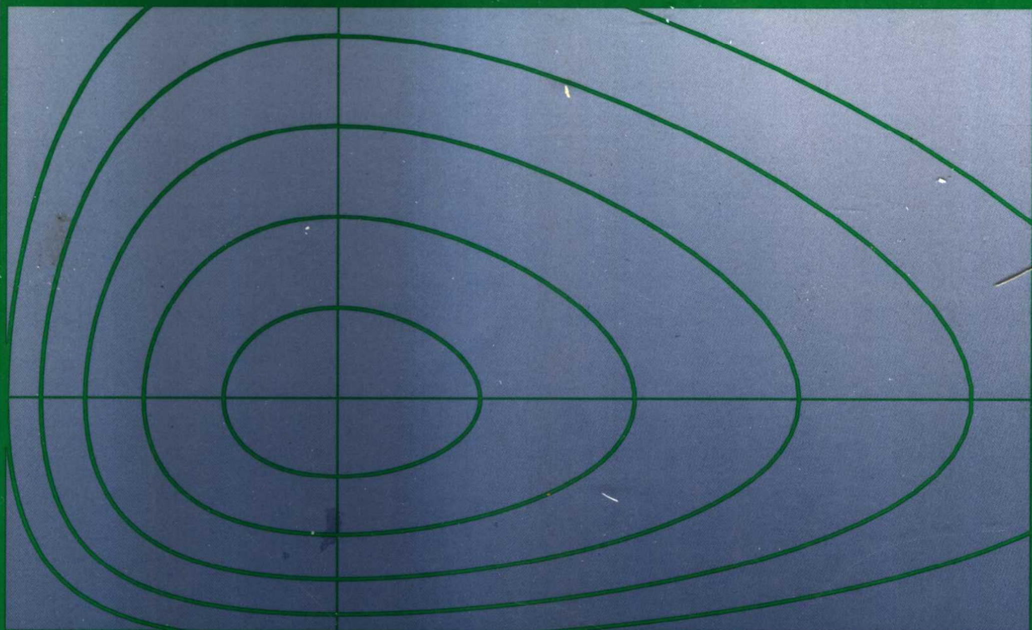
CALCULUS&*Mathematica*®

DERIVATIVES: Measuring Growth



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Calculus&Mathematica

DERIVATIVES: Measuring Growth

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Preface

Growth and change. These are the themes of life and measuring them is the theme of calculus. Learning how to measure rate of growth and learning how to use this measurement is what this part of Calculus&*Mathematica* is all about.

In this unit of Calculus&*Mathematica*, you'll work with the mathematics underlying measurements of growth rates and you'll use these measurements of growth rates to model processes naturally occurring in life. You'll have the chance to model the United States national debt, the United States population, growth of an animal, the blood alcohol level resulting from a given drinking schedule, the relationship between the growth of the height of an animal and the growth of the weight of the same animal, failures of O-rings on the space shuttle, credit card interest, personal finance, competing species, competing armies, spread of infection, and a lot more. Along the way, you'll get really good at *Mathematica* calculations and plotting. By the end of this part of Calculus&*Mathematica*, you'll be well on your way to mastering *Mathematica* as you begin to master calculus.

How to Use This Book

In Calculus&*Mathematica*, great care has been taken to put you in a position to learn visually. Instead of forcing you to attempt to learn by memorizing impenetrable jargon, you will be put in the position in which you will experience mathematics by seeing it happen and by making it happen. And you'll often be asked to describe what you see. When you do this, you'll be engaging in active mathematics as opposed to the passive mathematics you were probably asked to do in most other math courses. In order to take full advantage of this crucial aspect of Calculus&*Mathematica*, your first exposure to a new idea should be on the live computer screen where you can interact with the electronic text to your own satisfaction. This means that you should avoide "introductory lectures" and you should avoid reading this book at first. After you have some familiarity with new ideas as found on the computer screen, you should seek out others for discussion and you can refer to this book to brush up on a point or two after you leave the computer. In the final analysis, this book is nothing more than a partial record of what happens on the screen.

