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Introduction to Mathematical Logic and Resolution Principle

(Second Edition)

Guo-Jun Wang Hong-Jun Zhou

(数理逻辑引论与归结原理)

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Preface

Modern mathematics has acquired a significant growth level with the rapid progress of science and technology. Conversely we can also say that the development of modern mathematics serves to lay the foundations for the progress of science and technology. Mathematics till date has not only been a towering big tree having the luxuriant growth of leaves and branches but has also deeply rooted itself in the areas of morden science and technology. According to the Mathematics Subject Classification 2000 provided by the American Mathematical Society, the subjects have been numbered from 00, 01, $\cdots$, up to 97 except absence of a minority and each class has been further classified into tens of sorts of research directions. It is thus clear that the contents of mathematics are vast as the open sea and mathematicians having a good command of each branch like in the times of Euler no longer exist.

As stated above, modern mathematics has numerous branches, the research contents and methods of distinct branches are very different. Hence it is not realistic to expect mathematical researchers to be proficient in all branches. But it is, in our view, necessary for them to acquaint themselves to a certain extent with the contents and methods of mathematical logic. By 'acquaint themselves to a certain extent with' we primarily mean that they should understand the introduction to mathematical logic, i.e. the theory of logical calculi, including propositional and first order predicate calculi, because it is not only the common foundation of axiomatic set theory, model theory, proof theory and recursion theory in mathematical logic, but also the part in which non-logical experts are most interested. Particularly for scholars who are engaged in teaching and scientific research in specialized subjects of computer, applied mathematics, artificial intelligence and so on and for university students and graduate students who are studying in these specialities, a familiarity with logical calculi is necessary.

The theory of logical calculi is an effective tool. A familiarity with the methods and techniques in logical calculi will lay a foundation for further studying subjects such as resolution principle, logic programming and theorem automated proving, and the methods and techniques of resolution principle play a crucial role in logic programming and automated reasoning. If we could have a textbook which introduces commonly the theory of logical calculi and, based on this, presents clearly and precisely the theory of resolution principle, it would be of great value for teachers, students and researchers engaged in the specialities of computer, applied mathematics and artificial intelligence. This textbook is intended as an attempt in this direction.

The reference [12] is regarded as a classic one. It introduces several proof procedures which are based on Herbrand's theorem after examining in a great detail the theory of resolution principle, and provides basic contents such as problem solving and program design in theorem automated proving. The reference [12] is a good book and was cited by the related literature at home and abroad. It is a pity that the reference [12] lays special emphasis on the resolution principle, while the introduction to the theory of logical calculi is limited to only the part that is directly used later in the book. Important contents such as the equivalence of a prenex normal form to the original formula and the completeness of propositional and predicate calculi are not involved. Hence the contents of [12] are inadequate for the readers who expect to study logical calculi. The reference [15] makes a complement to [12], but the contents of logical calculi are still inadequate. The references on mathematical logic listed in this book are all masterpieces, in which the introduction to logical calculi is a high standard and is orthodox. For example, the proof for the completeness of propositional logic adopts the method of consistent extensions, and the proof for the completeness of the first order predicate logic adopts the traditional extension method by adding countably infinite individual constants[1] or by adding countably infinite variable symbols[22]. These methods are of course rigorous and the arguments are unassailable. However, these methods seem too professional. In addition, the related literature lacks in general the content of resolution principle. Hence it becomes necessary to publish a textbook as mentioned above, which introduces first the theory of logical calculi in a common way and, based on this, presents clearly and precisely the theory of resolution principle.

The authors found that:

(i) Although formalization and symbolization are the intrinsic characteristics of mathematical logic, we should remember to use formal symbols as little as possible or not use them if possible.

(ii) We should remember to describe abstract concepts as commonly as possible. These two points are very important and will help the reader to better understand logical notions. For example, in the semantics of propositional calculus, if we call a valuation mapping a 'judge', the set $\{0, 1\}$ of truth values a 'mark table', and the set of all valuations 'the panel of judges', it will produce fairly good effects, and this carries also a foreshadowing of the introduction to the semantics of many-valued logics.

