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Qinglin Roger Yu
Guizhen Liu

Graph Factors and Matching Extensions

图的因子和匹配可扩性

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Preface

Graph theory is one of the branches of modern mathematics which has shown impressive advances in recent years. An explosive growth of graph theory is witnessed due to its essential roles providing structural models and indispensable tools in computer science, communication networks and combinatorial optimization problems.

Graph theory has matured mathematically as indicated by an increasing number of deeper results, such as the Regularity Lemma, Hall's Theorem, Graph Minor Theorem, which have been successfully applied to many branches of mathematics. In the meantime, graph theory has grown stronger by introducing techniques from other branches of mathematics (e.g., probabilistic method, linear algebra, group theory, and topology).

Matching theory, or more generally factor theory, is one of the fundamental areas in graph theory. It studies the structures and properties of matchings and factors — the simplest nontrivial substructures of graphs. Matchings and factors have many applications in other areas of graph theory, and techniques from factor theory such as the alternating path and decomposition procedure, are used in systematic approaches to combinatorial problems. Matching and factor theory is also one of earliest topics to be studied in graph theory.

Since the appearance of the classic work *Matching Theory* by Lovász and Plummer in 1986, factor theory has flourished over the last two decades, and much new and interesting progress has been made in graph factors and matching extension theory. However, these new results are not well summarized in traditional graph theory textbooks. This book is intended to serve as a collection of recent results in this interesting and very active field, which could serve as a reference manual for researchers or an introduction for young graph theorists or graduate students. To this end, sections on more advanced topics are included, and a number of interesting and challenging open problems and conjectures are presented at the end of each chapter.

For two reasons, we do not intend to cover all aspects of factor theory. One reason is that the branch has progressed so much that we are unable to include all the old and recent developments in it; the second reason is that some topics are already covered elegantly in earlier books, e.g., Lovász and Plummer's book on matching theory. Our book concentrates mainly on the theoretical aspects of factor theory, not including the algorithmic aspects. Moreover, due to space limitation, some interesting and closely related topics were left out (e.g., factor factorizations, factors in random graphs, $(1, f)$-odd-factor, subgraph packing problems, etc.), which can be found in the original literature. We give priority to the topics which are still very active but lack of exposures, such as component factors, connected factors, matching extension, optimal brick decompositions, fractional factors and L-factors.

This book is based on lecture notes written for summer graduate schools at Nankai University and Shandong University in 2005. The selection of the material was of course heavily influenced by our personal interests, as well as the limitations of space, while trying to cope with the recent development in the areas not covered by any known book. The book is primarily aimed at researchers and graduate students in graph theory. However, most of the material discussed is accessible to anyone with an undergraduate level understanding of mathematics. Our main source of materials is from three sources: research articles, textbooks and survey papers. The complete list of articles is given at the end of the book. The textbooks include:

1. J. Akiyama and M. Kano, *Factors and Factorizations of Graphs*, Version 1.0, June 2007 (in press).
2. J. A. Bondy and U. S. R. Murty, *Graph Theory*, Graduate Texts in Mathematics 244, Springer, 2008.
3. L. Lovász and M. D. Plummer, *Matching Theory*, North-Holland Inc., Amsterdam, 1986.

The survey papers include:

1. J. Akiyama and M. Kano, Factors and factorizations of graphs – a survey, *J. Graph Theory* 9 (1985) 1–42.
2. M. Kouider and P. D. Vestergaard, Connected factors in graphs – a survey, *Graphs Combin.* 21 (2005) 1–26.
3. M. D. Plummer, Extending matchings in graphs – a survey, *Discrete Math.* 127 (1994) 277–292.
4. M. D. Plummer, Extending matchings in graphs – an update, *Congressus Numerantium* 116 (1996) 3–32.
5. M. D. Plummer, Graph factors and factorization, *Handbook on Graph Theory*, Eds.: J. Gross and R. Yellen, CRC Press, New York, 2003 403–430.

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We welcome any feedback regarding this book, corrections, and comments. Feel free to reach us by e-mail at `yu@tru.ca` (Yu) or `gzliu@sdu.edu.cn` (Liu). We will maintain a web site to provide an update list of errata and corrections. The Web page is

`http://www.tru.ca/faculty/yu/factor_book.html`

Thanks also to Mr. Tianfu Zhao, our editor at Higher Education Press for his expertise, dedication and encouragement. Our gratitude also goes to the editors at Springer for their professional assistants.

