The Physical Principles of Magnetism

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The Magnetic Field

1. Historical

The writings of Thales, the Greek, establish that the power of loadstone, or magnetite, to attract iron was known at least as long ago as 600 B.C. It has been claimed that the Chinese used the compass sometime before 2500 B.C. That magnetite can induce iron to acquire attractive powers, or to become magnetic, was mentioned by Socrates. Thus permanent and induced magnetism represent two of man's earliest scientific discoveries. However, the only real interest in magnetism in antiquity appears to be concerned with its use in the construction of the compass. For example, it is illuminating that it was not until many centuries later that Gilbert (1540–1603) realized that the earth was a huge magnet, even though the operation of the compass depends on this very fact.

The discovery of two regions called magnetic poles, or sometimes just "poles," which attracted a piece of iron more strongly than the rest of the magnetite, was made by P. Peregrines about 1269 A.D. Coulomb (1736–1806), in accurate quantitative experiments with the torsion balance, investigated the forces between magnetic poles of long thin steel rods. His results form the starting point of this treatise on magnetism.

2. The Magnetic Field Vector H

Coulomb found that there were two types of poles, now called positive or north, and negative, or south. Like poles repel one another and unlike poles attract one another. This force of attraction or repulsion is proportional to the product of the strength of the poles and inversely proportional to the square of the distance between them. This is Coulomb's law, which can be stated mathematically as

$$\mathbf{F} = k\,\frac{m_1 m_2}{r^2}\,\mathbf{r}_0, \qquad (1\text{-}2.1)$$

where **F** is the force,[1] m_1 and m_2 the pole strengths, r the distance between the poles, and $\mathbf{r}_0$ a unit vector directed along r. The constant of proportionality k that occurs permits a definition of pole strength. In the cgs system of units two like poles are of unit strength if they repel each other with a force of 1 dyne when they are 1 cm apart; that is, $k = 1$. Other systems of units and their relationship to the cgs system are discussed in Appendix I.

It is convenient to consider **F** as separated into two factors. One factor is just one of the poles, say m_2, usually called the test pole. The other factor depends on the other pole, called the source, and on the location with respect to it; it is called the magnetic field **H**. This field is defined as the force the pole exerts on a unit positive pole, or

$$\mathbf{H} = \frac{m}{r^2}\,\mathbf{r}_0. \qquad (1\text{-}2.2)$$

In addition to this use of **H** as the field at a point, we will employ the same symbol **H** as the set of values of the magnetic field at all points: no confusion should result, since the correct meaning will be clear from the context. The cgs unit of magnetic field is the oersted, although the term gauss is still frequently used. Should several poles be present, experiments show that the field is the vector sum of the forces on the test pole.

Instead of the vector field quantity **H**, it is often convenient to use a scalar potential φ. The quantity φ is defined so that its negative gradient is the magnetic field

$$\mathbf{H} = -\nabla\varphi, \qquad (1\text{-}2.3)$$

where the operator ∇ is

$$\nabla = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}.$$

Here **i**, **j**, **k**, are the unit vectors of a Cartesian coordinate system, and (x, y, z) are the coordinates at the point where the field or potential is under consideration.

[1] Boldface type indicates a vector quantity.

It follows immediately that for an isolated pole

$$\varphi = \frac{m}{r}. \tag{1-2.4}$$

The work done in bringing a unit test pole from infinity to the point (x, y, z) a distance r from m is found by integrating (1-2.2) along the path.[2] This turns out to be equal to φ and provides a simple physical meaning of the scalar potential.

