Contents

Introduction

Many important and significant problems in engineering, the physical sciences, and the social sciences, when formulated in mathematical terms, require the determination of a function satisfying an equation containing derivatives of the unknown function. Such equations are called ***differential equations***. Perhaps the most familiar example is Newton's law

$$m\frac{d^2u(t)}{dt^2} = F\left[t, u(t), \frac{du(t)}{dt}\right] \tag{1}$$

for the position $u(t)$ of a particle acted on by a force F, which may be a function of time t, the position $u(t)$, and the velocity $du(t)/dt$. To determine the motion of a particle acted on by a given force F it is necessary to find a function u satisfying Eq. (1). If the force is that due to gravity, then

$$m\frac{d^2u(t)}{dt^2} = -mg. \tag{2}$$

On integrating Eq. (2) we have

$$\frac{du(t)}{dt} = -gt + c_1,$$

$$u(t) = -\tfrac{1}{2}gt^2 + c_1t + c_2, \tag{3}$$

where c_1 and c_2 are constants. To determine $u(t)$ completely it is necessary to specify two additional conditions, such as the position and velocity of the particle at some instant of time. These conditions can be used to determine the constants c_1 and c_2.

In developing the theory of differential equations in a systematic manner it is helpful to classify different types of equations. One of the more obvious classifications is based on whether the unknown function depends on a single independent variable or on several independent variables. In the first case only ordinary derivatives appear in the differential equation and it is said to be an *ordinary differential equation*. In the second case the derivatives are partial derivatives and the equation is called a *partial differential equation*.

Two examples of ordinary differential equations, in addition to Eq. (1), are

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t), \tag{4}$$

for the charge $Q(t)$ on a condenser in a circuit with capacitance C, resistance R, inductance L, and impressed voltage $E(t)$; and the equation governing the decay with time of an amount $R(t)$ of a radioactive substance, such as radium,

$$\frac{dR(t)}{dt} = -kR(t), \tag{5}$$

where k is a known constant. Typical examples of partial differential equations are Laplace's (1749–1827) or the potential equation

$$\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0, \tag{6}$$

the diffusion or heat equation

$$\alpha^2 \frac{\partial^2 u(x, t)}{\partial x^2} = \frac{\partial u(x, t)}{\partial t}, \tag{7}$$

and the wave equation

$$a^2\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{\partial^2 u(x, t)}{\partial t^2}. \tag{8}$$

Here α^2 and a^2 are certain constants. The potential equation, the diffusion equation, and the wave equation arise in a variety of problems in the fields of electricity and magnetism, elasticity, and fluid mechanics. Each is typical of distinct physical phenomena (note the names), and each is representative of a large class of partial differential equations. While we will primarily be concerned with ordinary differential equations we will also consider partial differential equations, in particular the three important equations just mentioned.

1.1 ORDINARY DIFFERENTIAL EQUATIONS

The *order* of an ordinary differential equation is the order of the highest derivative that appears in the equation. Thus Eqs. (1) and (4) of the previous section are second order ordinary differential equations, and Eq. (5) is a first order ordinary differential equation. More generally, the equation

$$F[x, u(x), u'(x), \ldots, u^{(n)}(x)] = 0 \tag{1}$$

is an ordinary differential equation of the nth order. Equation (1) represents a relation between the independent variable x and the values of the function u and its first n derivatives $u', u'', \ldots, u^{(n)}$. It is convenient and follows the usual notation in the theory of differential equations to write y for $u(x)$, with $y', y'', \ldots, y^{(n)}$ standing for $u'(x), u''(x), \ldots, u^{(n)}(x)$. Thus Eq. (1) is written as

$$F(x, y, y', \ldots, y^{(n)}) = 0. \tag{2}$$

Occasionally, other letters will be used instead of y; the meaning will be clear from the context.