Once you have participated in the mathematics and science of each lesson, you can sharpen your hand skills and check up on your calculus literacy by trying the questions in the Literacy Sheet associated with each lesson. The Literacy Sheets appear at the end of each book.

Significant Changes from the Traditional Course

Your writing, plotting, and experimentation is the stock in trade of the course.

Experienced as a course in measurements heavily intertwined with other parts of science and the world.

Your emphasis is on linear and exponential growth from the beginning, before calculus begins. Linear functions are those with constant growth rates; exponentials are those with constant percentage growth rates.

You learn at the very beginning that exponential growth dominates power growth without appeal to the mysticism of L'Hopital's rule or any other calculus ideas.

You study functions not studied for their own sake, but rather for the measurements they make.

You learn about the derivative as a measurement of the instantaneous growth rate. As a result, the idea that functions with positive derivatives are increasing functions is available to you immediately without waiting for the Mean Value Theorem. The interpretation of the derivative as the slope of the tangent line is delayed.

You work with and analyze real world data on applications important to you.

Financial calculations recur on a regular basis.

You learn the meaning of the derivative as a measurement at the same time you're learning to calculate derivatives. This idea is reinforced by many plots that you produce and analyses of the graphs of $f[x]$ and $f'[x]$ on the same axes.

Although there is no formal "epsilon-delta" presentation of limits, you experience the limiting process visually by plotting the average growth rates $(f[x+h]-f[x])/h$ and the instantaneous growth rate $f'[x]$ as functions of x and watch what happens to the plots as they make h close in on 0.

The active form of the Mean Value Theorem, called the Race Track Principle, is introduced. Euler's method is explained in terms of the Race Track Principle.

Following Poincare, you do differentiation of functions of two variables with respect to each variable is done with no particular fanfare.

You do serious work with mathematical models involving derivatives. The benefits are twofold: Working with the models reinforces the idea of what the derivative is, and you can experience the tenacles of calculus outside the traditional calculus classroom.

Biological models are favored over physical models at the beginning because the derivative measures growth and growth is a natural biological process.

Linear dimension makes a decisive entrance into calculus.

Logistic growth is studied in some detail.

Qualitative analysis of the solutions of simple differential equations and the solutions of simple systems of differential equations enters a calculus course for the first time. Reasons: Studying them reinforces the meaning of the derivatives and they beautifully show the scope of calculus in science. You experiment with predator-prey, spread of infection, and Lanchester war models and try to explain the results in terms of derivatives.

Parametric plots in two and three dimensions are studied in the first course because they provide you with needed plotting freedom for what's to come.

Contents

DERIVATIVES: Measuring Growth

In normal use, the student engages in all the mathematics and the student engages in selected experiences in math and science as assigned by the individual instructor.

LESSON 1.01

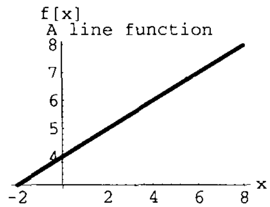
Growth

Basics

■ B.1) Growth of line functions

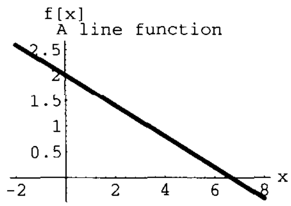
A line function $f[x]$ is any function whose formula is $f[x] = a\,x + b$ where a and b are constants. Here's a plot of a line function:

In[1]:=
```
Clear[f,x]
a = 0.5; b = 4; f[x_] = a x + b;
Plot[f[x],{x,-2,8},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","f[x]"},
PlotLabel->"A line function"];
```



There's steady growth as x advances from left to right. Here's another:

In[2]:=
```
Clear[f,x]
a = -0.3; b = 2; f[x_] = a x + b;
Plot[f[x],{x,-2,8},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","f[x]"},
PlotLabel->"A line function"];
```



Steady (negative) growth as x advances from left to right. Play with other choices of a and b until you get the feel of a line function.

B.1.a.i) The most important feature of a line function $f[x] = a\,x + b$ is revealed by the following calculation.

In[3]:=
```
Clear[f,a,b,x,h]; f[x_] = a x + b; Expand[f[x + h] - f[x]]
```
Out[3]=
```
a h
```

> What feature of line functions is revealed by this calculation?