Consider the work done in taking a unit pole from point 1 to 2 in a field **H** given by the line integral

$$\int_2^1 \mathbf{H} \cdot d\mathbf{s} = \int_2^1 H_s \, ds. \tag{1-2.5}$$

Here $d\mathbf{s}$ is a line element of the path from point 1 to point 2. Now, since $H_s = -\partial\varphi/\partial s$, we get

$$\int_2^1 \mathbf{H} \cdot d\mathbf{s} = \varphi_2 - \varphi_1. \tag{1-2.6}$$

This integral has the same value for any path having the same first and final point; that is, the work done is independent of the path. Mathematically this is stated as

$$\oint \mathbf{H} \cdot d\mathbf{s} = 0. \tag{1-2.7}$$

3. The Magnetization Vector M

Isolated magnetic poles have never been observed in nature, but occur instead in pairs, one pole being positive, the other negative. Such a pair is called a dipole. The magnetic moment of a dipole is defined as

$$\boldsymbol{\mu} = m\mathbf{d}, \tag{1-3.1}$$

where **d** is a vector pointing from the negative to the positive pole and equal in magnitude to the distance between the poles assumed to be points. If **d** approaches zero and m increases so that $\boldsymbol{\mu} = m\mathbf{d}$ is a constant, then in the limit in which $\mathbf{d} = 0$ the dipole is said to be ideal.

Atomic theory has shown that the magnetic dipole moments observed in bulk matter arise from one or two origins: one is the motion of electrons about their atomic nucleus (orbital angular momentum) and the other is the rotation of the electron about its own axis (spin angular momentum).

[2] This is the work done *against* the magnetic force; to compute the work done *by* it, the limits of the integration are reversed.

The nucleus itself has a magnetic moment. Except in special types of experiments, this moment is so small that it can be neglected in the consideration of the usual macroscopic magnetic properties of bulk matter.

It turns out that a magnetic field **H** interacts with the electrons of an atom in such a way that a magnetic moment is induced. This phenomenon is called *diamagnetism*. Since all matter contains electrons moving in orbits, diamagnetism occurs in all substances.

Depending on the electronic structure of an atom, it may or may not have a *permanent* magnetic moment. All magnetic effects other than diamagnetism result because of permanent atomic magnetic moments. If the coupling between the moments of different atoms is small or zero, the phenomenon of *paramagnetism* results. In the absence of an applied field **H** such materials will exhibit no net magnetic moment. If the

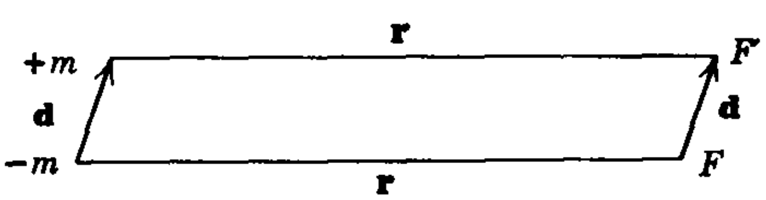

Fig. 1-3.1. F is the field point at which the potential produced by the dipole is calculated. F' is a point located a step **d** from F.

coupling between the atomic moments is very large, there are three important classifications. If the atomic moments are aligned parallel, the substance is said to be *ferromagnetic*. The magnetic moments may be aligned parallel within groups, usually two. If pairs of groups are aligned antiparallel and the atomic moments of the groups are equal, the substance is *antiferromagnetic*. However, the atomic moments of the groups may not be equal—for example, when two different elements are present; thus when they are aligned antiparallel there is a net moment. This phenomenon is called *ferrimagnetism*; some writers consider it to be just a special case of antiferromagnetism.

Because magnetic poles occur in pairs, it is of interest to calculate the magnetic field produced by such a combination. The dipole shown in Fig. 1-3.1 is considered to be almost but not quite ideal, so that $\mathbf{d} \ll \mathbf{r}$. The potential it produces at the field point F, φ_F, is due to both the positive and negative poles, that is,

$$\varphi_F = \varphi_F^{+} + \varphi_F^{-}.$$

Now let the point a step **d** from F be called F'. Except for sign, the potential $-m$ produces at F is the same $+m$ produces at F'. Hence

$$\begin{aligned}\varphi_F &= \varphi_F^{+} - \varphi_{F'}^{+} \\ &= -\mathbf{d} \cdot \nabla_F \varphi_F^{+}.\end{aligned}$$

