SPLINES AND VARIATIONAL METHODS

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PREFACE

This book introduces the numerical solution of boundary value problems by variational methods. Special emphasis is placed on the finite element and collocation methods. We have deliberately attempted to present the majority of the material in a very elementary manner, to make it accessible to serious students of engineering and mathematics who have no more mathematical background than elementary applied linear algebra and very basic advanced calculus or their equivalents. Hilbert space and more abstract functional analytic concepts have been introduced only when needed to put the theory on a firm mathematical footing or to unify a body of material and point out new directions of thinking about boundary value problems. Some will undoubtedly find this point of view distasteful. However, the approach seems to be in keeping with the development of a vast engineering literature on the subject. Engineers are unquestionably the pioneers in the development and research on finite elements, and a large number of their early successes were produced without benefit of abstract functional analysis; basic calculus and a strong physical sense of what will work were their only tools. (This is not to say that powerful mathematics is unnecessary to establish further successes and point out serious failures.) We have tried to reach the audience described, to introduce them gradually to the mathematician's way of thinking about such problems. In this sense, the book is no more than an introduction to more advanced works, as well as an access route to the enormous literature and great number of open research questions in the area.

A secondary direction we have tried to point out, hopefully with success, is the powerful application of approximation theoretic notions to very applied problems. It seems unfortunate that so few approximation theorists are interested in the dynamic application of their art to very difficult and important physical problems. We hope that this is a changing situation and that more of these individuals will turn their attentions and talents to some of the numerical problems facing the engineer. Although our emphasis is on splines as a fine approximating tool, it is clear that splines, as we know them, are [illegible] for all problems of numerical approximation. We feel that many fruitful new ideas are in the winds and that this is an exciting time for approximation theory. Perhaps the splines theorists, with their penchant for functional catholicism, will claim these stars of the unborn. Be that as it may, it is a fruitful era.

No attempt has been made to present the material from a physical viewpoint. This is recognized as a serious omission, since the womb or birthplace of the Rayleigh-Ritz-Galerkin method is the idea of minimizing the energies of physical systems over finite dimensional approximating spaces. Moreover, this outlook is still the most natural physical way of approaching many applied problems. Many excellent mathematics and physics texts take precisely this point of view, however; thus perhaps we can be forgiven our transgressions.

The book has been successfully used over a 4-year period to teach mixed audiences of first-year graduate students in engineering and mathematics, as well as industrial scientists. It has also been used for short expository lectures here and overseas. By giving reading assignments and covering only the pragmatic high points of Chapters 1 through 5 we have been able to go through the entire text in one quarter. This has entailed an attempt to present sufficient proofs to keep the mathematicians happy and to delete sufficient proofs to keep the engineers from becoming miserable. An integral part of the course has always included numerous large- and small-scale computational exercises and a lot of programming.

The author is grateful for help and encouragement from a number of individuals. Among these are Bob Rice and Dr. Ron Guenther of the Marathon Oil Corporation and Oregon State University, who thought a book on the subject that could be read by ordinary mortals would be advisable; Drs. Bob Russell and Neall Strand, true friends in need, who proofread the entire original version of the manuscript; my department chairman Dr. E. R. Deal, for providing understanding and typing assistance; Professor C. Jacobsz of the Council for Scientific and Industrial Research (CSIR) of the Republic of South Africa and to the CSIR for providing an appointment as a senior research scientist and bringing me to South Africa to deliver an expository lecture series and to work on my book and my research; to Professor Karl Nickel and the Technical University of Karlsruhe for bringing me to Germany to lecture from and work on the book and my research; to Dr. Erik Thompson of the Civil Engineering Department at Colorado State University for many helpful discussions and for introducing me to the engineering literature; to my students Dr. Tom Dence, Tim Simpson, Don De Gryse, A. Sato, and C. Chen, who pointed out errors in many portions of the manuscript, as did Dr. Don Jones of the University of Michigan; and finally to my many other students here and overseas whose interest and intolerance for obscur-

