduces the Fourier transform concept and presents a historical review of the FFT, the book is essentially divided into four subject areas:

1. The Fourier Transform

In Chapters 2 through 6 we lay the foundation for the entire book. We investigate the Fourier transform, its inversion formula, and its basic properties; graphical explanations of each discussion lends physical insight to the concept. Because of their extreme importance in FFT applications the transform properties of the convolution and correlation integrals are explored in detail: Numerous examples are presented to aid in interpreting the concepts. For reference in later chapters the concept of Fourier series and waveform sampling are developed in terms of Fourier transform theory.

2. The Discrete Fourier Transform

Chapters 6 through 9 develop the discrete Fourier transform. A graphical presentation develops the discrete transform from the continuous Fourier transform. This graphical presentation is substantiated by a theoretical development. The relationship between the discrete and continuous Fourier transform is explored in detail; numerous waveform classes are considered by illustrative examples. Discrete convolution and correlation are defined and compared with continuous equivalents by illustrative examples. Following a discussion of discrete Fourier transform properties, a series of examples is used to illustrate techniques for applying the discrete Fourier transform.

3. The Fast Fourier Transform

In Chapters 10 through 12 we develop the FFT algorithm. A simplified explanation of why the FFT is efficient is presented. We follow with the development of a signal flow graph, a graphical procedure for examining the FFT. Based on this flow graph we describe sufficient generalities to develop a computer flow chart and FORTRAN and ALGOL computer programs. The remainder of this subject area is devoted toward theoretical development of the FFT algorithm in its various forms.

4. Basic Application of the FFT

Chapter 13 investigates the basic application of the FFT, computing discrete convolution and correlation integrals. In general, applications of the FFT (systems analysis, digital filtering, simulation, power spectrum analysis, optics, communication theory, etc.) are based on a specific implementation of the discrete convolution or correlation integral. For this reason we describe in detail the procedures for applying the FFT to these discrete integrals.

A full set of problems chosen specifically to enhance and extend the presentation is included for all chapters.

I would like to take this opportunity to thank the many people who have contributed to this book. David E. Thouin, Jack R. Grisham, Kenneth W. Daniel, and Frank W. Goss assisted by reading various portions of the manuscript and offering constructive comments. Barry M. Rosenburg contributed the computer programs in Chapter 10 and W. A. J. Sippel was responsible for all computer results. Joanne Spiessbach compiled the bibliography. To each of these people I express my sincere appreciation.

A special note of gratitude goes to my wife, Vangee, who typed the entire manuscript through its many iterations. Her patience, understanding and encouragement made this book possible.

E. O. BRIGHAM, JR.

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CONTENTS

1

INTRODUCTION

In this chapter we describe briefly and tutorially the concept of transform analysis. The Fourier transform is related to this basic concept and is examined with respect to its basic analysis properties. A survey of the scientific fields which utilize the Fourier transform as a principal analysis tool is included. The requirement for discrete Fourier transforms and the historical development of the fast Fourier transform (FFT) are presented.

1-1 TRANSFORM ANALYSIS

Every reader has at one time or another used transform analysis techniques to simplify a problem solution.

The reader may question the validity of this statement because the term *transform* is not a familiar analysis description. However, recall that the logarithm is in fact a transform which we have all used.

To more clearly relate the logarithm to transform analysis consider Fig. 1-1. We show a flow diagram which demonstrates the general relationship between conventional and transform analysis procedures. In addition, we illustrate on the diagram a simplified transform example, the logarithm transform. We will use this example as a mechanism for solidifying the meaning of the term *transform* analysis.

From Fig. 1-1 the example problem is to determine the quotient $Y = X/Z$. Let us assume that extremely good accuracy is desired and a computer is not available. Conventional analysis implies that we must determine Y by long-hand division. If we must perform the computation of Y repeatedly,