We shall assume that it is always possible to solve a given ordinary differential equation for the highest derivative, obtaining

$$y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)}). \tag{3}$$

We will only study equations of the form (3). This is mainly to avoid the ambiguity that may arise because a single equation of the form (2) may correspond to several equations of the form (3). For example, the equation

$$y'^2 + xy' + 4y = 0$$

leads to the two equations

$$y' = \frac{-x + \sqrt{x^2 - 16y^2}}{2} \qquad \text{or} \qquad y' = \frac{-x - \sqrt{x^2 - 16y^2}}{2}.$$

The fact that we have written Eq. (3) does not necessarily mean that there is a function $y = \phi(x)$ which satisfies it. Indeed this is one of the questions that we wish to investigate. By a *solution* of the ordinary differential equation (3) on the interval $\alpha < x < \beta$ we mean a function ϕ such that $\phi', \phi'', \ldots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(x) = f[x, \phi(x), \phi'(x), \ldots, \phi^{(n-1)}(x)] \tag{4}$$

for every x in $\alpha < x < \beta$. Unless stated otherwise, we shall assume that the function f of Eq. (3) is a real-valued function, and we will be interested in obtaining real-valued solutions $y = \phi(x)$.

It is easily verified that the first order equation

$$\frac{dR}{dt} = -kR \tag{5}$$

has the solution

$$R = \phi(t) = ce^{-kt}, \qquad -\infty < t < \infty, \tag{6}$$

where c is an arbitrary constant. Similarly the functions $y_1(x) = \cos x$ and $y_2(x) = \sin x$ are solutions of

$$y'' + y = 0 \tag{7}$$

for all x.

One question that might come to mind is whether there are other solutions of Eq. (5) besides those given by Eq. (6), and whether there are other solutions of Eq. (7) besides $y_1(x) = \cos x$ and $y_2(x) = \sin x$. A question that might occur even earlier is the following: Given an equation of the form (3), how do we know whether it even has a solution? This is the question of the *existence* of a solution. Not all differential equations have solutions; nor is the question of existence purely mathematical. If a meaningful physical problem is correctly formulated mathematically as a differential equation, then the mathematical problem should have a solution. In this sense an engineer or scientist has some check upon the validity of his mathematical formulation.

Second, assuming a given equation has one solution, does it have other solutions? If so, what type of additional conditions must be specified in order to single out a particular solution? This is the question of *uniqueness*. Notice that there is an infinity of solutions of the first order equation (5) corresponding to the infinity of possible choices of the constant c in Eq. (6). If R is specified at some time t, this condition will determine a value for c; even so, however, we do not know yet that there may not be other solutions of Eq. (5) which also have the prescribed value of R at the prescribed time t. The questions of existence and uniqueness are difficult questions; they and related questions will be discussed as we proceed.

A third question, a more practical one, is: Given a differential equation of the form (3), how do we actually determine a solution? We might note that if we find a solution of the given equation we have at the same time answered the question of the existence of a solution. On the other hand, without knowledge of existence theory we might, for example, use a large computing machine to find an approximation to a "solution" that does not exist. Even though we may know that a solution exists, it may be that the solution is not expressible in terms of the usual elementary functions—polynomial, trigonometric, exponential, logarithmic, and hyperbolic functions. Unfortunately this is the situation for most differential equations. However, before we can consider difficult problems it is first necessary to master some of the elementary theory of ordinary differential equations.

Linear and Nonlinear Equations. A second important classification of ordinary differential equations is according to whether they are linear or nonlinear. The differential equation

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

is said to be *linear* if F is a linear function of the variables $y, y', \ldots, y^{(n)}$. Thus the general linear ordinary differential equation of order n is

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = g(x). \tag{8}$$

Equations (2), (4), and (5) of the previous section are linear ordinary differential equations. An equation which is not of the form (8) is a *nonlinear* equation. For example, the angle θ that an oscillating pendulum of length l

makes with the vertical direction (see Figure 1.1) satisfies the nonlinear equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0. \qquad (9)$$

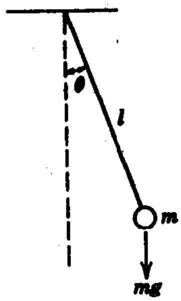

FIGURE 1.1

The mathematical theory and the techniques for solving linear equations are highly developed. In contrast, for nonlinear equations the situation is not as satisfactory. General techniques for solving nonlinear equations are largely lacking, and the theory associated with such equations is also more complicated than the theory for linear equations. In view of this it is fortunate that many significant problems lead to linear ordinary differential equations or, at least, in the first approximation to linear equations. For example, for the pendulum problem, if the angle θ is small, then $\sin \theta \cong \theta$ and Eq. (9) can be replaced by the linear equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \theta = 0.$$

On the other hand, there are important physical phenomena, such as the current flow in an electron tube, in which it is not possible to approximate the governing nonlinear differential equation by a linear one—the nonlinearity is crucial.