ing simple mathematical concepts with unnecessary abstract mathematical tools has tempered my outlook. The author cannot ignore her debt to the influence of Dr. I. J. Schoenberg of the Mathematics Research Center, University of Wisconsin, whose personal kindness and beautiful lectures first stimulated the author's interest in splines, and of Professor Mihklin of the USSR, whom the author has never met but whose mathematical outlook has obviously been most influential. Final thanks are also due to a sequence (rather long, but thankfully finite) of impeccable typists who have worked on various phases of the manuscript. Among these are Mrs. Evelyn Anderson and Mrs. Beth Murphy of CSU and Mrs. S. Van Wyk of the CSIR. It is hoped that there are not too many serious mathematical errors; for any that may appear, the author assumes full responsibility.

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1

INTRODUCTORY IDEAS

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1.1 A SIMPLY STATED PROBLEM

The need for good techniques for the approximation of functions arises in many settings; one of these is the numerical solution of differential equations. For example, suppose you are given the differential equation

$$\frac{d^2x}{dt^2} + a(t)\frac{dx}{dt} + b(t)x(t) = f(t), \qquad a \leq t \leq b \tag{1}$$

subject to the *boundary conditions*

$$x(a) = \alpha \qquad \text{and} \qquad x(b) = \beta,$$

where α and β are constant and $a(t)$, $b(t)$, and $f(t)$ are functions of t defined on the interval $[a,b]$. Moreover, suppose you know that this equation, subject to the boundary conditions, has a *unique* solution $x(t)$ which you would like to find. Our first problem then is

PROBLEM 1

How do we find $x(t)$?

As those who have worked to any extent in ordinary differential equations know, the answer to this query is decidedly gloomy. In particular

Answer. For most choices of $a(t)$, $b(t)$, and $f(t)$ we cannot find $x(t)$ exactly.

This being the case, we compromise. Since we cannot find $x(t)$ exactly, we can try to find it approximately. This leads us to a new problem.

PROBLEM 2

How do we find a good approximation $\tilde{x}(t)$ to the solution $x(t)$ of our differential equation?

This problem is far more tractable, and there are many ways of answering it. All possible solutions depend in some way on the answer to yet another problem.

PROBLEM 3

Given a function $x(t)$, what kind of functions $\tilde{x}(t)$ make good approximations to $x(t)$?

and to the companion problems

PROBLEM 4

What is meant by a *good approximation*?

and

PROBLEM 5

How does one compute a good approximation $\tilde{x}(t)$ to a given function $x(t)$?

Providing some answers to these simple questions and their two-dimensional analogs is precisely what this book is all about. Since *linear spaces*, *subspaces*, *norms*, and *basis* are notions fundamental to all we do in the sequel, we start with a minor digression to define these entities.

1.2 LINEAR SPACES

A *real linear space* X is simply a set of mathematical objects called *vectors* which add according to the usual laws of arithmetic and can be multiplied by real numbers in accord with the usual laws of arithmetic. Specifically to qualify as a real linear space, elements of X must satisfy the following conditions or *axioms*.

> For all x, y, and z, in X and for all real numbers α and β, αx is in X, $x+y$ is in X, $x+y=y+x$, $(x+y)+z=x+(y+z)$, $1\cdot x=x$, $(\alpha+\beta)x=\alpha x+\beta x$, $\alpha(\beta x)=(\alpha\beta)x$, and $\alpha(x+y)=\alpha x+\alpha y$. There exists a *zero vector* $\odot$ in X with the property that $x+\odot=x$ for all x. Finally, for each x there is a unique vector $-x$, called the *inverse of* x, such that $-x+x=\odot$.

If all these axioms are satisfied when multiplication is multiplication by complex numbers, we say X is a *complex linear space*. We usually deal with the real linear spaces.

Examples abound. Among these are the set E^2 of vectors in the plane with addition defined as coordinatewise addition and the set $C[a,b]$ of functions $f(t)$ continuous on the closed interval $[a,b]$. In each case we must first define addition and scalar multiplication.