Here ∇_F indicates differentiation with respect to the field coordinate (x, y, z) and not the source coordinate (x_i, y_i, z_i) since

$$\nabla_F \frac{1}{r} = -\nabla_s \frac{1}{r},$$

where

$$r = [(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2]^{1/2}.$$

Now, the point source $+m$ produces a potential at F given by

$$\varphi_F^{+} = \frac{m}{r}.$$

Therefore

$$\varphi_F = -m\mathbf{d} \cdot \nabla \frac{1}{r}.$$

Now suppose the dipole to be ideal, that is, $\mathbf{d} = 0$. In the limit we have

$$\varphi_F = -\boldsymbol{\mu} \cdot \nabla \frac{1}{r}.$$

Since

$$\nabla \frac{1}{r} = -\frac{1}{r^2}\left[\frac{(x - x_i)}{r}\mathbf{i} + \frac{(y - y_i)}{r}\mathbf{j} + \frac{(z - z_i)}{r}\mathbf{k}\right],$$

we get

$$\varphi_F = -\boldsymbol{\mu} \cdot \nabla \frac{1}{r} = \frac{1}{r^3}\boldsymbol{\mu} \cdot \mathbf{r} = \frac{1}{r^2}|\boldsymbol{\mu}| \cos \theta. \qquad (1\text{-}3.2)$$

The magnetic field $\mathbf{H}_F$ is then

$$\mathbf{H}_F = -\nabla\left(-\boldsymbol{\mu} \cdot \nabla \frac{1}{r}\right)$$

$$= -\frac{\boldsymbol{\mu}}{r^3} + \frac{(3\boldsymbol{\mu} \cdot \mathbf{r})\mathbf{r}}{r^5}. \qquad (1\text{-}3.3)$$

For substances that have a net magnetic moment it is usual to define a magnetization vector **M** as the ratio of the magnetic moment of a small volume at some point to that volume. The size of the volume chosen must be large enough so that a somewhat larger volume will still yield the same result for **M**; in this way we ensure that atomic fluctuations are negligible. If **M** is constant for the specimen, the material is said to be uniformly magnetized. From the definition of **M** it is clear that it is also the pole strength for a unit area perpendicular to **M**, that is

$$\sigma = \mathbf{M} \cdot \mathbf{n}, \qquad (1\text{-}3.4)$$

where σ is the pole strength per unit area and $\mathbf{n}$ is a unit vector normal to the surface.

Introduction of the vector $\mathbf{M}$ permits generalization of equation 1-3.2 for bulk material. Summing over the dipoles gives the total potential at a point external to the specimen as

$$\begin{aligned}\varphi_F &= -\sum \boldsymbol{\mu} \cdot \nabla \frac{1}{r} \\ &= -\int \mathbf{M} \cdot \nabla_F \frac{1}{r}\, dv_s. \end{aligned} \tag{1-3.5}$$

A special form of Green's theorem gives

$$\varphi_F = \int \frac{1}{r} \mathbf{M} \cdot \mathbf{n}\, dS - \int \frac{1}{r} \nabla_s \cdot \mathbf{M}\, dv \tag{1-3.6}$$

or

$$= \int \frac{\sigma\, dS}{r} - \int \frac{\rho}{r}\, dv,$$

where dS is an element of area. This result permits an interesting physical interpretation to be made. The magnetic potential can be considered to

(a) (b)

Fig. 1-3.2. In (*a*) each arrow represents a dipole, each with the same magnetic moment. The uncompensated charges for this dipole distribution are shown in (*b*). This illustrates the origin of volume charges for a simple case.

be due to two causes. One, the surface charge (or pole) density σ, and, two, a volume charge density ρ. The first of these can be easily pictured as arising from the uncompensated ends of the dipoles that end on the surface. The volume density may be pictured as the uncompensated poles that arise from an inhomogenity of the distribution of the moments, as illustrated in Fig. 1-3.2.