Answer: The calculation reveals that when you take a line function $f[x] = a\,x+b$, then you find that

$$f[x + h] - f[x] = a\,h.$$

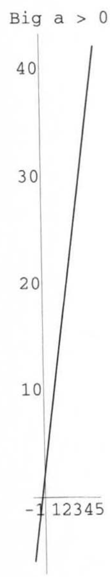
This tells you that when x advances by h units, then $f[x]$ grows by $a\,h$ units. Consequently a line function $f[x] = a\,x + b$ has constant growth rate of a units on the $f[x]$-axis for each unit on the x-axis.

B.1.a.ii) As you saw above, the growth rate of a line function $f[x] = a\,x + b$ measures out to a units on the $f[x]$ axis per unit on the x-axis.

> What is the significance of a?

Answer: Big positive a's force big-time fast growth as the following true scale plot shows:

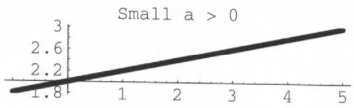
In[4]:=
```
Clear[f,x]
a = 8;
b = 2;
f[x_] = a x + b;
Plot[f[x],{x,-1,5},
PlotStyle->{{Red,Thickness[0.015]}},
PlotLabel->"Big a > 0",
AspectRatio->Automatic];
```



Small positive a's force slow growth as the following true scale plot shows:

In[5]:=

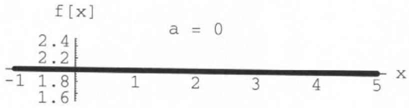
```
Clear[f,x]; a = 0.2; b = 2; f[x_] = a x + b;
Plot[f[x],{x,-1,5},
PlotStyle->{{Red,Thickness[0.015]}},
PlotLabel->"Small a > 0", AspectRatio->Automatic];
```



$a = 0$ forces no growth at all as the following true scale plot shows:

In[6]:=

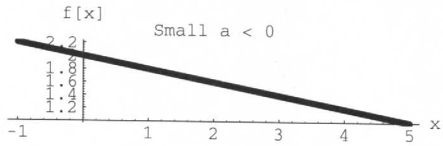
```
Clear[f,x]; a = 0; b = 2; f[x_] = a x + b;
Plot[f[x],{x,-1,5},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","f[x]"},
PlotLabel->"a = 0", AspectRatio->Automatic];
```



Small negative a's force slow (negative) growth as the following true scale plot shows:

In[7]:=

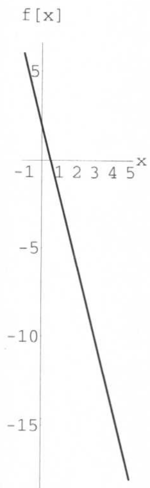
```
Clear[f,x]
a = -0.2; b = 2; f[x_] = a x + b;
Plot[f[x],{x,-1,5},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","f[x]"},
PlotLabel->"Small a < 0",
AspectRatio->Automatic];
```



Big negative a's force fast (negative) growth as the following true scale plot shows:

In[8]:=

```
Clear[f,x]
a = -4;
b = 2;
f[x_] = a x + b;
Plot[f[x],{x,-1,5},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","f[x]"},
AspectRatio->Automatic];
```

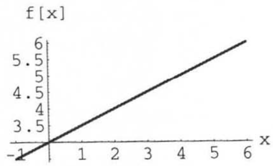


Not a handsome plot, but the message gets through.

B.1.b) Plot $f[x] = 0.5\,x + 3$ for $-1 \leq x \leq 6$ in true scale. $f[x]$ goes up how many times faster than x?

Answer: Here is a true scale plot:

In[9]:=
```
Clear[f,x]
f[x_] = 0.5 x + 3;
Plot[f[x],{x,-1,6},AxesLabel->{"x","f[x]"},
AspectRatio->Automatic,
PlotStyle->{{GrayLevel[0.4],Thickness[0.01]}}];
```

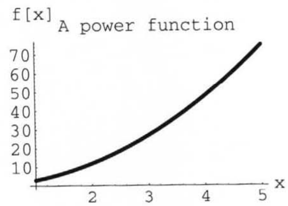


As you can see, $f[x] = 0.5\,x + 3$ goes up 0.5 units on the $f[x]$-axis per unit on the x-axis. This is in harmony with the fact that $f[x] = 0.5\,x + 3$ has growth rate 0.5. As a result, $f[x]$ goes up 0.5 times as fast as x goes up.