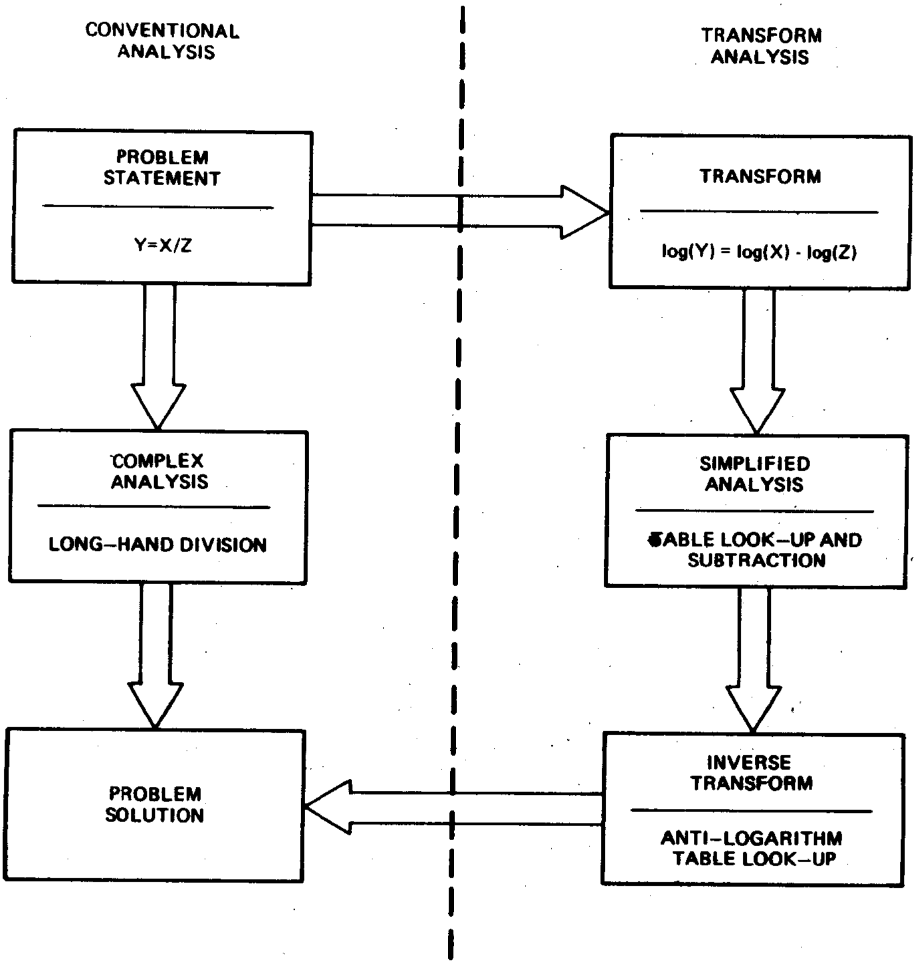

Figure 1-1. Flow diagram relationship of conventional and transform analysis.

then conventional analysis (long-hand division) represents a time consuming process.

The right-hand side of Fig. 1-1 illustrates the basic steps of transform analysis. As shown, the first step is to convert or transform the problem statement. For the example problem, we choose the logarithm to transform division to a subtraction operation.

Because of this simplification, transform analysis then requires only a table look-up of log (X) and log (Z), and a subtraction operation to determine log (Y). From Fig. 1-1, we next find the inverse transform (anti-logarithm) of log (Y) by table look-up and complete the problem solution. We note that by using transform analysis techniques we have reduced the complexity of the example problem.

In general, transforms often result in simplified problem solving analysis. One such transform analysis technique is the Fourier transform. This transform has been found to be especially useful for problem simplification in many fields of scientific endeavor. The Fourier transform is of fundamental concern in this book.

1-2 BASIC FOURIER TRANSFORM ANALYSIS

The logarithm transform considered previously is easily understood because of its single dimensionality; that is, the logarithm function transforms a single value X into the single value $\log(X)$. The Fourier transform is not as easily interpreted because we must now consider functions defined from $-\infty$ to $+\infty$. Hence, contrasted to the logarithm function we must now transform a function of a variable defined from $-\infty$ to $+\infty$ to the function of another variable also defined from $-\infty$ to $+\infty$.

A straightforward interpretation of the Fourier transform is illustrated in Fig. 1-2. As shown, the essence of the Fourier transform of a waveform is to decompose or separate the waveform into a sum of sinusoids of different frequencies. If these sinusoids sum to the original waveform then we have determined the Fourier transform of the waveform. The pictorial representation of the Fourier transform is a diagram which displays the amplitude and frequency of each of the determined sinusoids.

Figure 1-2 also illustrates an example of the Fourier transform of a simple waveform. The Fourier transform of the example waveform is the two sinusoids which add to yield the waveform. As shown, the Fourier transform diagram displays both the amplitude and frequency of each of the sinusoids. We have followed the usual convention and displayed both positive and negative frequency sinusoids for each frequency; the amplitude has been halved accordingly. The Fourier transform then decomposes the example waveform into its two individual sinusoidal components.

The Fourier transform identifies or distinguishes the different frequency sinusoids (and their respective amplitudes) which combine to form an arbitrary waveform. Mathematically, this relationship is stated as

$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}\,dt \tag{1-1}$$

where $s(t)$ is the waveform to be decomposed into a sum of sinusoids, $S(f)$ is the Fourier transform of $s(t)$, and $j = \sqrt{-1}$. An example of the Fourier transform of a square wave function is illustrated in Fig. 1-3(a). An intuitive justification that a square waveform can be decomposed into the set of sinusoids determined by the Fourier transform is shown in Fig. 1-3(b).