(iii) The theory of Boolean algebras is closely related to the theory of logical calculi and is the basic knowledge in which students major in mathematics and computer must be proficient. Hence it is natural and easily comprehensible to prove the completeness of propositional and predicate logics by proceeding from the theory of Boolean algebras. This book takes full note of these three aspects mentioned above when introducing the logical calculi. The algebraic proofs for the completeness were first given in the reference [10], but the notations there are different from the ones currently used, and the proof for the completeness of first order predicate logic was fairly scattered. The reader needs reading tens of pages in order to find the completeness theorem. This book will propose concisely an alternative algebraic proof for the completeness in terms of the quotient algebra.

Resolution principle was proposed by John Alan Robinson in 1965, which is one of the important tools in theorem automated proving (see, e.g., [24],[25]). In particular, one can drive a light carriage on a familiar road to study the important content of logic programming[20] in computer science after having an intimate knowledge of a set of formalized methods in resolution principle. This book, on the basis of introducing systematically logical calculi in a common way, presents the basic contents of resolution principle as clearly and precisely as possible. For example, unification is a crucial notion in resolution principle and in logic programming, we, however, cannot find anywhere the complete proof for the associative law of composition of substitutions related to it. This book provides a rigorous and complete proof for the associative law. As for the proof of the first theorem of Herbrand, the reference [12] cited the König's lemma of which the source cannot be found out (it was only marked with 'Knuth, 1968' and did not appear in the reference at all), and it seems other books omitted König's lemma. This book gives an alternative lemma, i.e. Lemma 5.5.2, from which a rigorous proof for the first theorem of Herbrand follows. For one more example, the reference [12], when proving the completeness of PI-resolution as well as that of lock resolution, gave first only a relatively rigorous proof for the case of ground clauses but used rather indefinite statements such as "it is easy to see ..." and "using ... a process similar to that given in the proof of ..." for the general case. This book provides rigorous proofs for these theorems by introducing the notion of resolution-preserving extension and by establishing corresponding lemmas. Moreover, in order to find the clause set of a formula one should find first the Skolem standard form of the formula in general. Searching the Skolem standard form of a formula of the form $A \wedge B$, however, is much more complicated than searching that of each conjunct. The reference [12] always turned without any declaration to compute the respective Skolem standard forms of A and B when computing that of a complex formula of the form $A \wedge B$. Section 6.5 in this book will give a special discussion on this issue. In addition, this book pays attention to discuss the theory in concern as clearly as possible. For not easily understandable notions and theorems, this book gives appropriate examples to explain them. For example, the proof of the lifting lemma on resolvents is very long and is difficult to understand, and this book constructs first an example (see Example 6.4.2) identical with the proof of the lemma, and then gives the proof. This book frequently reminds the reader or summarizes the problems in previous contents worthy our notice by means of "remarks" by stages, and arranges plenty of exercises. We expect these will help the reader to better understand the contents of the book.

In view of parallels between many and two-valued logical calculi, and particularly for extensive applications of many-valued logics in approximate reasoning, this book arranges Chapter 8 which provides an overview of many-valued logical calculi by introducing Łukasiewicz propositional fuzzy logic and the new formal deductive system $\mathcal{L}^*$ proposed by the first author. Quantitative logic which is our research work in recent years is arranged as the last chapter.