■ B.2) Growth of power functions $f[x] = a\,x^k$

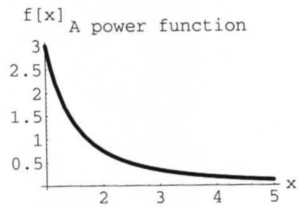
A power function $f[x]$ is any function whose formula has the form $f[x] = a\,x^k$ where a and k are constants. Here's a plot of a power function:

In[10]:=
```
Clear[f,x]
a = 3; k = 2; f[x_] = a x^k;
Plot[f[x],{x,1,5},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","f[x]"},
PlotLabel->"A power function"];
```



The growth increases as x advances from left to right. Here's another:

In[11]:=
```
Clear[f,x]
a = 3; k = -2; f[x_] = a x^k;
Plot[f[x],{x,1,5},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","f[x]"},
PlotLabel->"A power function"];
```



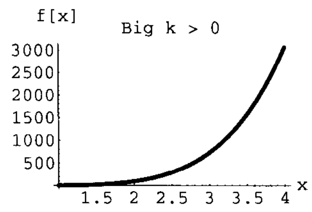
The (negative) growth decreases as x advances from left to right.

B.2.a) When you take a power function $f[x] = a\,x^k$, what is the significance of k?

Answer: Here is what a large positive k forces:

In[12]:=

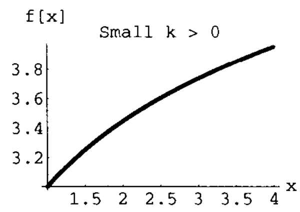
```
Clear[f,x]
a = 3; k = 5; f[x_] = a x^k;
Plot[f[x],{x,1,4},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","f[x]"},
PlotLabel->"Big k > 0",
PlotRange->All];
```



Look at those numbers on the vertical axis! Big positive k's force big-time growth. Here is what a small positive k forces:

In[13]:=

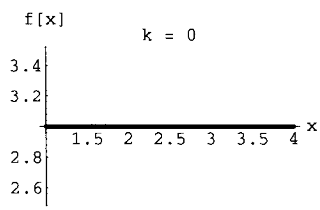
```
Clear[f,x]
a = 3; k = 0.2; f[x_] = a x^k;
Plot[f[x],{x,1,4},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","f[x]"},
PlotLabel->"Small k > 0",
PlotRange->All];
```



A small positive k forces small-time growth. Here is what $k = 0$ forces:

In[14]:=

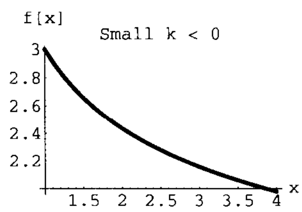
```
Clear[f,x]
a = 3; k = 0; f[x_] = a x^k;
Plot[f[x],{x,1,4},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","f[x]"},
PlotLabel->"k = 0",
PlotRange->All];
```



$k = 0$ forces no growth. Here is what a small negative k forces:

In[15]:=

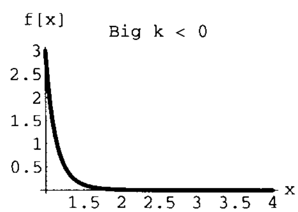
```
Clear[f,x]
a = 3; k = -0.3; f[x_] = a x^k;
Plot[f[x],{x,1,4},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","f[x]"},
PlotLabel->"Small k < 0",
PlotRange->All];
```



A small negative k forces small-time negative growth. Here is what a big negative k forces:

In[16]:=

```
Clear[f,x]
a = 3; k = -8; f[x_] = a x^k;
Plot[f[x],{x,1,4},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","f[x]"},
PlotLabel->"Big k < 0", PlotRange->All];
```



A big negative k forces big-time (negative) growth until x reaches the point at which $f[x] = a\,x^k$ peters out.