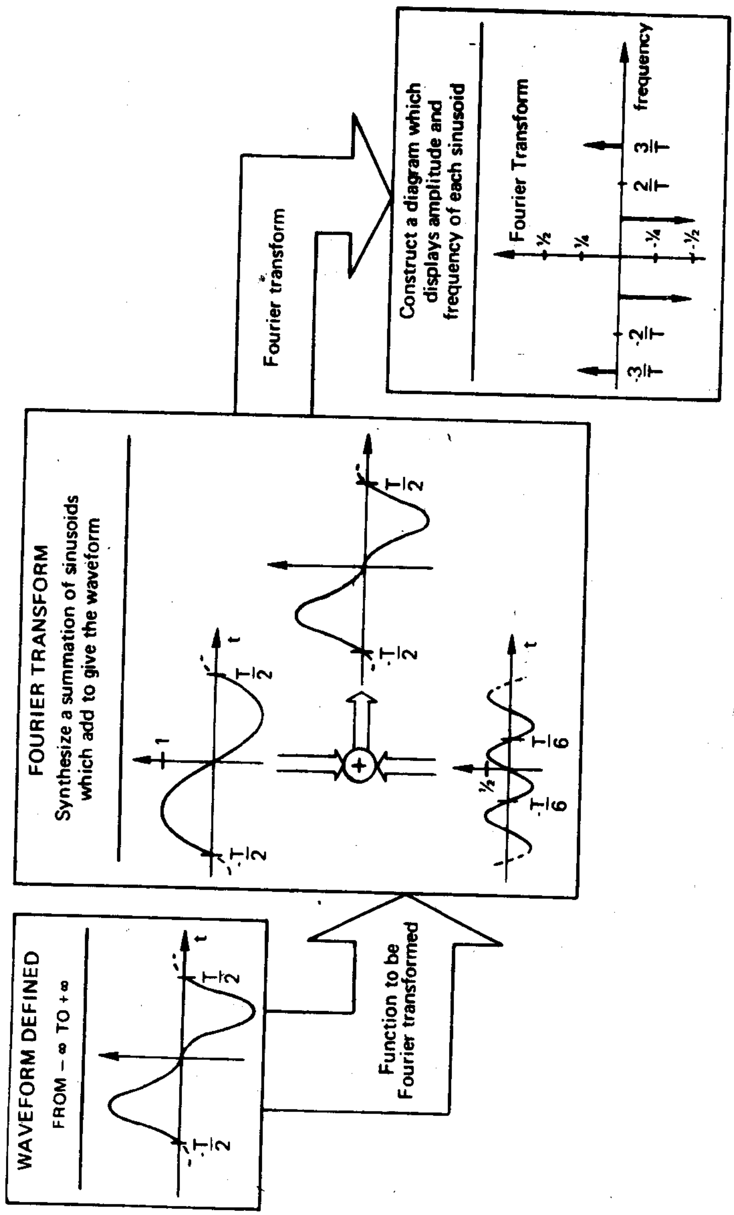

Figure 1-2. Interpretation of the Fourier transform.