This book is divided into a total of 8 chapters. Chapter 1 provides preliminaries and particularly the theory of Boolean algebras. Chapter 2 studies two-valued propositional calculus. Chapters 3 and 4 introduce the semantics and the syntax of first order predicate logical calculus, respectively, from which first order systems with equality are excluded. Chapter 5 systematically introduces Skolem standard form and Herbrand's theorem. In Section 5.4 we propose the theory of regular function systems which can be viewed as a generalization of Herbrand universe. Chapters 6 and 7 discuss resolution principle and its refinements, respectively. Chapter 8 introduces many-valued logical calculi. The last chapter, Chapter 9, serves as an introduction to quantitative logic which is fairly young. We hope it can develop towards the right direction and expect its possible applications in related disciplines. Lastly, we would like to remind the reader that Sections 4.5 and 5.4 are independent of the main part of the book and hence are marked with $*$. The reader can choose to skip over these two sections, while still following the main thrust of the book.

The Chinese edition of the book was used as teaching material a number of times for graduate students and visiting scholars. They corrected the slips in writing of the first version and proposed many helpful suggestions. They all deserve our thanks. Prof. Dao-Wu Pei at Zhejiang Sci-Tech University and Prof. Hong-Bo Wu at Shaanxi Normal University deserve a very special note of thanks. They discussed with the first author the completeness and the simplification of axioms of the system $\mathcal{L}^*$ for many times. These helped in compiling Chapter 8. We would like to thank Postdoctoral researcher Hua-Wen Liu at Shandong University, and Graduate students of Shaanxi Normal University, especially San-Min Wang, Feng Qin, Xiang-Nan Zhou, Xiao-Yan Qin, Cheng Han, Li Fu, Fang Ren, Xiao-Jing Hui, Mu-Cong Zheng, Yong-Jian Xie, Qing-Jun Luo, Li-Feng Li, Dong-Xiao Zhang, Dong-Li Liu, Yong-Mei Ru, Hua-Rong Zhang, Yu-Jing Song, Xiang-Yun Wang, He-Jun Yuan, Long-Chun Wang, Qing-Yan Song, Rong Lan, Wen-Yan Xu, Ren-Suo Su, Yan Ren, Xiao-Jue Ma, Jing-Jie Wu, Ye Jiao, Yan-Bin Luo, Heng-Yang Wu, Meng-Ke Siqin, Jian-Hua Zhang, Miao Liu, Kai-Min Wang, Zheng-Bo Zhao, Jia-Lu Zhang, Qing-Sheng Yuan, Jian-Cheng Zhang and Xing-Fang Zhang for their helpful comments and useful suggestions. We would like to give our special thanks to Jun Li, Yan-Hong She, Bang-He Han, Hui-Xian Shi, Li-Na Ma and Bi-Jing Li. They did superb jobs of proofreading the draft of this English edition. Besides catching many typographical errors, they gave us numerous valuable suggestions on refining the language and on improving the readability of the book. We are also grateful to Xin-Ling Huang for her help in typing the TEX files.

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We apologize to colleagues who helped us in the preparation of the book but are not mentioned here due to our inadvertent omission.

Contents

Chapter 1

Preliminaries

According to the viewpoint of the School of Bourbaki, there are three mother structures in mathematics from which all other mathematical structures can be generated, and which are not reducible one to the other. These are precisely: algebraic structure, topological structure, and order structure. The present chapter is devoted to a brief introduction of the theory of order structure, and the latter part provides an overview of algebraic structure. All these preliminaries will help the reader understand the logical calculi better.

1.1 Partially ordered sets

1.1.1 Preordered sets

Definition 1.1.1 Let X be a non-void set. An **n-ary relation** R on X is a subset of the Cartesian product set X^n. If an n-tuple $(x_1, \cdots, x_n)$ of elements of X belongs to R, i.e. $(x_1, \cdots, x_n) \in R$, then we say that $(x_1, \cdots, x_n)$ satisfies R, denoted by $R(x_1, \cdots, x_n) = 1$; otherwise, we say that $(x_1, \cdots, x_n)$ does not satisfy R, denoted by $R(x_1, \cdots, x_n) = 0$. We write xRy instead of $R(x, y) = 1$ in the case of $n = 2$. A unary relation R on X is nothing but a subset of X.