In an elementary text it is natural to emphasize the discussion of linear equations. The greater part of this oook is therefore devoted to linear equations and to various methods for solving them. However, Chapters 8 and 9, as well as a large part of Chapter 2, are concerned with nonlinear equations. Throughout the text we attempt to show why nonlinear equations are, in general, more difficult, and why many of the techniques that are useful in solving linear equations cannot be applied to nonlinear equations.

1.2 HISTORICAL REMARKS

Without knowing something about differential equations and methods of solving them it is difficult to discuss the history and development of this important branch of mathematics. Further, the development of the theory of differential equations is intimately interwoven with the general development of mathematics and cannot be divorced from it. In these brief comments we will just mention a few, certainly not all, of the famous mathematicians of the seventeenth and eighteenth centuries who made important contributions in this area. We will follow closely the brief historical survey given by Ince* [Appendix A] in his authoritative treatise on ordinary differential equations, and the discussion by Cajori.

The theory of differential equations dates back to the beginnings of the calculus with Newton (1642–1727) and Leibniz (1646–1716) in the seventeenth century. Indeed Ince states,

> Yet our hazy knowledge of the birth and infancy of the science of differential equations condenses upon a remarkable date, the eleventh day of November, 1675, when Leibniz first set down on paper the equation
>
> $$\int y\,dy = \tfrac{1}{2}y^2,$$
>
> thereby not merely solving a simple differential equation, which was in itself a trivial matter, but what was an act of great moment, forging a powerful tool, the integral sign.

Following Newton and Leibniz come the names of the Bernoulli brothers Jakob (1654–1705) and Johann (1667–1748) and Johann's son Daniel (1700–1782). These are just three of eight members of the Bernoulli family who were prominent scientists and mathematicians in their time. With the aid of the calculus, they formulated as differential equations and solved a number of problems in mechanics, including that of the determination of the curve of most rapid descent for the motion of a particle under the influence of gravity. It was Daniel whose name is associated with the famous Bernoulli equation in fluid mechanics. In 1690 Jakob Bernoulli published the solution of the differential equation, written in differential form, $(b^2y^2 - a^3)^{1/2}\,dy = (a^3)^{1/2}\,dx$. Today this is a simple exercise but at that time to go from the equation $y' = [a^3/(b^2y^2 - a^3)]^{1/2}$ to the differential form, and then to assert that the integrals† of each side must be equal except for a constant was a major step. Indeed, for example, while Johann Bernoulli knew that $ax^p\,dx = d[ax^{p+1}/(p+1)]$ was not meaningful for $p = -1$, he did not know that $dx/x = d(\ln x)$. Nevertheless he was able to show that the equation $dy/dx = y/ax$, which we could solve by writing it as

$$a\frac{dy}{y} = \frac{dx}{x},$$

has the solution $y^a/x = c$ where c is a constant of integration. See Section 2.4.

By the end of the seventeenth century most of the elementary methods of solving first order ordinary differential equations (Chapter 2) were known, and attention was centered on higher order ordinary differential equations and partial differential equations. Riccati (1676–1754), an Italian mathematician, considered equations of the form $f(y, y', y'') = 0$ (Section 3.1). He

* References are listed at the end of each chapter.

† Jakob Bernoulli appears to be the first person to have used the word integral.

also considered the nonlinear equation known as the Riccati equation, $dy/dx = a_0(x) + a_1(x)y + a_2(x)y^2$, though not in such a general form.

Euler,* one of the greatest mathematicians of all time, also lived during the eighteenth century. Of particular interest here is his work on the formulation of problems in mechanics in mathematical language, and his development of methods of solving these mathematical problems. Lagrange (1736–1813) said of Euler's work in mechanics, "The first great work in which analysis is applied to the science of movement." Euler also considered such questions as the possibility of reducing second order equations to first order equations by a suitable change of variables; he introduced the concept of an integrating factor (Section 2.6), and he gave a general treatment of linear ordinary differential equations with constant coefficients (Chapters 3 and 5) in 1739. Later in the eighteenth century the great French mathematicians Lagrange and Laplace made important contributions to the theory of ordinary differential equations and gave the first scientific treatment of partial differential equations. Indeed, possibly the most famous partial differential equation in mathematical physics, $u_{xx} + u_{yy} = 0$, where subscripts denote partial derivatives, is known as Laplace's equation. The student who is interested in the history of the theory of differential equations might wish to refer to one of the many books† dealing with the development of mathematics.