EXAMPLE 1 THE REAL PLANE E^2

The set $X=E^2$ is simply the set

$$E^2=\{(x_1,x_2):\ x_1 \text{ and } x_2 \text{ are real numbers}\}$$

of all ordered pairs of real numbers or vectors in a plane. Addition is coordinatewise addition, and multiplication is coordinatewise multiplication. In particular, given a real number α and vectors $x=(x_1,x_2)$ and $y=(y_1,y_2)$, we define

$$x+y=(x_1+y_1,x_2+y_2)$$

$$\alpha x=(\alpha x_1,\alpha x_2).$$

The reader can easily verify that all the axioms of a linear space are satisfied where $\odot=(0,0)$ is the zero vector (see Figure 1.1).

The real linear space $C[a,b]$ is much more interesting.

EXAMPLE 2 THE SPACE $C[a,b]$

The space $C[a,b]$ is simply the set of all functions continuous on $[a,b]$. To define addition and scalar multiplication, let α be any real number and let $f(t)$ and $g(t)$ be two continuous functions from $C[a,b]$. We define $f+g$ and αf in the usual way. That is,

$$(f+g)(t)=f(t)+g(t), \qquad a<t<b$$

and

$$(\alpha f)(t)=\alpha\cdot f(t), \qquad a<t<b.$$

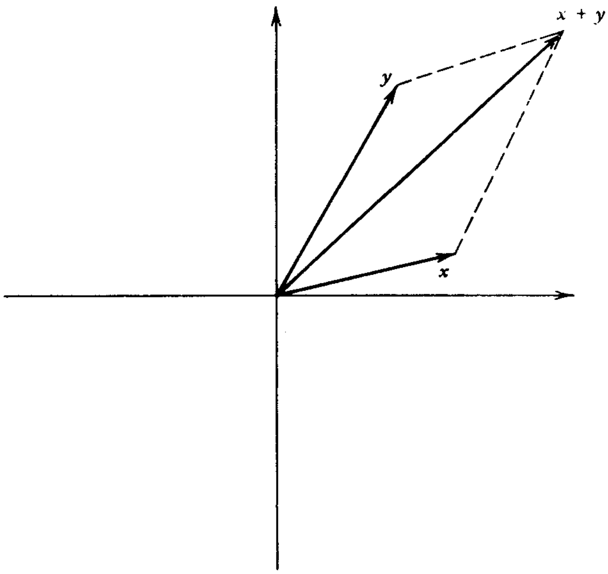

Figure 1.1 Addition of vectors in the plane E^2.

Noting that the sum of any two continuous functions is a continuous function and that a constant times a continuous function is again continuous, it is easy to verify that $C[a,b]$ forms a real linear space. The *zero vector* of this space is the function that vanishes identically on the interval $[a,b]$.

A space to which we shall take frequent recourse is

EXAMPLE 3 $P_n[a,b]$, POLYNOMIALS OF DEGREE n.

A real polynomial $p(t)$ of exact degree n or less in one variable t is a function

$$p(t)=a_nt^n+a_{n-1}t^{n-1}+\cdots+a_1t+a_0,$$

where $a_0,a_1,\ldots,a_n$ are given real numbers and t is a real variable. The polynomial $p(t)$ is said to have *exact degree* n, if and only if $a_n\neq 0$. Thus $2t^3+5t-1$ is of degree 4 but is of exact degree 3. Let

$$P_n[a,b]=\left\{ a_nt^n+\cdots+a_1t+a_0: \quad \begin{array}{l}\text{each } a_i \text{ is a real constant}\\ \text{and } a\leqslant t\leqslant b\end{array}\right\}.$$

Thus $P_n[a,b]$ is the set of all polynomials of exact degree n or less, defined on the interval $[a,b]$. Of course adding two polynomials of degree n or less produces a polynomial of degree n or less, whereas multiplying a polynomial of degree n or less by a constant gives a polynomial of degree n or less. Knowing this, it is easily checked that $P_n[a,b]$ is a real linear space. The zero in this space is again the zero function, which is the same as the zero polynomial. Note too that since every polynomial is continuous, $P_n[a,b] \subset C[a,b]$ (read $P_n[a,b]$ is a *subset* of $C[a,b]$).