Example 1.1.1 Define R on the unit interval $[0, 1]$ by xRy if and only if $y = x^2$. Then R is a binary relation on $[0, 1]$. In general, let $f : [0, 1] \to [0, 1]$ be a unary function. Then the graph $R = \{(x, f(x)) \mid x \in [0, 1]\}$ of f is a binary relation on $[0, 1]$. More generally, let $f : X^n \to X$ be an n-ary function on X. Then the graph $R = \{(x_1, \cdots, x_n, f(x_1, \cdots, x_n)) \mid (x_1, \cdots, x_n) \in X^n\}$ is an $(n+1)$-ary relation on X. But an $(n+1)$-ary relation on X is not necessarily a graph of an n-ary function on X. For instance, define on [0,1] xRy if and only if $x \leqslant y$, then R is a binary relation on $[0, 1]$, which, however, is not a graph of any function on $[0, 1]$.

Definition 1.1.2 Let X be a non-void set, and $\prec$ a binary relation on X. Then $\prec$ is called a **preorder** if it satisfies the following two conditions:

(i) $x \prec x \quad (x \in X)$;

(ii) $x \prec y$ and $y \prec z$ imply $x \prec z \quad (x, y, z \in X)$.

The ordered pair $(X, \prec)$ is called a **preordered set** . These conditions are referred to (i) and (ii), respectively, as **reflexivity** and **transitivity**.

Example 1.1.2 (i) Let X be the set of all triangles on the Euclidean plane $\mathbf{R}^2$. Denote by $m(x)$ the area of the triangle x, and define $x \prec y$ if and only if $m(x) \leqslant m(y)$, i.e. the area of x is not greater than that of y. Then $\prec$ is a preorder on X, and $(X, \prec)$ is a preordered set.

(ii) Let $\mathcal{P}_f(N)$ be the set of all finite subsets of the natural number set $\mathbf{N}$. Denote by $|A|$ the number of elements of A, and define $A \prec B$ if and only if $|A| \leqslant |B| \quad (A, B \in \mathcal{P}_f(N))$. Then $(\mathcal{P}_f(N), \prec)$ is a preordered set.

(iii) Let $\mathbf{R}$ be the set of all real numbers. Denote by $|x|$ the absolute value of x, and define $x \prec y$ if and only if $|x| \leqslant |y|$ $(x, y \in \mathbf{R})$. Then $(\mathbf{R}, \prec)$ is a preordered set.

1.1.2 Partially ordered sets

Definition 1.1.3 Let $(P, \prec)$ be a preordered set. $\prec$ is called a **partial order** if it satisfies, besides the reflexivity and the transitivity, the condition of **anti-symmetry**, i.e.

$$x \prec y \text{ and } y \prec x \text{ imply } x = y.$$

The ordered pair $(P, \prec)$ is called a **partially ordered set** (**poset,** for short). '$x \prec y$' is read as 'x is less than or equal to y', or 'y is larger than or equal to x'. If either $x \prec y$ or $y \prec x$ holds for each $x, y \in P$, then $(P, \prec)$ is called a **totally ordered set**, also called a **linearly ordered set**.

Example 1.1.3 (i) None of the three preordered sets given in Example 1.1.2 is a poset.

(ii) Let P be the set of all triangles on the Euclidean plane $\mathbf{R}^2$. For every triangle $x, y \in P$, define $x \prec y$ if and only if x is contained in y, then $(P, \prec)$ is a poset. More generally, let $\mathcal{P}(X)$ be the powerset of X, for all subsets A and B of X, define $A \prec B$ if and only if $A \subset B$. Then $(\mathcal{P}(X), \subset)$ is a poset, where $A \subset B$ means that every element of A is also an element of B.

(iii) Let $C_{[0,1]}$ be the set of all continuous functions defined on [0,1]. For every $f, g \in C_{[0,1]}$, define $f \prec g$ if and only if $f(x) \leqslant g(x)$ for every $x \in [0, 1]$. Then $(C_{[0,1]}, \prec)$ is a poset.