4. Magnetic Induction, the Vector B

The magnetic forces that must exist inside a ferro- or ferrimagnetic medium pose some special problems. Such forces have meaning only if it is possible to specify a method of measuring them. The approach adopted by Maxwell was to consider the medium as a continuum and to make a cavity around the point at which the force on the test pole was to be determined. However, the force per unit pole depends on the shape of the cavity, since this force depends partly on the pole distribution around the cavity, so that there exist an infinite number of ways that the field could be defined. In fact, two particular cavity shapes are chosen.

The field **H** is defined as the field vector in a needle-shaped cavity with an infinitesimally small diameter. The reason for this choice is that the field defined in this way satisfies equation 1-2.7. The field vector obtained when the cavity is a disk of infinitesimally small height is called the magnetic induction **B**; the reason for this choice is that Maxwell's equation $\nabla \cdot \mathbf{B} = 0$ is then satisfied. With the aid of Gauss's theorem, we can show that **B** and **H** are related by[3]

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}; \tag{1-4.1}$$

B is said to be in the units of gauss in the cgs system.

The magnetic flux Φ is defined as the flux of the vector **B** through a surface of area A; that is,

$$\Phi = \int_A \mathbf{B} \cdot \mathbf{n}\, dS. \tag{1-4.2}$$

The unit of flux is called the maxwell. Thus the induction in gauss, at some field point, is equal to the flux density, the number of maxwells per square centimeter. The foregoing definition of flux is possible only because $\nabla \cdot \mathbf{B} = 0$, one of Maxwell's equations. Often a graphical meaning is given to the flux. It can be represented as lines or tubes whose density is equal to B and direction is along **B**.

When the vectors **B**, **H**, and **M** are parallel, it is useful to define the permeability μ by

$$B = \mu H \tag{1-4.3}$$

and the susceptibility χ by

$$M = \chi H. \tag{1-4.4}$$

The susceptibility per unit mass χ_ρ is defined as χ/ρ, where ρ is the density. The atomic or molar susceptibility χ_A or χ_m then is found by multiplying χ_ρ by the atomic or molecular weight. From (1-4.1) it follows immediately that

$$\mu = 1 + 4\pi\chi. \tag{1-4.5}$$

[3] A good treatment of this problem for the analogous electrical case may be found in C. J. F. Böttcher, *Theory of Electric Polarization*, Elsevier Publishing Co., New York (1952), Ch. II.

If the magnetic vectors are not parallel, the components of **B** and **H** relative to an arbitrarily chosen Cartesian coordinate system can be related by the set of equations:

$$B_x = \mu_{11}H_x + \mu_{12}H_y + \mu_{13}H_z,$$
$$B_y = \mu_{21}H_x + \mu_{22}H_y + \mu_{23}H_z,$$
$$B_z = \mu_{31}H_x + \mu_{32}H_y + \mu_{33}H_z.$$

The quantities μ_{ij} are components of the permeability tensor $\tilde{\boldsymbol{\mu}}$. Similarly, the relationship between M and H can be expressed with the aid of a susceptibility tensor $\boldsymbol{\chi}$.

It is an experimental result that χ is negative for diamagnetic materials and positive for the other types of magnetism, being very large for ferri- and ferromagnetic substances.

The magnetization of dia-, para-, and antiferromagnetic substances disappears if the applied field is removed. This is in contrast to the behavior of ferro- and ferrimagnetic materials, which usually retain at least part of their induced magnetic moment in the absence of an applied field. For these materials the susceptibility is a function of the applied field, the temperature, and the history of the samples. Discussion of the temperature and field dependence of the susceptibility is left until later.