In more recent years part of the effort of mathematicians in the areas of ordinary and partial differential equations has been to develop a rigorous, systematic (but general) theory. The goal is not so much to construct solutions of particular differential equations, but rather to develop techniques suitable for treating classes of equations.

PROBLEMS

1. For each of the following differential equations determine its order and whether or not the equation is linear.

(a) $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 2y = \sin x$

(b) $(1 + y^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = e^x$

(c) $\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 1$

(d) $\dfrac{dy}{dx} + xy^2 = 0$

(e) $\dfrac{d^2y}{dx^2} + \sin(x + y) = \sin x$

(f) $\dfrac{d^3y}{dx^3} + x\dfrac{dy}{dx} + (\cos^2 x)y = x^3$

* Euler (1707–1783) was a prolific mathematician. His collected works fill over sixty volumes. Even though blind during the last seventeen years of his life his work continued undiminished.

† The books by Ince and Cajori have already been mentioned. See also Bell and Struik.

2. Verify for each of the following that the given function or functions are solutions of the differential equation.

(a) $y'' - y = 0;\ y_1(x) = e^x, y_2(x) = \cosh x$

(b) $y'' + 2y' - 3y = 0;\ y_1(x) = e^{-3x},\ y_2(x) = e^x$

(c) $y'''' + 4y''' + 3y = x;\ y_1(x) = x/3, y_2(x) = e^{-x} + x/3$

(d) $2x^2y'' + 3xy' - y = 0, x > 0;\ y_1(x) = x^{1/2}, y_2(x) = x^{-1}$

(e) $x^2y'' + 5xy' + 4y = 0, x > 0;\ y_1(x) = x^{-2}, y_2(x) = x^{-2}\ln x$

(f) $y'' + y = \sec x, 0 < x < \pi/2;\ y = \phi(x) = (\cos x)\ln\cos x + x\sin x$

(g) $y' - 2xy = 1;\ y = \phi(x) = e^{x^2}\int_0^x e^{-t^2}\,dt + e^{x^2}$

3. Determine for what values of r each of the following linear differential equations has solutions of the form $y = e^{rx}$.

(a) $y' + 2y = 0$

(b) $y'' - y = 0$

(c) $y'' + y' - 6y = 0$

(d) $y''' - 3y'' + 2y' = 0$

4. Determine for what values of r each of the following linear differential equations has solutions of the form $y = x^r$.

(a) $x^2y'' + 4xy' + 2y = 0, \quad x > 0$

(b) $x^2y'' - 4xy' + 4y = 0, \quad x > 0$

5. The *order* of a partial differential equation is defined as the order of the highest partial derivative that appears in the equation. Similarly the equation is said to be *linear* if the equation is linear in the dependent function and its derivatives. For each of the following equations determine the order and whether or not the equation is linear.

(a) $u_{xx} + u_{yy} + u_{zz} = 0$

(b) $\alpha^2u_{xx} = u_t$

(c) $a^2u_{xx} = u_{tt}$

(d) $u_{xx} + u_{yy} + uu_x + uu_y + u = 0$

(e) $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$

(f) $u_t + uu_x = 1 + u_{xx}$

6. Verify for each of the following differential equations that the given function or functions are solutions.

(a) $u_{xx} + u_{yy} = 0; \quad u_1(x, y) = x^2 - y^2, \quad u_2(x, y) = \cos x\cosh y$

(b) $\alpha^2u_{xx} = u_t; \quad u_1(x, t) = e^{-\alpha^2t}\sin x,$
$u_2(x, t) = e^{-\alpha^2\lambda^2t}\sin\lambda x, \quad \lambda$ a real constant

(c) $a^2u_{xx} = u_{tt}; \quad u_1(x, t) = \sin\lambda x\sin\lambda at, \quad \lambda$ a real constant,
$u_2(x, t) = \sin(x - at)$