(iv) Let $\prec$ be the natural order on $\mathbf{R}$. Then $(R, \prec)$ is a poset, and is also a totally ordered set. Furthermore, let $\mathbf{C}$ be the set of all complex numbers. Define $a + b\mathrm{i} \prec c + d\mathrm{i}$ if and only if $a < c$ or $a = c$ and $b \leqslant d$, then $(\mathbf{C}, \prec)$ is a totally ordered set. However, if define $a + b\mathrm{i} \prec c + d\mathrm{i}$ if and only if $a \leqslant c$ and $b \leqslant d$, then $(\mathbf{C}, \prec)$ is only a poset, but not totally ordered.

(v) In Figure 1.1.1 below, let P be a set of elements represented by black points of the Euclidean plane $\mathbf{R}^2$, and let the element corresponding to the lower vertex of a line segment be less than or equal to that corresponding to the other vertex as well as itself. Then the $P's$ in (a) to (e) are all posets.

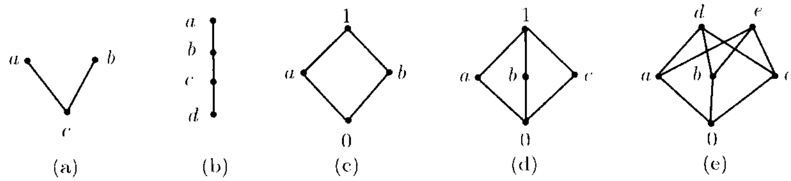

Figure 1.1.1

From now on, we denote by $\leqslant$ the order $\prec$ on a poset P.

1.1.3 Supremum and infimum

Definition 1.1.4 Let $(P, \leqslant)$ be a poset, $X \subset P, a \in P$. The element a is called **a(n) lower (upper) bound** of X if $a \leqslant x$ $(x \leqslant a)$ for every $x \in X$. Let a be a(n) lower (upper) bound. a is called the **infimum (supremum)** of X if $b \leqslant a$ $(a \leqslant b)$ for each lower (upper) bound b of X. In this case, we write $a = \inf X (a = \sup X)$ or $a = \wedge X (a = \vee X)$.

Example 1.1.4 (i) For the poset P in Figure 1.1.1(a), the subset $\{a, b\}$ has the infimum c, but it has no upper bounds. For the P in Figure 1.1.1(b), the supremum and the infimum of the subset $\{a, b, c\}$ are a and c, respectively. For the P in Figure 1.1.1(e), $X = \{a, b, c\}$ has the infimum 0, and two upper bounds d and e, but it has no supremum.

(ii) In the poset $(\mathcal{P}(X), \subset)$, let $\{A_i \mid i \in I\} \subset \mathcal{P}(X)$. Then the supremum of $\{A_i \mid i \in I\}$ as well as its infimum does exist, they are the set-union and the set-intersection of $\{A_i \mid i \in I\}$, respectively, i.e. $\sup\{A_i \mid i \in I\} = \bigcup_{i \in I} A_i$, and $\inf\{A_i \mid i \in I\} = \bigcap_{i \in I} A_i$.

(iii) In the poset $(C_{[0,1]}, \leqslant)$, let $X = \{h \mid h(x) = x^n, n = 1, 2, \cdots\}$. Then $\sup X = f$, where $f(x) = x (x \in [0, 1])$, and $\inf X = g$, where $g(x) = 0 (x \in [0, 1])$. If we give up the continuity, then the equation $\sup X = f$ still holds, whereas the $\inf X$ is defined by $(\inf X)(1) = 1$ and $(\inf X)(x) = 0$ for $x \in [0, 1)$. Moreover, let $Y = \{\bar{m} \mid m \in \mathbf{Z}\}$, where $\bar{m}$ is a constant function with value m on $[0, 1]$. Then neither the lower bound nor the upper bound exists.