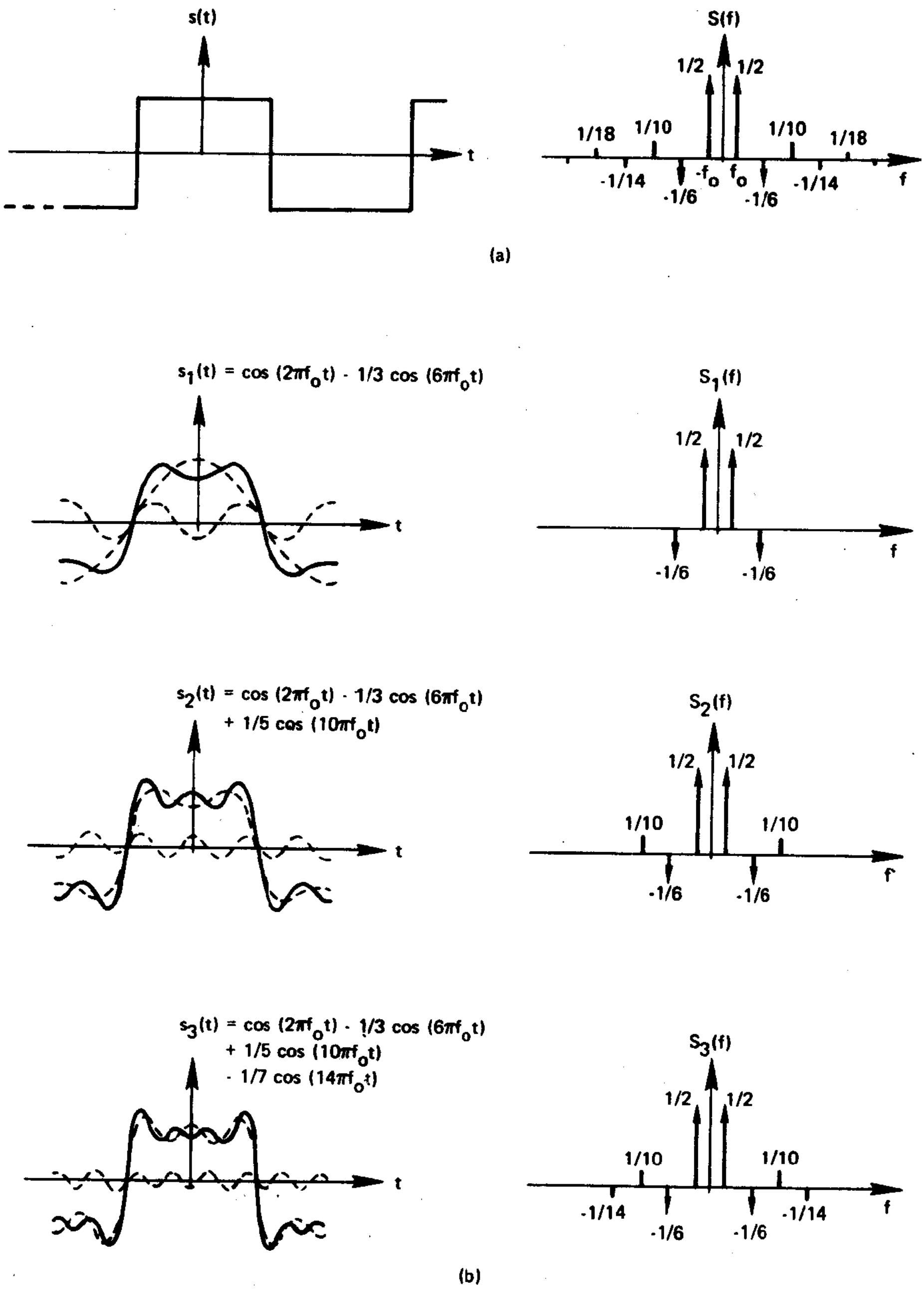

Figure 1-3. Fourier transform of a square wave function.

We normally associate the analysis of periodic functions such as a square wave with Fourier series rather than Fourier transforms. However, as we will show in Chapter 5, the Fourier series is a special case of the Fourier transform.

If the waveform $s(t)$ is not periodic then the Fourier transform will be a continuous function of frequency; that is, $s(t)$ is represented by the summation of sinusoids of all frequencies. For illustration, consider the pulse waveform and its Fourier transform as shown in Fig. 1-4. In this example

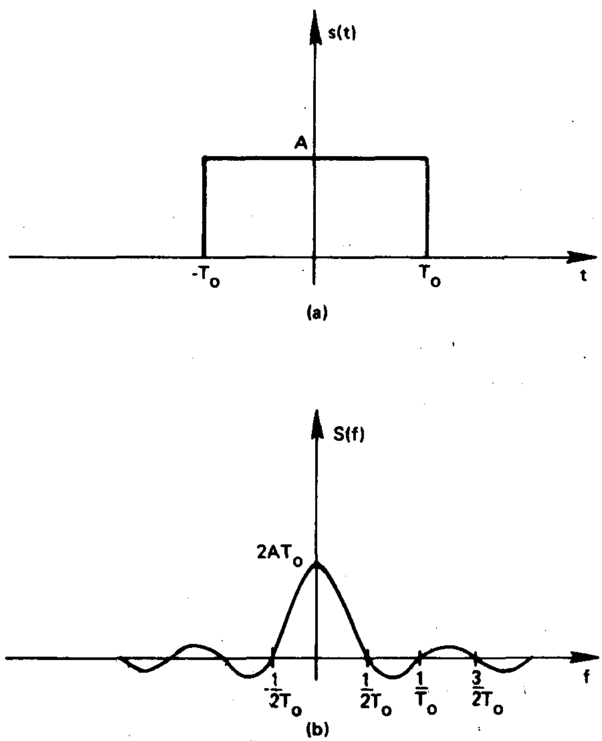

Figure 1-4. Fourier transform of a pulse waveform.

the Fourier transform indicates that one sinusoid frequency becomes indistinguishable from the next and, as a result, all frequencies must be considered.

The Fourier transform is then a frequency domain representation of a function. As illustrated in both Figs. 1-3(a) and 1-4, the Fourier transform frequency domain contains exactly the same information as that of the original function; they differ only in the manner of presentation of the information. Fourier analysis allows one to examine a function from another point of view, the transform domain. As we will see in the discussions to