We are interested in special subsets of linear spaces, which are known as *subspaces*. A subset M of a linear space X is called a *linear subspace* (or simply a *subspace*) of X if

> M is a subset of X and for each x and y in M and for any pair of scalars α and β, $\alpha x + \beta y$ belongs to M.

For example, a real line passing through the origin is a linear subspace of the plane E^2. Also, the set $P_n[a,b]$ is a subspace of $C[a,b]$. We use these concepts repeatedly in the sequel.

EXERCISES

1. Prove that $C[a,b]$ is a linear space.
2. Let P.C.$[a,b]$ denote the set of all functions that are piecewise continuous on $[a,b]$. In particular, a function $f(t)$ belongs to P.C.$[a,b]$ if and only if it has at most a finite number of discontinuities on $[a,b]$ and $\int_a^b [f(t)]^2\,dt < \infty$. Is P.C.$[a,b]$ a linear space? Prove your answer. If $f \in$ P.C.$[a,b]$, does $f'(t) \in$ P.C.$[a,b]$? Why? If $f(t)$ is a simple jump function, such as

$$f(t) = \begin{cases} 0 & \text{when} \quad a \leqslant t < \dfrac{b-a}{2} \\ 1 & \text{when} \quad \dfrac{b-a}{2} \leqslant t \leqslant b, \end{cases}$$

does either $f(t)$ or $f'(t)$ belong to $C[a,b]$? Explain.

3. Let $X = \{(x_0, y_0) + (x, y)$: x and y are real and x_0 and y_0 are constants$\}$. Is X a linear subspace of E^2? Explain.
4. Let $X = \text{P.C.}^p[a,b] = \{f(t)\colon f^{(p-1)} \in C[a,b] \text{ and } f^{(p)} \in \text{P.C.}[a,b]\}$. Is X a linear space? Why? If $f \in X$, does the fundamental theorem of calculus hold? In particular, does

$$f^{(p-1)} - f^{(p-1)}(t) = \int_{t_0}^{t} f^{(p)}(s)\,ds$$

for all t, t_0 in $[a,b]$? Explain.

1.3 NORMED LINEAR SPACES

Actually, we are interested in more than just linear spaces. We want linear spaces in which we can assign a notion of *length* $\|x\|$ to each vector x in X. Such linear spaces are known as *normed linear spaces*, and the real number $\|x\|$ is referred to as the *norm of x*. In particular, a *norm* is a real valued function defined on a linear space X having the following properties for each real number α and each pair of vectors x and y from X:

1. $\|x\|>0$ unless $x=0$
2. $\|\alpha x\|=|\alpha|\cdot\|x\|$
3. $\|x+y\|\leqslant\|x\|+\|y\|$ (triangle inequality).

The reader can check that these properties of a norm do coincide with usual geometric properties of length. Each of the linear spaces we have cited in our examples is easily made a normed linear space through an appropriate choice for $\|\ \|$. For example, if $X=E^2$, the usual choice for the length or norm $\|\ \|_2$ of a vector $x=(x_1,x_2)$ in the real plane is

$$\|x\|_2=\sqrt{x_1^2+x_2^2}\,.$$

But there are many other choices available. For example, the function $\|\ \|_1$ defined by

$$\|x\|_1=|x_1|+|x_2|$$

is a norm as is the function $\|\ \|_\infty$, defined by

$$\|x\|_\infty=\max\{|x_1|,|x_2|\}.$$

To prove that each of these function is a norm, you must verify that each one satisfies each of the conditions 1, 2, and 3 required of a norm. For example, to see that $\|\ \|_1$ satisfies condition 1, note that $\|x\|_1=0$ implies $|x_1|+|x_2|=0$. But this is possible if and only if $x_1=x_2=0$. Thus $\|x\|_1=0$ if and only if $x=(0,0)$, the zero vector. Moreover, since $\|x\|_1$ is clearly nonnegative for all x, we see that condition 1 is satisfied. Conditions 2 and 3 follow readily from elementary properties of absolute values of real numbers.