5. The Demagnetization Factor D

The field H' inside a specimen is different from the applied field H because of the magnetization or equivalently, the poles. Consider a ferromagnet with ellipsoidal shape in a uniform external field **H**. As discussed later, the magnetization of the ellipsoidal specimen will also be uniform. The poles that appear on the surface, indicated in Fig. 1-5.1, produce a uniform internal field, $H' - H$, opposite in direction to **H**. For specimens with an ellipsoidal shape it is usual to write

$$\mathbf{H}' = \mathbf{H} - D\mathbf{M}, \tag{1-5.1}$$

where D is called the demagnetization factor. D depends on the geometry of the specimen. For diamagnets $H' > H$; for all other magnets $H' < H$. The difference in the field H' and H can usually be neglected for dia- and paramagnets, but it can be very large for ferro- and ferrimagnets. From the reasoning of Section 1-4, it can be seen that for a disk $D = 4\pi$ for the direction perpendicular to the plane of the disk. In general the demagnetizing factor is a tensor **D**.

In the easiest general case to calculate **M** is uniform. In equation 1-3.6,

$$\varphi = \int \frac{1}{r} \mathbf{M} \cdot \mathbf{n}\, dS - \int \frac{1}{r} \nabla \cdot \mathbf{M}\, dv, \tag{1-5.2}$$

the second term is zero, and the potential, and therefore the field, is due only to the surface pole distribution. Also, in expression 1-3.5 for φ, **M** can be taken outside the integral sign, and we have

$$\varphi = -\mathbf{M} \cdot \int \nabla \frac{1}{r}\, dv. \tag{1-5.3}$$

Now $-\int \nabla(1/r)\, dv$ is the gravitational force due to a volume of uniform unit mass density (the gravitational constant $G = 1$ here) or the electric force due to a volume of unit charge density. We therefore have the

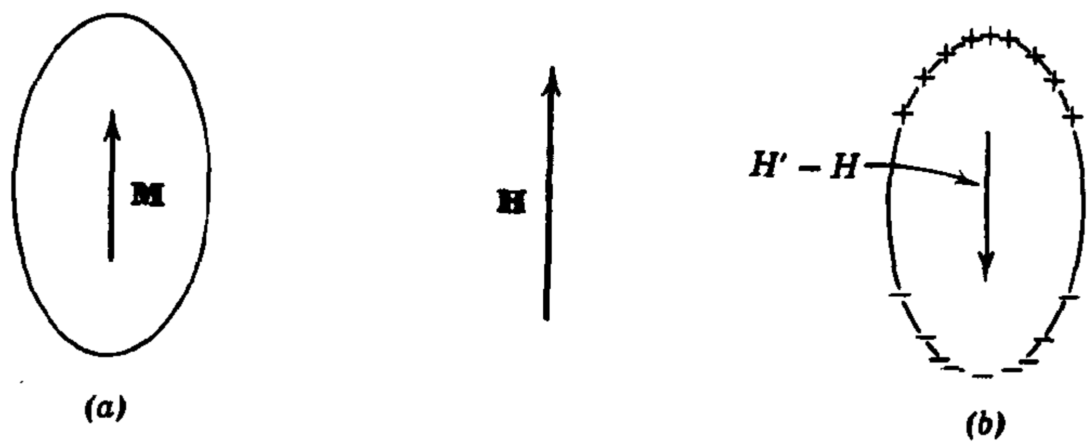

Fig. 1-5.1. (*a*) A uniformly magnetized ellipsoid and (*b*) the equivalent poles. The uncompensated surface poles produce a uniform internal field $H' - H$ and an external field that is identical to that of an equivalent dipole positioned at the ellipsoid's center. **H** is a uniform applied field.

important result that equations derived in potential theory may be used to find φ, next $H' - H$, and finally D.

It is instructive to consider the derivation of the foregoing result by a simple physical argument.[4] Let Ψ be the potential due to gravitation, or the electric charge of the body assumed of uniform density ρ. Now, if the body is moved a distance $-\delta x$ in the direction of x, the change of the potential at any point will be $-(d\Psi/dx)\,\delta x$. Instead, if we consider the body to be moved δx, and its original density ρ changed to $-\rho$, then $-(d\Psi/dx)\,\delta x$ is the resultant potential due to the two bodies (Fig. 1-5.2).