7. Verify for each of the following differential equations that the given function is a solution.

(a) $u_{xx} + u_{yy} + u_{zz} = 0$; $u = \phi(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$, $(x, y, z) \neq (0, 0, 0)$

(b) $\alpha^2 u_{xx} = u_t$; $u = \phi(x, t) = (\pi/t)^{1/2} e^{-x^2/4\alpha^2 t}$, $t > 0$

(c) $a^2 u_{xx} = u_{tt}$; $u = f(x - at) + g(x + at)$, where f and g are twice differentiable functions

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CHAPTER 2

First Order Differential Equations

2.1 LINEAR EQUATIONS

This chapter deals with differential equations of first order, that is, equations of the form

$$y' = f(x, y), \tag{1}$$

where f is a given function of two variables. Any function $y = \phi(x)$, which with its derivative y' identically satisfies Eq. (1), is called a solution, and our object is to try to determine whether such functions exist and, if so, how to find them. In order to gain some familiarity with differential equations and their solutions, we will first consider the linear first order equation

$$y' + p(x)y = g(x), \tag{2}$$

where p and g are given continuous functions on some interval $\alpha < x < \beta$. In this section we will be concerned with methods for solving Eq. (2). More theoretical questions involving the existence and uniqueness of solutions in general will be discussed in Section 2.2.

Let us begin with the equation

$$y' + ay = 0, \tag{3}$$

where a is a real constant. This equation can be solved by inspection. What function has a derivative which is a multiple of the original function? Clearly $y = e^{-ax}$ satisfies Eq. (3); furthermore

$$y = ce^{-ax}, \tag{4}$$

where c is an arbitrary constant, also does so. Since c is arbitrary, Eq. (4) represents infinitely many solutions of the differential equation (3). It is natural to ask whether Eq. (3) has any solutions other than those given by Eq. (4). We will show in the next section that there are no other solutions, but for the time being this question remains open.

Geometrically, Eq. (4) represents a one-parameter family of curves, called *integral curves* of Eq. (3). For $a = 1$ several members of this family

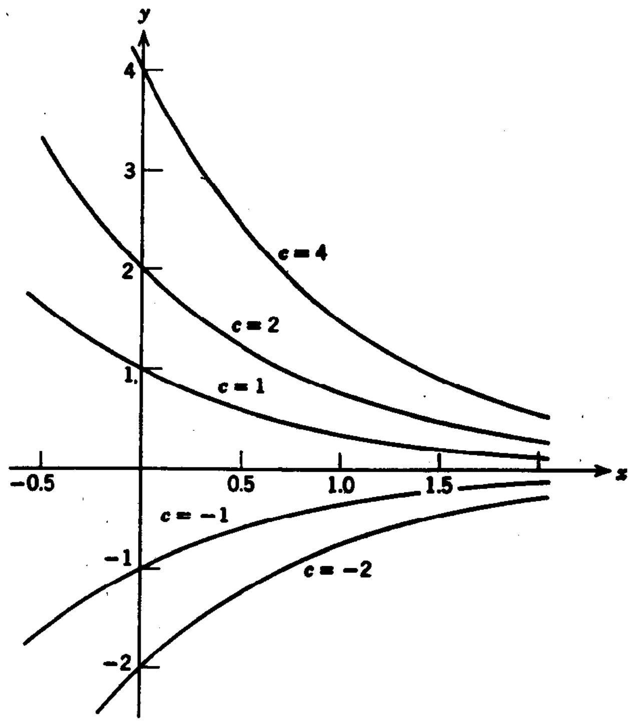

FIGURE 2.1

are sketched in Figure 2.1. Each integral curve is the geometric representation of the corresponding solution of the differential equation. Specifying a particular solution is equivalent to picking out a particular integral curve from the one-parameter family. It is usually convenient to do this by prescribing a point (x_0, y_0) through which the integral curve must pass; that is, we seek a solution $y = \phi(x)$ such that

$$\phi(x_0) = y_0.$$

Such a condition is called an *initial condition.* Since y stands for $\phi(x)$ we could also write

$$y = y_0 \quad \text{at} \quad x = x_0.$$

However, it is common practice to express the initial condition in the form

$$y(x_0) = y_0, \tag{5}$$

and this is the notation we will usually use in this book. A first order differential equation together with an initial condition form an *initial value problem.**

* This terminology is suggested by the fact that the independent variable often denotes time, the initial condition defines the situation at some fixed instant, and the solution of the initial value problem describes what happens later.