We are especially interested in the *Tchebycheff* or *uniform norm* on the space $C[a,b]$.

DEFINITION TCHEBYCHEFF NORM

Let $f(t)\in C[a,b]$. The real-valued function $\|\ \|$ defined by

$$\|f\|=\max_{a\leqslant t\leqslant b}|f(t)|$$

is known as the *Tchebycheff norm of f*.

One must, of course, prove that this function actually is a norm. This is easily accomplished because of elementary properties of absolute value of real numbers. In particular, $\|f\| \geq 0$, since $|f(t)| \geq 0$ for all t in $[a,b]$. Also $\|f\| = 0$ only if $|f(t)| = 0$ for all t in $[a,b]$. Thus $\|f\| > 0$ unless $f(t) \equiv 0$ on $[a,b]$. Moreover, $|\alpha f(t)| = |\alpha| \cdot |f(t)|$ implies

$$\max_{a \leq t \leq b} |\alpha f(t)| = \max_{a \leq t \leq b} |\alpha| \cdot |f(t)| = |\alpha| \max_{a \leq t \leq b} |f(t)| = |\alpha| \cdot \|f\|.$$

Thus 1 and 2 are clearly true. Condition 3 follows from the triangle inequality $|f(t) + g(t)| \leq |f(t)| + |g(t)|$ for absolute values of real numbers. Thus the Tchebycheff norm is indeed a norm.

A second norm on $C[a,b]$ which is also very important to us is the so-called L_2 *norm* $\| \; \|_2$. In particular

DEFINITION L_2 NORM

Let $f(t) \in C[a,b]$. The real valued function $\| \; \|_2$ defined by

$$\|f\|_2 = \sqrt{\int_a^b [f(t)]^2 \, dt}$$

is known as the L_2 *norm of f*.

Verification that $\| \; \|_2$ is a norm on $C[a,b]$ is left as an exercise (see Exercise 2).

Defining a norm on a linear space X introduces the companion notion of the *distance* $\|x - y\|$ between two points x and y belonging to X. For example, if $X = E^2$, $x = (1,3)$ and $y = (-2,4)$, then

$$\|x - y\| = \sqrt{9+1} = \sqrt{10}$$

$$\|x - y\|_1 = 3 + 1 = 4$$

$$\|x - y\|_\infty = \max\{3, 1\} = 3.$$

On the other hand, if $X = C[a,b]$ with the Tchebycheff norm, and $f(t)$ and $g(t)$ are any two functions belonging to X,

$$\|f - g\| = \max_{a \leq t \leq b} |f(t) - g(t)|.$$

To take a specific case, let $f(t)=\cos\pi t$, $g(t)=t^2$ and $X=C[0,1]$. Then

$$\|f-g\|=2$$

$$\|f\|=\|f-0\|=1$$

$$\|g\|=\|g-0\|=1$$

(see Figure 1.2).

Given a continuous function $f(t)$ from $C[a,b]$, it is helpful to consider the set of all functions $g(t)$ from $C[a,b]$ for which $\|f-g\|<\epsilon$. Such collections of functions, known as ϵ-neighborhoods of f, are nice things—we can draw pictures of them. Suppose, for example, $f(t)=\sin 2t+2$, where $0\leqslant t\leqslant\pi$, and $X=C[0,\pi]$. Then $\{g\in C[0,\pi]:\|\sin 2t+2-g\|<\epsilon\}$ is the family of all continuous functions $g(t)$ from $C[0,\pi]$ which thread through a tube of width 2ϵ, symmetric about the graph of $\sin 2t+2$ (see Figure 1.3). It is clear that $\|f-g\|<\epsilon$ if and only if the graph of $g(t)$ threads through the shaded region about the graph of $f(t)$. If $g(t)$ were a function approximating $f(t)$ and $\|f-g\|<\epsilon$, it appears g would be a good approximation to f if ϵ were small. We borrow this simple geometry to give *an initial definition of a good approximation.*

DEFINITION

Let $f\in C[a,b]$ with the Tchebycheff norm. A function g belonging to $C[a,b]$ is a *good approximation* to f provided $\|f-g\|<\epsilon$ for a sufficiently small ϵ.