(iv) Let $P = [0, 1]$, $\leqslant$ the natural order on P, and X the empty set of P. Then $\sup X = 0$ and $\inf X = 1$. The reason is that the empty set takes every element of

P as its lower (upper) bound, and hence the supremum as the least upper bound is 0 and the infimum as the greatest lower bound is 1 (note that the condition 'for any $x \in \varnothing, x \leqslant a (a \leqslant x)$' holds for every element $a \in P$).

1.1.4 Directed sets

Definition 1.1.5 A subset Y of a preordered set $(X, \leqslant)$ is called a **directed subset** if for every element $a, b \in Y$, there exists $c \in Y$ such that $a \leqslant c$ and $b \leqslant c$. In particular, $(X, \leqslant)$ is said to be **directed** if $Y = X$.

Example 1.1.5 (i) The three preordered sets given in Example 1.1.2, and the posets in Example 1.1.3 (ii), (iii) and (iv) as well as those in Example 1.1.4 (ii),(iii) and (iv) are all directed.

(ii) Every poset having a greatest element is directed, so is every totally ordered set.

(iii) Consider the poset $(P, \leqslant)$ in Example 1.1.3(ii). Suppose that X is the set of all triangles contained in the unit circle with the origin of the Euclidean plane as its center. Then X is not a directed subset of P. Indeed, take a diameter of the unit circle as the base side and two points on the circle but not on the diameter to construct two triangles. Then there are no triangles inside the unit circle which are larger than or equal to these two triangles simultaneously.

(iv) Let X be an infinite set, and $\mathcal{P}_f(X)$ the set of all finite subsets of X. Then $(\mathcal{P}_f(X), \subset)$ is directed.

(v) Let X be a nonempty set, $\mathcal{F}(X)$ the set of all fuzzy subsets of X, i.e. $\mathcal{F}(X) = \{f \mid f : X \to [0,1] \text{ is a function}\}$, and the order on $\mathcal{F}(X)$ defined pointwisely, i.e. $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for every $x \in X$. Then $[\mathcal{F}(X), \leqslant]$ is a poset. Let $\mathcal{F}_<(X)$ be the set of fuzzy sets such that the membership degree of each element $x \in X$ is strictly less than 1 [i.e. $f(x) < 1$ for every $x \in X$]. Then $[\mathcal{F}_<(X), \leqslant]$ is a directed subset of $\mathcal{F}(X)$. Let $\mathcal{F}_0(X)$ be the set of fuzzy sets such that the membership degrees of at most finite elements of X are not 0. Then $[\mathcal{F}_0(X), \leqslant]$ is a directed subset of $\mathcal{F}(X)$.

Exercise 1.1

1. Give two examples of preordered sets which are not posets.

2. Let $(\mathcal{U}, \subset)$ be the poset generated by the set of all open sets of the real line $\mathbf{R}$ under the set-inclusion order, and

$$X = \left\{ \left(-2 - \frac{1}{n}, 2 + \frac{1}{n} \right) \mid n = 1, 2, \cdots \right\}.$$

Find $\sup X$ and $\inf X$. More generally, let $\mathcal{A}$ be a collection of open sets. Do the equations $\sup \mathcal{A} = \cup\{A \mid A \in \mathcal{A}\}$ and $\inf \mathcal{A} = \cap\{A \mid A \in \mathcal{A}\}$ hold?

3. Give an example of a directed poset which has no greatest element and is not totally ordered.

1.2 Lattices

1.2.1 Lattices

Definition 1.2.1 Let $(L, \leqslant)$ be a poset. $(L, \leqslant)$ is called a **lattice** if both $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all $a, b \in L$. $\sup\{a, b\}$ and $\inf\{a, b\}$ are often denoted by $a \vee b$ and $a \wedge b$, respectively.

It is easy to check the following proposition, hence it is left as an exercise for the reader.

Proposition 1.2.1 Let $(L, \leqslant)$ be a lattice. Then:

(i) Let X be a non-void finite subset of L. Then both $\sup X$ and $\inf X$ exist.

(ii) $a \vee b = b \vee a, a \wedge b = b \wedge a$.

(iii) $(a \vee b) \vee c = a \vee (b \vee c), a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

(iv) $a \leqslant b$ if and only if $a \vee b = b$ if and only if $a \wedge b = a$.

Definition 1.2.2 Let $(L, \leqslant)$ be a poset. $(L, \leqslant)$ is said to be **complete** if both $\sup X$ and $\inf X$ exist for all subsets $X \subset L$.

Every complete lattice $(L, \leqslant)$ has a greatest element $\sup L$, denoted by 1_L, and a least element $\sup \varnothing$, denoted by 0_L. We often write 1 and 0 in place of 1_L and 0_L, respectively, if no confusion arises.

Example 1.2.1 (i) $(\mathbf{R}, \leqslant)$ is a lattice which is not complete. $([0,1], \leqslant)$ is a complete lattice. More generally, every totally ordered set is necessarily a lattice because the infimum (supremum) of a and b is the smaller (greater) one of a and b. In particular, the totally ordered set $\{0, 1\}$ of two elements 0 and 1 is a lattice.

(ii) Both $(\mathcal{P}(X), \subset)$ and $(\mathcal{F}(X), \leqslant)$ are complete lattices.

(iii) Every lattice $(L, \leqslant)$ with a finite non empty underlying set L is complete.

(iv) The posets given in Figure 1.1.1 (b),(c) and (d) are lattices with finite underlying sets, and therefore are complete.

1.2.2 Distributive lattices

Definition 1.2.3 Let $(L, \leqslant)$ be a lattice. $(L, \leqslant)$ is said to be **distributive** if it satisfies the **distributive laws:**

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \tag{1.2.1}$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \tag{1.2.2}$$

for all $a, b, c \in L$.

Before giving examples of distributive lattices, we prove several commonly used propositions.

Proposition 1.2.2 Let $(L, \leqslant)$ be a lattice. If one of the identities (1.2.1) and (1.2.2) holds, then so does the other one, and therefore $(L, \leqslant)$ is distributive.

Proof Assume that the identity (1.2.1) holds. Indeed, since $a \leqslant a \vee b, a \wedge c \leqslant a$, it follows from Proposition 1.2.1(iv) that $(a \vee b) \wedge a = a$ and $a \vee (a \wedge c) = a$. Then by (1.2.1),

$$\begin{aligned}(a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) \\ &= a \vee ((a \vee b) \wedge c) \\ &= a \vee (a \wedge c) \vee (b \wedge c) \\ &= a \vee (b \wedge c).\end{aligned}$$

This shows that (1.2.2) holds. By duality, (1.2.2) implies (1.2.1) too. □

Proposition 1.2.3 Let $\{(L_i, \prec_i) \mid i \in I\}$ be a collection of preordered sets, $I \neq \varnothing$. Let $L = \prod\limits_{i \in I} L_i$ be the Cartesian product of $L_i' s$. Define on L

$$(a_i)_{i \in I} \prec (b_i)_{i \in I} \text{ if and only if, for every } i \in I, a_i \prec_i b_i. \tag{1.2.3}$$

Then:

(i) $(L, \prec)$ is a preordered set.

(ii) If every $(L_i, \prec_i)$ is a poset, then so is $(L, \prec)$.

(iii) If every $(L_i, \prec_i)$ is a (complete) lattice, then so is $(L, \prec)$.

(iv) If every $(L_i, \prec_i)$ is distributive, then so is $(L, \prec)$.

Proof (i) and (ii) trivially hold. Let $(L_i, \prec_i)$ be a lattice, $i \in I$, and $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ two elements of L. Then it follows from (1.2.3) that

$$(a_i)_{i \in I} \vee (b_i)_{i \in I} = (a_i \vee b_i)_{i \in I}, \tag{1.2.4}$$

$$(a_i)_{i \in I} \wedge (b_i)_{i \in I} = (a_i \wedge b_i)_{i \in I}. \tag{1.2.5}$$

Thus $(L, \prec)$ is a lattice. In a similar way one can check that $(L, \prec)$ is complete whenever each $(L_i, \prec_i)$ is complete $(i \in I)$. Therefore (iii) holds. Lastly, let $(L_i, \prec_i)$ be distributive for each $i \in I$. Then it follows from (1.2.4) and (1.2.5) that

$$\begin{aligned}(a_i)_{i \in I} \wedge ((b_i)_{i \in I} \vee (c_i)_{i \in I}) &= (a_i \wedge (b_i \vee c_i))_{i \in I} \\ &= ((a_i \wedge b_i) \vee (a_i \wedge c_i))_{i \in I} \\ &= (a_i \wedge b_i)_{i \in I} \vee (a_i \wedge c_i)_{i \in I} \\ &= ((a_i)_{i \in I} \wedge (b_i)_{i \in I}) \vee ((a_i)_{i \in I} \wedge (c_i)_{i \in I}).\end{aligned}$$

This shows that (iv) is true. The proof is complete. □

Proposition 1.2.4 Every totally ordered set is a distributive lattice.

Proof Let $(L, \leqslant)$ be a totally ordered set. For $a, b, c \in L$, without loss of generality, we can assume that $a \leqslant b \leqslant c$. Then it follows from $a \wedge (b \vee c) = a \wedge c = a$ and $(a \wedge b) \vee (a \wedge c) = a \vee a = a$ that (1.2.1) holds. Therefore $(L, \leqslant)$ is distributive. □

Example 1.2.2 (i) Both $(\mathbf{R}, \leqslant)$ and $([0,1], \leqslant)$ are distributive lattices.

(ii) $(\mathcal{F}(X), \leqslant)$ is a distributive lattice. Indeed, $\mathcal{F}(X) = \prod_{i \in X} L_i$, where $L_i = [0,1], i \in X$. By Example 1.1.5(v), $\leqslant$ is just the pointwise order as defined in Proposition 1.2.3. Hence $(\mathcal{F}(X), \leqslant)$ is distributive.

(iii) $(\mathcal{P}(X), \subset)$ is distributive, i.e.

$$\begin{cases} A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \end{cases} \tag{1.2.6}$$

Let us limit ourselves to the first identity, in a similar way we can show the second one. $x \in A \cap (B \cup C)$ if and only if $x \in A$ and either $x \in B$ or $x \in C$ if and only if either $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$, i.e. $x \in (A \cap B) \cup (A \cap C)$. Therefore $(\mathcal{P}(X), \subset)$ is distributive.

(iv) In Figure 1.1.1, both the posets (b) and (c) are distributive lattices, but (d) is not. Indeed, $a \wedge (b \vee c) = a \wedge 1 = a$, whereas $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0 \neq a$, hence (1.2.1) does not hold.

1.2.3 Infinite distributive laws

Definition 1.2.4 Let $(L, \leqslant)$ be a complete lattice. Then the following identities (1.2.7) and (1.2.8) are called the **first infinite distributive law** and the **second infinite distributive law**, respectively:

$$a \wedge \left(\bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \wedge b_i), \tag{1.2.7}$$

$$a \vee \left(\bigwedge_{i \in I} b_i \right) = \bigwedge_{i \in I} (a \vee b_i). \tag{1.2.8}$$

Example 1.2.3 (i) It is straightforward to check that $(\mathcal{P}(X), \subset)$ satisfies the first and the second infinite distributive laws.

(ii) $(\mathcal{F}(X), \leqslant)$ also satisfies the first and the second infinite distributive laws. Since every totally ordered set which is complete satisfies the first and the second infinite distributive laws, similarly to the proof of Proposition 1.2.3, one can show that $(\mathcal{F}(X), \leqslant)$ satisfies these two infinite distributive laws too.