For example, the differential equation (3),

$$y' + ay = 0,$$

and the initial condition

$$y(0) = 2, \tag{6}$$

form an initial value problem. As noted above, all solutions of the differential equation (3) are given by Eq. (4). The particular solution satisfying the initial condition (6) is found by substituting $x = 0$ and $y = 2$ in Eq. (4). Then c equals 2 and the desired solution is the function

$$y = \phi(x) = 2e^{-ax}. \tag{7}$$

This is the unique solution of the given initial value problem. More generally, it is possible to show that the initial value problem composed of the differential equation (2) and the initial condition (5) will have a unique solution whenever the coefficients p and g are continuous functions. This is discussed in the next section.

In order to develop a systematic method for solving first order linear equations it is convenient first to work backward. Thus we rewrite the solution (4) of the differential equation (3) in the form

$$ye^{ax} = c. \tag{8}$$

On differentiating the left side of Eq. (8) we obtain

$$(ye^{ax})' = y'e^{ax} + aye^{ax} = e^{ax}(y' + ay), \tag{9}$$

and hence Eq. (8) implies that

$$e^{ax}(y' + ay) = 0. \tag{10}$$

Cancellation of the positive factor e^{ax} yields the differential equation (3). It is important to note that the solution of Eq. (3) can be constructed by reversing the above process, that is, we multiply Eq. (3) by e^{ax}, obtaining Eq. (10), from which Eq. (8) follows by using Eq. (9). Finally, solving Eq. (8) for y gives Eq. (4).

The same procedure can be used to solve the more general equation

$$y' + ay = g(x). \tag{11}$$

Multiplying by e^{ax} gives

$$e^{ax}(y' + ay) = e^{ax}g(x),$$

or, using Eq. (9),

$$(ye^{ax})' = e^{ax}g(x).$$

Hence

$$ye^{ax} = \int^{x} e^{at}g(t)\,dt + c,$$

where c is an arbitrary constant. Hence a solution of Eq. (11) is the function

$$y = \phi(x) = e^{-ax}\int^x e^{at}g(t)\,dt + ce^{-ax}. \tag{12}$$

In Eq. (12) and elsewhere in this book we use the notation $\int^x f(t)\,dt$ to denote an antiderivative of the function f, that is, $F(x) = \int^x f(t)\,dt$ designates some particular representative of the class of functions whose derivatives are equal to f. All members of this class are included in the expression $F(x) + c$, where c is arbitrary.

Thus, for a given function g the problem of determining a solution of Eq. (11) is reduced to that of evaluating the antiderivative in Eq. (12). The difficulty involved in this depends on g; nevertheless Eq. (12) gives an explicit formula for the solution $y = \phi(x)$. The constant c can be determined if an initial condition is prescribed.

Now let us turn to the general first order linear equation (2),

$$y' + p(x)y = g(x).$$

By analogy with the foregoing process we would like to choose a function μ so that if Eq. (2) is multiplied by $\mu(x)$, the left-hand side of Eq. (2) becomes the derivative* of $\mu(x)y$. That is, we want to choose μ, if possible, so that

$$\begin{aligned}\mu(x)[y' + p(x)y] &= [\mu(x)y]' \\ &= \mu(x)y' + \mu'(x)y.\end{aligned}$$

Thus $\mu(x)$ must satisfy

$$p(x)y\mu(x) = y\mu'(x).$$

Assuming for the moment that $\mu(x) > 0$, we obtain†

$$\frac{\mu'(x)}{\mu(x)} = [\ln \mu(x)]' = p(x). \tag{13}$$

Hence

$$\ln \mu(x) = \int^x p(t)\,dt,$$

and finally

$$\mu(x) = \exp\left[\int^x p(t)\,dt\right]. \tag{14}$$

* A function μ having this property is called an integrating factor. Integrating factors are discussed more fully in Section 2.6.

† Recall that $\int^x \frac{dt}{t} = \ln|x|$.