In some situations this definition may not be a sufficient measure of goodness of approximation. To see this, consider the function $f(t)=\frac{1}{2}\epsilon\cos 4\pi t$ on the interval $[0,1]$, where ϵ is a very small constant. Then the function $g(t)\equiv 0$ on $[0,1]$ (the zero vector in $C[0,1]$) is a "good approximation" to f by our definition, since

$$\|f-g\|=\max_{0\leqslant t\leqslant 1}\left|\frac{\epsilon}{2}\cos 4\pi t\right|=\frac{\epsilon}{2}<\epsilon.$$

However, note that $y'(t)\equiv 0$, and $f'(t)=(-4\pi\epsilon/2)\sin 4\pi t$. Thus $\|f'-g'\|=\|f'\|=2\pi\epsilon$. Similarly $\|f''-g''\|=8\pi^2\epsilon$, $\|f'''-g'''\|=32\pi^3\epsilon$, $f^{(n)}-g^{(n)}\|=(4\pi)^n\epsilon/2$, and so forth. Thus if you require a "good approximation" g to a given function f to be one that makes each of the quantities $\|f-g\|$, $\|f'-g'\|,\ldots,\|f^{(n)}-g^{(n)}\|$ small, the definition just given simply will not do, since $(4\pi)^n\epsilon/2$ could be quite a large number even though ϵ was quite a small number. In this case, the way out of the difficulty is to retain the essential character of the definition of a good approximation, namely.

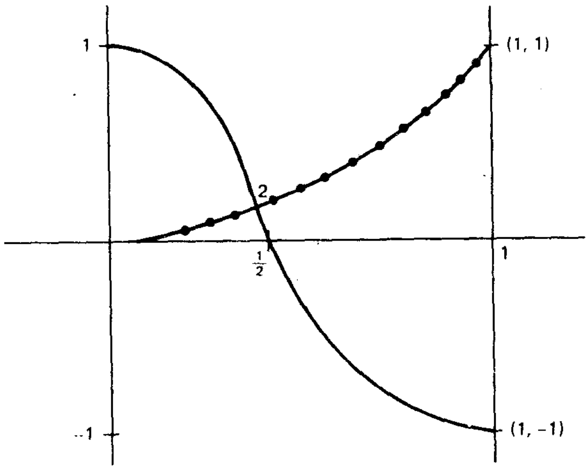

Figure 1.2 Dotted line: $g(t) = t^2$; solid line: $f(t) = \cos \pi t$. $\|f - g\| = 2$. Graph of f and g.

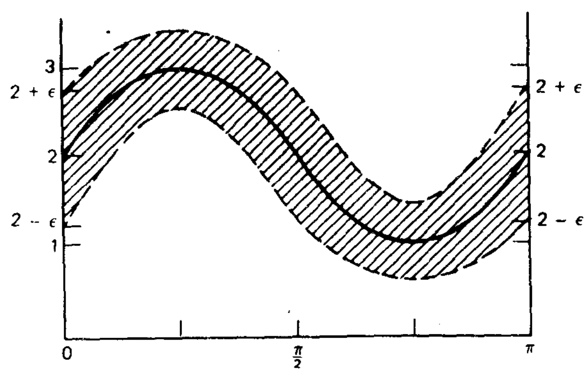

Figure 1.3 Graph of a "tube" of width 2ϵ symmetric about $\sin 2t + 2$.

$\|f - g\|$ small, but change the linear space and the norm. For the example we have cited, you could take $X = C^n[a,b]$, the space of n times continuously differentiable functions defined on $[a,b]$. For each $f \in C^n[a,b]$ define the norm $\| \ \|_n$ of f as

$$\|f\|_n = \|f\| + \|f'\| + \|f''\| + \cdots + \|f^{(n)}\|.$$

You must, of course, check that this function really is a norm and that $C^n[a,b]$ with addition and scalar multiplication defined as in $C[a,b]$ really