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Jinan,
May 2009

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Contents

Chapter 1
Matchings and Perfect Matchings

As a young branch of mathematics, Graph Theory has experienced the explosion growth as the same phenomenon that has been taking place in computing science and communication networking. In the mean time, there are many new terminologies and knowledge accumulated in the process. So there are often more than one names or notions defined for a same entity. We list the terms and notions frequently used in this book and hope to provide readers with a consistent reference.

Notions

Let G be a graph, with vertex set $V(G)$ and edge set $E(G)$. Let f be a positive integer-valued function defined on vertex set $V(G)$.

$|S|$: order of a set S;
$\lfloor x \rfloor$: the largest integer not greater than x;
$\lceil x \rceil$: the smallest integer not less than x;
$V(G)$: the vertex set of G, sometimes we use $|G|$ for the order of G, i.e., $|G| = |V(G)|$;
$E(G)$: the edge set of G, occasionally we write $||G||$ for the size of G, i.e., $||G|| = |E(G)|$;
$A \triangle B$: symmetric difference of two sets A and B, i.e., $A \triangle B = (A - B) \cup (B - A)$;
$d_G(v)$: the degree of a vertex v in G;
$\delta(G)$: the minimum degree of G;
$\Delta(G)$: the maximum degree of G;
$\nabla(v)$: the edges incident with the vertex v;
$\nabla(S)$: the set of edges with exactly one end-vertex in the set S;
$N_G(S)$: the set of vertices adjacent to a vertex of S, i.e., $N_G(S) = \{ x \mid xy \in E(G) \text{ and } y \in S\}$, or in short $N(S)$ if no confusion is arisen;
$E_G(S,T)$: the edges with one end in S and another end in T and $e_G(S,T) = |E_G(S,T)|$;

$\sigma_k(G)$: degree sum of k independent vertices;
$\mu(G)$: the matching number of G;
$\mu_f(G)$: the fractional matching number of G;
$\eta(G)$: the maximum cardinality of joins in G;
$\xi(\Sigma)$: the smallest integer m such that no graph G embeddable in surface Σ is m-extendable;
$ext(G)$: the extendability number of a graph G;
$b(G)$: the number of bricks of G in a decomposition;
$\hat{c}(G)$: the number of critical $\{K_2, K_3\}$-components in G;
$p(G)$: the number of Petersen subgraphs;
$i\mu(G)$: the induced matching number;
G^c: the complement of G;
$G[S]$: the induced subgraph of G by a vertex set S;
$o(G)$: the number of odd components of G;
$\omega(G)$: the number of components of G;
$c(G)$: the circumference of G;
$\beta^*(G)$: the vertex covering number of G;
$I(G)$: the set of isolated vertices in G;
$i(G)$: the number of isolated vertices in G;
$D(G)$: the set of all vertices in G which are not saturated by at least one maximum matching of G;
$A(G)$: the set of vertices in $V(G) - D(G)$ adjacent to at least one vertex in $D(G)$;
$C(G) = V(G) - D(G) - A(G)$;
$fc(G)$: the number of factor-critical components in G;
$tc(G)$: the number of triangle clusters in G;
$oc(G)$: the number of those components of G that are odd cacti;
$\Phi(G)$: the number of 1-factors in G;
G_δ: the subgraph induced by the vertices with minimum degree in G;
G_Δ: the subgraph induced by the vertices with maximum degree in G;
$\kappa(G)$: the vertex connectivity of G;
$\kappa'(G)$: the edge connectivity of G;
$\lambda(C)$: the characteristic of a cut C;
$c\lambda(G)$: the cyclic connectivity of G;
$\alpha(G)$: the independent number of G;
$\gamma(G)$: the dominating number;
$\gamma^*(G)$: the genus of G;
$\chi'(G)$: the edge chromatic number of G;
$\chi'_c(G)$: the edge-cover number of G;
$\tau_2(G)$: the size of a minimum 2-covering of G;
$G \vee H$: the join of two graphs G and H;
G^r: the rth power of a graph G;
$t(G)$: the toughness of G introduced by Chvátal [151],

$$t(G) = \min\left\{\frac{|S|}{\omega(G-S)} \mid S \subseteq V(G),\ \omega(G-S) \geq 2\right\};$$

$\tau(G)$: a variation of toughness introduced by Enomoto [184],

$$\tau(G) = \min\left\{\frac{|S|}{\omega(G-S)-1} \mid S \subseteq V(G),\ \omega(G-S) \geq 2\right\}.$$

We define, for all disjoint subsets S and T of $V(G)$,

$$\delta_G(S,T) = \begin{cases} k|S| + \sum_{x\in T} d_G(x) - k|T| - e_G(S,T) - q_G(S,T) & \text{for } k\text{-factor;} \\ f(S) + \sum_{x\in T} d_G(x) - f(T) - e_G(S,T) - q_G(S,T) & \text{for } f\text{-factor;} \\ f(S) + \sum_{x\in T} d_G(x) - g(T) - e_G(S,T) - q_G(S,T) & \text{for } (g,f)\text{-factor,} \end{cases}$$

where $q_G(S,T)$ is defined for k-factors, f-factors and (g,f)-factors in Theorems 2.1.1, 2.1.2 and 2.1.3, respectively.

1.1 Definitions and terminologies

To understand the developments of matchings and factors in graphs, we would like to review the history briefly by listing the most significant milestones in this fascinating area of graph theory.

The first attempt on the study of factors was made by Danish mathematician Petersen (1891), who proved that every graph of even degrees can be decomposed into the union of edge-disjoint 2-factors. This was motivated from the study of an algebraic factorization problem. He also showed that every 2-connected cubic graph has a 1-factor. These two results can be viewed as a forerunner of modern graph factor theory.

For matchings in bipartite graphs, König (1931) and Hall (1935) obtained the so-called König-Hall Theorem (sometimes, known as Hall's Theorem). Due to its wide applications in many graph theory problems and other branches of mathematics, König-Hall Theorem remains one of most influential graph-theoretic results.

In 1947, Tutte gave a characterization (i.e., so-called Tutte's 1-Factor Theorem) for the existence of perfect matchings in arbitrary graphs and it has become a cornerstone of factor theory. Till now, this elegant theorem is still one of the most fundamental results in factor theory. Subsequently, Tutte (1952) extended the techniques in the proof of 1-Factor Theorem to obtain a sufficient and necessary condition for a graph to have an f-factor.

Gallai (1964) and Edmonds (1965), independently, investigated a canonical decomposition of arbitrary graphs in terms of its maximum matchings and thus revealed the structure of graphs related to matchings. This deep and important result is referred as Gallai-Edmonds Structure Theorem. The existence of the canonical decomposition and an efficient method to find such a decomposition allowed Edmonds to obtain the first polynomial algorithm for finding maximum matchings in graphs.

The most general degree-constrained factors, (g, f)-factors, were studied by Lovász (1970). He gave necessary and sufficient conditions for a graph to have a (g, f)-factor. This theorem generalized the criteria of all other factors, such as 1-factors, k-factors, f-factors and $[a, b]$-factors.

For the more comprehensive account of history on matching theory and graph factors, readers can refer to Preface of Lovász and Plummer's *Matching Theory* or Biggs, Lloyd and Wilson's *Graph Theory 1736-1936.*

In this book, we mainly deal with factors in *finite undirected simple* graphs. Some results also hold for graphs with multiple edges, which we point out accordingly.

The *degree* of a vertex v in a graph G, denoted by $d_G(v)$, is the number of edges of G incident with v, each loop counting as two edges. In particular, if G is a simple graph, $d_G(v)$ is the number of neighbors of v in G. A vertex of degree zero or one are called an *isolated vertex* and a *leaf,* respectively. A *tree* is a connected graph containing no cycles. Thus every tree has at least two leaves. It is well-known that every connected graph G contains a tree. For a tree T in a connected graph, we always assume that T is a *spanning* connected subgraph without cycle.

We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of the vertices of G.

A *cut-edge* of a graph is one whose deletion results in one more component. Sometimes, a cut-edge is also called a *bridge.* Clearly, every edge of a tree is a cut-edge.

A *matching* M of a graph G is a subset of $E(G)$ such that any two edges of M have no end-vertices in common. A matching of k edges is called a *k-matching.* Let d be a non-negative integer. A matching is called a *defect-d matching* if it covers exactly $|V(G)| - d$ vertices of G. A defect-0 matching is called a *perfect matching* and defect-1 matching is called *near-perfect matching.*

A perfect matching is also referred as *1-factor* since it is a 1-regular subgraph. If a graph has 1-factors, it is called *1-factorable.* There are probably equal numbers of people using "perfect matchings" rather than "1-factors", so we will use them non-discriminatively.

If an edge e is contained in a 1-factor, we called e an *allowed edge.* If an edge f does not lie in any 1-factor, then f is *forbidden.*

A *covering* of a graph G is a subset S of $V(G)$ such that every edge of G has at least one end in S. A covering S^* is a *minimum covering* if G has no covering S with $|S| < |S^*|$. The order of a minimum covering is called *covering number* and denoted by $\beta^*(G)$.

A graph G is said to be *bicritical* if for every pair of distinct vertices u and v, $G - \{u, v\}$ has a 1-factor. A graph G is *factor-critical* if $G - u$ has a 1-factor for every $u \in V(G)$ (see Fig. 1.1).

(a) (b)

Fig. 1.1 (a) a factor-critical graph and (b) a bicritical graph.

A graph G is called *n-factor-critical* if the subgraph $G - S$ has a 1-factor for all n-subset S of $V(G)$.

Clearly, the concept of n-factor-criticality is a generalization of bicriticality and factor-criticality. That is, factor-critical graphs and bicritical graphs are n-factor-critical graph when $n = 1$ and 2, respectively.

From the definitions, we can see the following result easily.

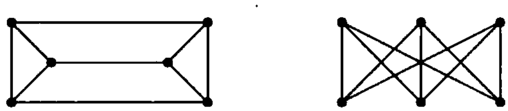
Proposition 1.1.1.

(a) *A graph G is factor-critical if and only if the join of G and K_1, $G \vee K_1$, is bicritical;*

(b) *a graph G is bicritical if and only if $G - v$ is factor-critical for every $v \in V(G)$.*

Let G be a connected graph with a 1-factor and $|V(G)| \geq 2k + 2$. If each k-matching of G is contained in a 1-factor, we call G a *k-extendable* graph. For convenience, a 0-extendable graph means a graph which has a 1-factor.

Clearly bicritical graphs are 1-extendable. A 3-connected bicritical graph is called a *brick*. A *brace* is a 2-extendable bipartite graph (i.e., 3-connected 2-extendable bipartite graph). The examples of a brick and a brace are given in Fig. 1.2.



(a) (b)

Fig. 1.2 (a) a brick and (b) a brace.

Factors in a graph are spanning subgraphs satisfying some given properties. A *k-factor* of a graph G is a spanning subgraph H such that $d_H(x) = k$ for every $x \in V(G)$.

Let f be an integer-valued function defined on $V(G)$, i.e.,

$$f : V(G) \to \mathbb{Z}^+ = \{0, 1, 2, \dots\},$$

then a spanning subgraph H such that $d_H(x) = f(x)$ for all $x \in V(G)$ is called an *f-factor*.

A more general factor is so-called (g, f)-factors: for two functions $g, f : V(G) \to \mathbb{Z}$, a spanning subgraph H is called a *(g, f)-factor* if $g(x) \leq d_H(x) \leq f(x)$ for all $x \in V(H)$ (see an example in Fig. 1.3). In addition, if $g(x) \equiv f(x) \pmod 2$ for all $x \in V(G)$, a (g, f)-factor F with $d_F(x) \equiv f(x)$ is called a *(g, f)-parity-factor*.

For two positive integers a, b, let $g(x) = a$ and $f(x) = b$ for all vertices $x \in V$. Then (g, f)-factors are called *$[a, b]$-factors*.

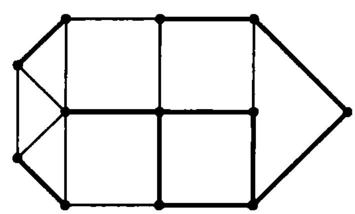


Fig. 1.3 A (g, f)-factor with $g(x) = 1$ and $f(x) = 3$ for all $x \in V$.

A set $S \subset V$ is a *dominating set* of G if every vertex in V is either in S or is adjacent to a vertex in S. If A and B are subsets of V, we say that A *dominates* B if every vertex of B has a neighbor in A or is a vertex of A; we also say that B is *dominated* by A. The *dominating number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set. A set S is *2-dominating* if every vertex of V is either in S or is within distance 2 to S.

We only define the most often used terms of the book in this section. For other undefined terms, we follow Bollobás [83].

1.2 Matchings in bipartite graphs and augmenting path

In this section, we focus on matchings and perfect matchings in bipartite graphs, which is the starting point of matching theory. This is not surprising since matchings in bipartite graphs are closely related to other disciplines of mathematics (e.g., matrix theory and set theory) and many application problems (e.g., the job assignment problem). Furthermore, the properties and techniques of matchings in bipartite graphs also motivate the research of matchings in non-bipartite graphs.

For a matching M, a vertex x of G is called *saturated* or *covered* by M if x is incident to an edge of M. A matching M is called a *maximum matching* if $|M| \geq |M'|$ for any matching M' of G. Clearly, a perfect matching (i.e., a 1-factor) is a maximum matching saturating every vertex of a graph. Of course, any graph with perfect matchings must have even number of vertices.

In many applications, one wishes to find a matching in a bipartite graph $G = (X, Y)$ which covers every vertex in X. Necessary and sufficient conditions for the existence of such a matching were proven by Hall (1935). To this day, Hall's Theorem remains one of the most widely applied graph-theoretic results. The proof here is due to Rado.

Theorem 1.2.1 (Hall's Theorem, Hall (1935)). *Let $G = (X, Y)$ be a bipartite graph. Then G has a matching saturating all vertices of X if and only if for any $S \subseteq X$,*

$$|N(S)| \geq |S|. \tag{1.1}$$

Proof. The necessity is clear.

Let G be a minimal bipartite graph satisfying the condition (1.1). It suffices to show that G consists of $|X|$ independent edges.

Otherwise, G contains two edges, x_1y and x_2y, where $x_1, x_2 \in X$, $x_1 \neq x_2$ and $y \in Y$. From the minimality, the deletion of either of these edges invalidates (1.1). Thus there are two subsets $X_1, X_2 \subset X$ such that for $i = 1, 2$ we have $|N(X_i)| = |X_i|$, and x_i is the only vertex of X_i adjacent to y. Then

$$\begin{aligned}|N(X_1) \cap N(X_2)| &\geq |N(X_1 - \{x_1\}) \cap N(X_2 - \{x_2\})| + 1 \\ &\geq |N(X_1 \cap X_2)| + 1 \geq |X_1 \cap X_2| + 1.\end{aligned}$$

But this implies the following contradiction:

$$\begin{aligned}|N(X_1 \cup X_2)| &= |N(X_1) \cup N(X_2)| \\ &= |N(X_1)| + |N(X_2)| - |N(X_1) \cap N(X_2)| \\ &\leq |X_1| + |X_2| - |X_1 \cap X_2| - 1 \\ &= |X_1 \cup X_2| - 1.\end{aligned}$$

□

Note that Hall's Theorem also holds for multigraphs. The above result is the usual version of matching characterization in bipartite graphs. However, we present the next version in terms of isolated vertices, which has a strong similarity to Tutte's 1-Factor Theorem referred to later. Recall that $i(G)$ denotes the number of isolated vertices in G.

Theorem 1.2.2 (Hall's Matching Theorem, Hall (1935)). *Let $G = (X, Y)$ be a bipartite graph. Then G has a perfect matching if and only if $|X| = |Y|$ and for any $S \subseteq X$,*

$$i(G - S) \leq |S|.$$

Remark 1.2.1. For every bipartite graph $G = (X, Y)$, with $|X| \geq |Y|$ and maximum degree Δ, König (1916) showed that there exists Δ-regular bipartite supergraph $G' = (X, Y')$ (a multigraph is allowed) such that $Y \subseteq Y'$ and G is a subgraph of G'.

Corollary 1.2.1 (König (1916)). *Every r-regular bipartite graph G can be decomposed into r edge-disjoint 1-factors. In particular, G has a 1-factor.*

Corollary 1.2.2. *Every bipartite graph G has a maximum matching which covers all the vertices of maximum degree.*

Proof. By Remark 1.2.1, there exists a $\Delta(G)$-regular supergraph G' containing G. From Corollary 1.2.1, G' has a 1-factor and so G has a matching covers all the vertices of maximum degree. In turn, this matching can be augmented to a maximum matching of G. □

A family of graphs which share many properties with bipartite graphs was discovered by Fulkerson, Hoffman and McAndrew [219]. A graph G is said to have the *odd cycle property* (OCP in short) if every pair of odd cycles in G either has a vertex in common or is joined by an edge. We shall discuss the relation between this property and the existence of factors in much more details in Section 2.2.

The following is a generalization of Corollary 1.2.1.

Theorem 1.2.3. *If G is a r-regular graph of even order with the odd cycle property, then G has a 1-factor.*

Let M be a matching in a graph G. An *M-alternating path* or *cycle* in G is a path or cycle whose edges are alternately in M and $E(G) - M$. Note that an M-alternating path might or might not start or end with edges of M. If neither its origin nor its terminus is covered by M, the M-alternating path is called an *M-augmenting path* (see Fig. 1.4).

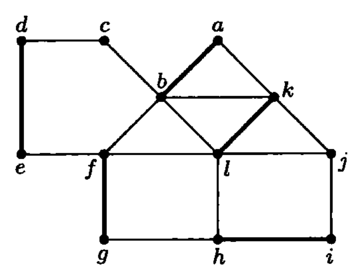


Fig. 1.4 An M-alternating cycle $C = ablk$, an M-alternating path $P = defghi$ and an M-augmenting path $P' = cdefghij$.

The idea of M-augmenting path is first introduced by Berge (1957). Today, it has become a very matured and standard technique in the study of factor theory. There are many variations of augmenting path defined for different problems, we shall see all kinds of variations in the coming chapters. It is not an overstatement that augmenting path technique is the most powerful technique not only in the study of matchings but also for the study of all factors.

Berge (1957) used augmenting path to give a characterization of maximum matching.

Theorem 1.2.4 (Berge (1957)). *A matching M in a graph G is maximum if and only if G has no M-augmenting paths.*

Proof. Let M be a matching in G. Suppose that G contains an M-augmenting path P. Then $M' := M \triangle E(P)$ is a matching in G, and $|M'| = |M| + 1$. Thus M is not a maximum matching.

Conversely, suppose that M is not a maximum matching, and let M'' be a maximum matching in G, then $|M''| > |M|$. Set $H := G[M \triangle M'']$. Then each vertex of H has degree one or two in H, for it can be incident with at most one edge of M and one edge of M''. Consequently, each component of H is either an even cycle with edges alternately in M and M'', or else a path with edges alternately in M and M''.

Because $|M'| > |M|$, the subgraph H contains more edges of M'' than of M, and therefore some path-component P of H must start and end with edges of M''. The origin and terminus of P, being covered by M'', are not covered by M. The path P is thus an M-augmenting path in G, a contradiction. □

Hall's Theorem can be restated in a more general version. We denote the matching number, the size of a maximum matching, by $\mu(G)$.

Theorem 1.2.5. *Let $G = (X, Y)$ be a bipartite graph, M a matching in G, and U the set of M-unsaturated vertices in X. Then*

(a) *for any subset S of X, $|U| \geq |N(S)| - |S|$; and*
(b) *$|U| = |N(S)| - |S|$ if and only if M is a maximum matching of G.*

Furthermore, the matching number of G is given by

$$\mu(G) = |X| - \max\{|S| - |N(S)| \mid S \subseteq X\}.$$

The expression for $\mu(G)$ is known as the König-Ore Formula.

Next, we investigate the properties of matching transformation in bipartite graphs. That is, given two matchings, we can start with one and transform gradually to another one through the operation of symmetric difference.

Theorem 1.2.6. *A maximum matching M of a graph G can be obtained from any other maximum matching M' by a sequence of transfers along alternating cycles and paths of even lengths.*

Proof. From the proof of Theorem 1.2.4, every component of $G[M' \triangle M]$ is either an alternating cycle or an alternating even path relative to M'. Changing M' in each component in turn will transform M' into M. □