■ B.3) Growth of exponential functions $f[x] = a\,e^{rx}$

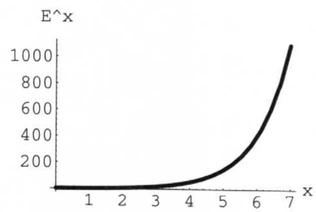
One difference between serious science and old-time classroom algebra is the significance of a certain squirrelly number called e.

In[17]:=
```
N[E,100]
```
Out[17]=
```
2.718281828459045235360287471352662497757247093699959574966 9\
  676277240766303535475945713821785251664 27
```

B.3.a) Plot e^x and then plot e^{-x} and describe what you see.

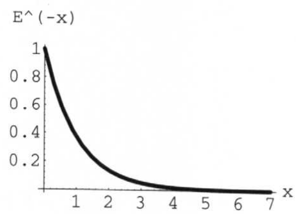
Answer: Here is a plot of e^x:

In[18]:=
```
Clear[x]
Plot[E^x,{x,0,7},
PlotStyle->{{Blue,Thickness[0.015]}},
AxesLabel->{"x","E^x"}];
```



Pristine exponential growth. Here is a plot of e^{-x}:

In[19]:=
```
Clear[x]
Plot[E^(-x),{x,0,7},
PlotStyle->{{Blue,Thickness[0.015]}},
AxesLabel->{"x","E^(-x)"}];
```



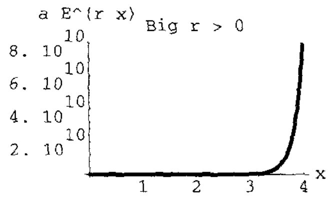
Pristine exponential decay.

B.3.b) When you take an exponential function $f[x] = a\,e^{rx}$, what is the significance of r?

Answer: Here is what a large positive r forces:

In[20]:=

```
Clear[f,x]
a = 3; r = 6; f[x_] = a E^(r x);
Plot[f[x],{x,0,4},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","a E^(r x)"},
PlotLabel->"Big r > 0",
PlotRange->All];
```

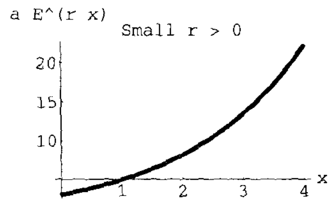


Look at those numbers on the vertical axis! Big positive r's force astoundingly big-time growth. Folks call this by the name "exponential growth."

Here is what a small positive r forces:

In[21]:=

```
Clear[f,x]
a = 3; r = 0.5; f[x_] = a E^(r x);
Plot[f[x],{x,0,4},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","a E^(r x)"},
PlotLabel->"Small r > 0",
PlotRange->All];
```

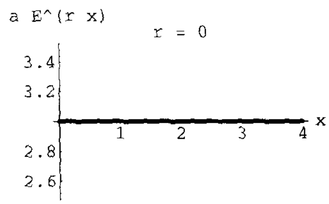


A small positive r forces small-time growth at first.

Here is what $r = 0$ forces:

In[22]:=

```
Clear[f,x]
a = 3; r = 0; f[x_] = a E^(r x);
Plot[f[x],{x,0,4},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","a E^(r x)"},
PlotLabel->"r = 0",
PlotRange->All];
```

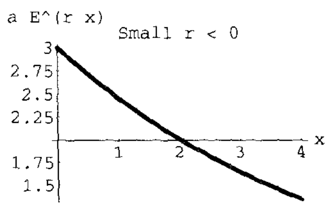


$r = 0$ forces no growth.

Here is what a small negative r forces:

In[23]:=

```
Clear[f,x]
a = 3; r = -0.2; f[x_] = a E^(r x);
Plot[f[x],{x,0,4},
PlotStyle->{{Red,Thickness[0.015]}},
AxesLabel->{"x","a E^(r x)"},
PlotLabel->"Small r < 0",
PlotRange->All];
```



A small negative r forces small-time negative growth.

Here is what a big negative r forces: