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数学建模方法与分析

（英文版·第3版）

MATHEMATICAL MODELING

THIRD EDITION

MARK M. MEERSCHAERT

（美）Mark M. Meerschaert 著

机械工业出版社
China Machine Press

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Preface

Mathematical modeling is the link between mathematics and the rest of the world. You ask a question. You think a bit, and then you refine the question, phrasing it in precise mathematical terms. Once the question becomes a mathematics question, you use mathematics to find an answer. Then finally (and this is the part that too many people forget), you have to reverse the process, translating the mathematical solution back into a comprehensible, no–nonsense answer to the original question. Some people are fluent in English, and some people are fluent in calculus. We have plenty of each. We need more people who are fluent in both languages and are willing and able to translate. These are the people who will be influential in solving the problems of the future.

This text, which is intended to serve as a general introduction to the area of mathematical modeling, is aimed at advanced undergraduate or beginning graduate students in mathematics and closely related fields. Formal prerequisites consist of the usual freshman–sophomore sequence in mathematics, including one–variable calculus, multivariable calculus, linear algebra, and differential equations. Prior exposure to computing and probability and statistics is useful, but is not required.

Unlike some textbooks that focus on one kind of mathematical model, this book covers the broad spectrum of modeling problems, from optimization to dynamical systems to stochastic processes. Unlike some other textbooks that assume knowledge of only a semester of calculus, this book challenges students to use *all* of the mathematics they know (because that is what it takes to solve real problems).

The overwhelming majority of mathematical models fall into one of three categories: optimization models; dynamic models; and probability models. The type of model used in a real application might be dictated by the problem at hand, but more often, it is a matter of choice. In many instances, more than one type of model will be used. For example, a large Monte Carlo simulation model may be used in conjunction with a smaller, more tractable deterministic dynamic model based on expected values.

This book is organized into three parts, corresponding to the three main categories of mathematical models. We begin with optimization models. A five–step method for mathematical modeling is introduced in Section 1 of Chapter 1, in the context of one–variable optimization problems. The remainder of the first chapter is an introduction to sensitivity analysis and robustness. These fundamentals of mathematical modeling are used in a consistent way throughout the rest of the book. Exercises at the end of each chapter require students to master them as well. Chapter 2, on multivariable optimization, introduces decision variables, feasible and optimal solutions, and constraints. A review of the method of Lagrange multipliers is provided for the benefit of those students who were not exposed to this important technique in multivariable calculus. In the section on sensitivity analysis for problems with constraints, we learn that Lagrange multipliers represent shadow prices (some authors call them dual variables). This sets the stage for our discussion of linear programming later in Chapter 3. At the end of Chapter 3 is a section on discrete optimization that was added in the second edition. Here we give a practical introduction to integer programming using the branch–and–bound method. We also explore the connection between linear and integer programming problems, which allows an earlier introduction to the important issue of discrete versus continuous models. Chapter 3 covers some important computational techniques, including Newton's method in one and several variables, and linear and integer programming.

In the next part of the book, on dynamic models, students are introduced to the concepts of state and equilibrium. Later discussions of state space, state variables, and equilibrium for stochastic processes are intimately connected to what is done here. Nonlinear dynamical systems in both discrete and continuous time are covered. There is very little emphasis on exact analytical solutions in this part of the book, since most of these models admit no analytic solution. At the end of Chapter 6 is a section on chaos and fractals that was added in the second edition. We use both analytic and simulation methods to explore the behavior of discrete and continuous dynamic models, to understand how they can become chaotic under certain conditions. This section provides a practical and accessible introduction to the subject. Students gain experience with sensitive dependence to initial conditions, period doubling, and strange attractors that are fractal sets. Most important, these mathematical curiosities emerge from the study of real–world problems.

Finally, in the last part of the book, we introduce probability models. No prior knowledge of probability is assumed. Instead we build upon the material in the first two parts of the book, to introduce probability in a natural and intuitive way as it relates to real–world problems.

Each chapter in this book is followed by a set of challenging exercises. These exercises require significant effort, as well as a certain amount of creativity, on the part of the student. I did not invent the problems in this book. They are real problems. They were not designed to illustrate the use of any particular mathematical technique. Quite the opposite. We will occasionally go over some new mathematical techniques in this book *because the problem demands it.* I was determined that there would be no place in this book where a student could look up and ask, "What is all of this for?" Although typically oversimplified or grossly unrealistic, story problems embody the fundamental challenge in applying mathematics to solve real problems. For most students, story problems present plenty of challenge. This book teaches students how to solve story problems. There is a general method that can be applied successfully by any reasonably capable student to solve any story problem. It appears in Chapter 1, Section 1. This same general method is applied to problems of all kinds throughout the text.

Following the exercises in each chapter is a list of suggestions for further reading. This list includes references to a number of UMAP modules in applied mathematics that are relevant to the material in the chapter. UMAP modules can provide interesting supplements to the material in the text, or extra credit projects. All of the UMAP modules are available at a nominal cost from the Consortium for Mathematics and Its Applications (`www.comap.com`).

One of the major themes of this book is the use of appropriate technology for solving mathematical problems. Computer algebra systems, graphics, and numerical methods all have their place in mathematics. Many students have not had an adequate introduction to these tools. In this course we introduce modern technology in context. Students are motivated to learn because the new technology provides a more convenient way to solve real–world problems. Computer algebra systems and 2–D graphics are useful throughout the course. Some 3–D graphics are used in Chapters 2 and 3 in the sections on multivariable optimization. Students who have already been introduced to 3–D graphics should be encouraged to use what they know. Numerical methods covered in the text include, among others, Newton's method, linear programming, the Euler method, and linear regression.

The text contains numerous computer–generated graphs, along with instruction on the appropriate use of graphing utilities in mathematics. Computer algebra systems are used extensively in those chapters where significant algebraic calculation is required. The text includes computer output from the computer algebra systems Maple and Mathematica in Chapters 2, 4, 5, and 8. The chapters on computational techniques (Chapters 3, 6, and 9) discuss the appropriate use of numerical algorithms to solve problems that admit no analytic solution. Sections 3.3 and 3.4 on linear-integer programming include computer output from the popular linear programming package LINDO. Sections 8.3 and 8.4 on linear regression and time series include output from the commonly used statistical package Minitab.

Students need to be provided with access to appropriate technology in order to take full advantage of this textbook. We have tried to make it easy for instructors to use this textbook at their own institution, whatever their situation. Some will have the means to provide students with access to sophisticated computing facilities, while others will have to make do with less. The bare necessities include: (1) a software utility to draw 2–D graphs; and (2) a machine on which students can execute a few simple numerical algorithms. All of this can be done, for example, with a computer spreadsheet program or a programmable graphics calculator. The ideal situation would be to provide all students access to a good computer algebra system, a linear programming package, and a statistical computing package. The following is a partial list of appropriate software packages that can be used in conjunction with this textbook.

Computer Algebra Systems:

- Derive, Soft Warehouse, Inc., `www.derive.com`
- Maple, Waterloo Maple, Inc., `www.maplesoft.com`
- Mathcad, Mathsoft, Inc., `www.mathsoft.com`
- Mathematica, Wolfram Research, Inc., `www.wolfram.com`
- MATLAB, The MathWorks, Inc., `www.mathworks.com`

Statistical Packages:

- Minitab, Minitab, Inc., `www.minitab.com`
- SAS, SAS Institute, Inc., `www.sas.com`
- SPSS, SPSS Inc., `www.spss.com`
- S-PLUS, Insightful Corp., `www.insightful.com`
- R, R Foundation for Statistical Computing, `www.r-project.org`

Linear Programming Packages:

- LINDO, LINDO Systems, Inc., `www.lindo.com`
- MPL, Maximal Software, Inc., `www.maximal-usa.com`
- AMPL, AMPL Optimization, LLC, `www.ampl.com`
- GAMS, GAMS Development Corp., `www.gams.com`

The numerical algorithms in the text are presented in the form of pseudocode. Some instructors will prefer to have students implement the algorithms on their own. On the other hand, if students are not going to be required to program, we want to make it easy for instructors to provide them with appropriate software. All of the algorithms in the text have been implemented on a variety of computer platforms that can be made available to users of this textbook at no additional cost. If you are interested in obtaining a copy, please contact the author, or go to `www.stt.msu.edu/~mcubed/modeling.html` where you can download these implementations. Also, if you are willing to share your own implementation with other instructors and students, please send us a copy. With your permission, we will make copies available to others at no charge.

The third edition of this text reflects the insightful comments and suggestions of a number of students and faculty. Most significant are two new sections that were widely requested. At the end of Chapter 7, Introduction to Probability Models, is a new section on diffusion. Here we give a gentle introduction to partial differential equations by focusing on the diffusion equation. We provide a simple derivation of the point source solution to this partial differential equation, using Fourier analytic methods, to arrive at the normal density. Then we connect the diffusion model to the central limit theorem introduced in the previous Section 7.3, Introduction to Statistics. This new section on diffusion grew out of a class taught at the University of Nevada for beginning graduate students in the earth sciences. The applications are to contaminant migration in the atmosphere and ground water. At the end of Chapter 8, Stochastic Models, is a new section on time series. This section also serves as an introduction to multivariate regression models with more than one predictor. As a natural follow-up to the discussion in Section 8.3, Linear Regression, the new section on time series introduces the important idea of correlation. It also shows how to recognize correlated variables and include the dependence structure in a time series model. The discussion is focused on autoregressive models, since these are the most generally useful time series models. They are also the most convenient, in that they can be handled using widely available linear regression software. For the benefit of students with access to a statistical package, this section illustrates the proper application and interpretation of advanced methods including autocorrelation plots and sequential sums of squares. However, the entire section can also be covered using only a basic implementation of regression that allows multiple predictors and outputs the two basic measures: R^2 and the residual standard deviation s. This can all be done with a good spreadsheet or hand calculator.

The third edition also reflects the evolution of technology. The text includes output from the newest version of the computer algebra systems Maple and Mathematica. Spreadsheet implementations of linear and integer programming solvers are introduced in Chapter 3, along with output from the popular linear programming package LINDO. The sections on linear regression and time series contain computer outputs from the popular statistical package Minitab. All of the computer graphics and all discussions of the appropriate use of technology have been updated.

Support for instructors is now stronger than ever. A complete and detailed solutions manual for instructors is available from the author or the publisher, for instructors who adopt the text for classroom use. Computer implementations of the algorithms used in the text can be downloaded for a variety of platforms, along with the computer files used to produce all of the graphics and computer outputs included in the text. These downloads are all available at `www.stt.msu.edu/~mcubed/modeling.html`

The response to the first two editions of the text was gratifying. The best part of this job is interacting with students and instructors who use this book. Please feel free to contact me with any comments or suggestions.

Wednesday 25 April 2007

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Part I
OPTIMIZATION MODELS

Chapter 1

ONE VARIABLE OPTIMIZATION

Problems in optimization are the most common applications of mathematics. Whatever the activity in which we are engaged, we want to maximize the good that we do and minimize the unfortunate consequences or costs. Business managers attempt to control variables in order to maximize profit or to achieve a desired goal for production and delivery at a minimum cost. Managers of renewable resources such as fisheries and forests try to control harvest rates in order to maximize long–term yield. Government agencies set standards to minimize the environmental costs of producing consumer goods. Computer system managers try to maximize throughput and minimize delays. Farmers space their plantings to maximize yield. Physicians regulate medications to minimize harmful side effects. What all of these applications and many more have in common is a particular mathematical structure. One or more variables can be controlled to produce the best outcome in some other variable, subject in most cases to a variety of practical constraints on the control variables. Optimization models are designed to determine the values of the control variables which lead to the optimal outcome, given the constraints of the problem.

We begin our discussion of optimization models at a place where most students will already have some practical experience. One–variable optimization problems, sometimes called maximum–minimum problems, are typically discussed in first–semester calculus. A wide variety of practical applications can be handled using just these techniques. The purpose of this chapter, aside from a review of these basic techniques, is to introduce the fundamentals of mathematical modeling in a familiar setting.

1.1 The Five–Step Method

In this section we outline a general procedure that can be used to solve problems using mathematical modeling. We will illustrate this procedure, called the *five–*

step method, by using it to solve a one–variable maximum–minimum problem typical of those encountered by most students in the first semester of calculus.

Example 1.1. A pig weighing 200 pounds gains 5 pounds per day and costs 45 cents a day to keep. The market price for pigs is 65 cents per pound, but is falling 1 cent per day. When should the pig be sold?

The mathematical modeling approach to problem solving consists of five steps:

1. Ask the question.
2. Select the modeling approach.
3. Formulate the model.
4. Solve the model.
5. Answer the question.

The first step is to ask a question. The question must be phrased in mathematical terms, and it often requires a good deal of work to do this. In the process we are required to make a number of assumptions or suppositions about the way things really are. We should not be afraid to make a guess at this stage. We can always come back and make a better guess later on. Before we can ask a question in mathematical terms we need to define our terms. Go through the problem and make a list of variables. Include appropriate units. Next make a list of assumptions about these variables. Include any relations between variables (equations and inequalities) that are known or assumed. Having done all of this, we are ready to ask a question. Write down in explicit mathematical language the objective of this problem. Notice that the preliminary steps of listing variables, units, equations and inequalities, and other assumptions are really a part of the question. They frame the question.

In Example 1.1 the weight w of the pig (in lbs), the number of days t until we sell the pig, the cost C of keeping the pig t days (in dollars), the market price p for pigs ($/lb), the revenue R obtained when we sell the pig ($), and our resulting net profit P ($) are all variables. There are other numerical quantities involved in the problem, such as the initial weight of the pig (200 lbs). However, these are not variables. It is important at this stage to separate variables from those quantities that will remain constant.

Next we need to list our assumptions about the variables identified in the first stage of step 1. In the process we will take into account the effect of the constants in the problem. The weight of the pig starts at 200 lbs and goes up by 5 lbs/day so we have

$$(w \text{ lbs}) = (200\,\text{lbs}) + \left(\frac{5\,\text{lbs}}{\text{day}}\right)(t\,\text{days}).$$

Variables:	t = time (days)
	w = weight of pig (lbs)
	p = price for pigs (\$/lb)
	C = cost of keeping pig t days (\$)
	R = revenue obtained by selling pig (\$)
	P = profit from sale of pig (\$)
Assumptions:	$w = 200 + 5t$
	$p = 0.65 - 0.01t$
	$C = 0.45t$
	$R = p \cdot w$
	$P = R - C$
	$t \geq 0$
Objective:	Maximize P

Figure 1.1: Results of step 1 of the pig problem.

Notice that we have included units as a check that our equation makes sense. The other assumptions inherent in our problem are as follows:

$$\left(\frac{p \text{ dollars}}{\text{lb}}\right) = \left(\frac{0.65 \text{ dollars}}{\text{lb}}\right) - \left(\frac{0.01 \text{ dollars}}{\text{lb} \cdot \text{day}}\right)(t \text{ days})$$

$$(C \text{ dollars}) = \left(\frac{0.45 \text{ dollars}}{\text{day}}\right)(t \text{ days})$$

$$(R \text{ dollars}) = \left(\frac{p \text{ dollars}}{\text{lb}}\right)(w \text{ lbs})$$

$$(P \text{ dollars}) = (R \text{ dollars}) - (C \text{ dollars})$$

We are also assuming that $t \geq 0$. Our objective in this problem is to maximize our net profit, P dollars. Figure 1.1 summarizes the results of step 1, in a form convenient for later reference.

The three stages of step 1 (variables, assumptions, and objective) need not be completed in any particular order. For example, it is often useful to determine the objective early in step 1. In Example 1.1, it may not be readily apparent that R and C are variables until we have defined our objective, P, and we recall that $P = R - C$. One way to check that step 1 is complete is to see whether our objective P relates all the way back to the variable t. The best general advice about step 1 is to *do something.* Start by writing down whatever is immediately apparent (e.g., some of the variables can be found simply by reading over the problem and looking for nouns), and the rest of the pieces will probably fall into place.

Step 2 is to select the modeling approach. Now that we have a problem stated in mathematical language, we need to select a mathematical approach to use to get an answer. Many types of problems can be stated in a standard form for which an effective general solution procedure exists. Most research in applied mathematics consists of identifying these general categories of problems and inventing efficient ways to solve them. There is a considerable body of literature in this area, and many new advances continue to be made. Few, if any, students in this course will have the experience and familiarity with the literature to make a good selection for the modeling approach. In this book, with rare exceptions, problems will specify the modeling approach to be used. Our example problem will be modeled as a one–variable optimization problem, or maximum–minimum problem.

We outline the modeling approach we have selected. For complete details we refer the reader to any introductory calculus textbook.

> We are given a real–valued function $y = f(x)$ defined on a subset S of the real line. There is a theorem that states that if f attains its maximum or minimum at an interior point $x \in S$, then $f'(x) = 0$, assuming that f is differentiable at x. This allows us to rule out any interior point $x \in S$ at which $f'(x) \neq 0$ as a candidate for max–min. This procedure works well as long as there are not too many exceptional points.

Step 3 is to formulate the model. We need to take the question exhibited in step 1 and reformulate it in the standard form selected in step 2, so that we can apply the standard general solution procedure. It is often convenient to change variable names if we will refer to a modeling approach that has been described using specific variable names, as is the case here. We write

$$\begin{aligned} P &= R - C \\ &= p \cdot w - 0.45t \\ &= (0.65 - 0.01t)(200 + 5t) - 0.45t. \end{aligned}$$

Let $y = P$ be the quantity we wish to maximize and $x = t$ the independent variable. Our problem now is to maximize

$$\begin{aligned} y &= f(x) \\ &= (0.65 - 0.01x)(200 + 5x) - 0.45x \end{aligned} \tag{1.1}$$

over the set $S = \{x : x \geq 0\}$.

Step 4 is to solve the model, using the standard solution procedure identified in step 2. In our example we want to find the maximum of the function $y = f(x)$ defined by Eq. (1.1) over the set $x \geq 0$. Figure 1.2 shows a graph of the function $f(x)$. Since f is quadratic in x, the graph is a parabola. We compute that

$$f'(x) = \frac{(8 - x)}{10},$$

so that $f'(x) = 0$ at the point $x = 8$. Since f is increasing on the interval $(-\infty, 8)$ and decreasing on $(8, \infty)$, the point $x = 8$ is the global maximum. At