To any element of volume, mass, or charge $\rho\, dv$, there will correspond an element of the shifted body of $-\rho\, dv$ a distance $-\delta x$ away. Hence the dipole moment of these two elements is $\rho\, dv\, \delta x$, and the magnetization

[4] J. C. Maxwell, *Treatise on Electricity and Magnetism*, 3rd ed., Oxford University Press, Oxford (1891), vol. ii, p. 66. [Reprinted Dover Publications, New York (1954).]

is $\rho\,\delta x$. Therefore if $-(d\Psi/dx)\,\delta x$ is the magnetic potential of the body of magnetization $\rho\,\delta x$, then $-d\Psi/dx$ is the potential for a body of magnetization $\rho(= M)$.

In the volume common to the two bodies the density is effectively zero. A shell of positive charge or density resides on one side of the matter and one of negative on the other, each of density $\rho \cos \epsilon$, ϵ being the angle between the outward normal and the axis x. This then corresponds exactly to the first term of equation 1-5.2 and gives an immediate physical picture of the mathematics.

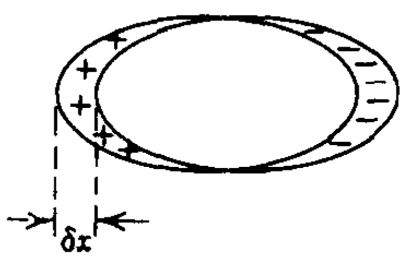

Fig. 1-5.2. Uncompensated charges or poles for two ellipsoids of opposite charge densities and a distance δx apart.

The x-component of $\mathbf{H}' - \mathbf{H}$ is $-(\partial\varphi/\partial x) = M_x(\partial^2\Psi/\partial x^2)$. Therefore, for $\mathbf{M}$ to be uniform, which implies that $\mathbf{H}'$ is uniform, Ψ must be a quadratic function of the coordinates. From potential theory[5] this occurs only when the body is bounded by a surface of second degree. The only physically possible body is then an ellipsoid.

The derivation of the expressions for Ψ is beyond the scope of this book and belongs to potential theory.[5] For completeness, and because of their practical usefulness, some of the important formulas for the demagnetization factor of ellipsoids of revolution are given.

We define a as the polar semiaxis and b as the equatorial semiaxis with $m = a/b$. Then for the prolate spheroid ($m > 1$)

$$D_a = \frac{4\pi}{(m^2-1)}\left\{\frac{m}{(m^2-1)^{1/2}} \ln\left[m + (m^2-1)^{1/2}\right] - 1\right\} \tag{1-5.4}$$

and

$$D_b = \tfrac{1}{2}(4\pi - D_a), \tag{1-5.5}$$

where D_a is the demagnetization factor for a and D_b along b.

In terms of the eccentricity $\epsilon^2 = 1 - (b/a)^2$

$$D_a = 4\pi \frac{(1-\epsilon^2)}{\epsilon^2}\left(\frac{1}{2\epsilon} \ln \frac{1+\epsilon}{1-\epsilon}\right) - 1. \tag{1-5.6}$$

For the oblate spheroid ($m < 1$)

$$D_a = \frac{4\pi}{1-m^2}\left[1 - \frac{m}{(1-m^2)^{1/2}} \cos^{-1} m\right] \tag{1-5.7}$$

or

$$D_a = \frac{4\pi}{\epsilon^2}\left[1 - \frac{(1-\epsilon^2)^{1/2}}{\epsilon} \sin^{-1}\epsilon\right]. \tag{1-5.8}$$

[5] W. Thomson and P. G. Tait, *Treatise on Natural Philosophy*, 2nd ed., Vol. i, Part ii, Cambridge University Press, Cambridge (1883).