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I First-Order Differential Equations

1 Introduction

A *differential equation* is an equation between specified derivatives of a function, its values, and known quantities. Many laws of physics are most simply and naturally formulated as differential equations (or DE's, as we shall write for short). For this reason, DE's have been studied by the greatest mathematicians and mathematical physicists since the time of Newton.

Ordinary differential equations are DE's whose unknowns are functions of a single variable; they arise most commonly in the study of dynamic systems and electric networks. They are much easier to treat than partial differential equations, whose unknown functions depend on two or more independent variables.

Ordinary DE's are classified according to their order. The *order* of a DE is defined as the largest positive integer, n, for which an nth derivative occurs in the equation. This Chapter will be restricted to *real first-order* DE's of the form

$$\phi(x, y, y') = 0. \tag{1}$$

Given the function ϕ of three real variables, the problem is to determine all real functions $y = f(x)$ which satisfy the DE, that is, all solutions of (1) in the following sense.

■ DEFINITION. *A* solution *of* (1) *is a differentiable function* $f(x)$ *such that* $\phi(x, f(x), f'(x)) = 0$ *for all* x *in the interval where* $f(x)$ *is defined.*

EXAMPLE 1. In the first-order DE

$$x + yy' = 0, \tag{2}$$

the function ϕ is a polynomial function $\phi(x, y, z) = x + yz$ of the three variables involved. The solutions of (2) can be found by considering the identity $d(x^2 + y^2)/dx = 2(x + yy')$. From this identity, one sees that $x^2 + y^2 = C$ is a constant if $y = f(x)$ is any solution of (2).

The equation $x^2 + y^2 = C$ defines y *implicitly* as a two-valued function of x, for any positive constant C. Solving for y, we get *two* solutions, the (single-valued†) functions $y = \pm\sqrt{C - x^2}$, for each positive constant C. The *graphs* of these solutions, the so-called *solution curves*, form two families of semicircles, which fill the upper half-plane $y > 0$ and the lower half-plane $y < 0$, respectively.

On the x-axis, where $y = 0$, the DE (2) implies that $x = 0$. Hence the DE has no solutions which cross the x-axis, except possibly at the origin. This fact is easily overlooked, because the solution curves *appear* to cross the x-axis to form full circles, as in Figure 1.1. However, these

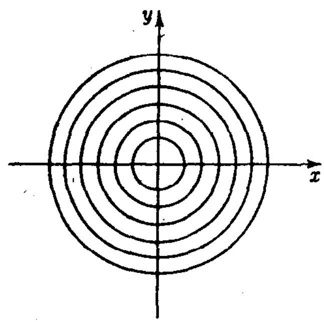

FIGURE 1.1 Integral Curves of $x + yy' = 0$.

circles have infinite slope where they cross the x-axis; hence y' does not exist, and the DE (2) is *not* satisfied there.

The preceding difficulty also arises if one tries to solve the DE (2) for y'. Dividing through by y, one gets $y' = -x/y$, an equation which cannot be satisfied if $y = 0$. The preceding difficulty is thus avoided if one restricts attention to regions where the DE (1) is normal, in the following sense.

■ DEFINITION. *A* normal *first-order* DE *is one of the form*

$$y' = F(x, y). \tag{3}$$

† In this book, the word "function" will always mean single-valued function, unless the contrary is expressly specified.

In the normal form $y' = -x/y$ of the DE (2), the function $F(x, y)$ is continuous in the upper half-plane $y > 0$ and in the lower half-plane where $y < 0$; it is undefined on the x-axis.

2 Fundamental theorem of the calculus

The most familiar class of differential equations consists of the first-order DE's of the form

$$y' = g(x). \tag{4}$$

Such DE's are normal; their solutions are described by the fundamental theorem of the calculus, which reads as follows.

■ FUNDAMENTAL THEOREM OF THE CALCULUS. *Let the function $g(x)$ in* DE (4) *be continuous in the interval $a \leq x \leq b$. Given a number c, there is one and only one solution $f(x)$ of the* DE (4) *in the interval such that $f(a) = c$. This solution is given by the definite integral*

$$f(x) = c + \int_a^x g(t)\,dt, \qquad c = f(a). \tag{5}$$

This basic result serves as a model of rigorous formulation in several respects. First, it specifies the region under consideration, as a vertical strip $a \leq x \leq b$ in the xy-plane. Second, it describes in precise terms the class of functions $g(x)$ considered. And third, it asserts the *existence* and *uniqueness* of a solution, given the "initial condition" $f(a) = c$.

We recall that the definite integral

$$\int_a^x g(t)\,dt = \lim_{\max \Delta t_k \to 0} \sum g(t_k)\,\Delta t_k, \qquad \Delta t_k = t_k - t_{k-1}, \tag{5'}$$

is defined for each fixed x as a limit of Riemann sums; it is not necessary to find a formal expression for the indefinite integral $\int g(x)\,dx$ to give meaning to the definite integral $\int_a^x g(t)\,dt$, provided only that $g(t)$ is continuous. Such functions as the *error function* $\operatorname{erf} x = (2/\sqrt{\pi}) \int_0^x e^{-t^2}\,dt$ and the *sine integral function* $\mathrm{SI}(x) = \int_x^\infty [(\sin t)/t]\,dt$ are indeed commonly *defined* as definite integrals; cf. Chapter 3, §1.

To formulate and prove analogous theorems for more general first-order normal DE's, we need some technical concepts. We define a *domain*† as a nonempty *open connected* set. A function $\phi = \phi(x_1, \ldots, x_r)$ is said to be *of class* $\mathscr{C}^n$ in a domain D, when all its derivatives $\partial\phi/\partial x_i$, $\partial^2\phi/\partial x_i\,\partial x_j$, $\cdots$ of orders $1, \cdots, n$ exist and are continuous in D. One writes this condition in symbols as $\phi \in \mathscr{C}^n$ in D, or $\phi \in \mathscr{C}^n(D)$. When ϕ

† Some authors say *region* where we say domain. We will call the closure of a domain a *closed domain.*

is merely assumed to be continuous in D, one writes $\phi \in \mathscr{C}$ in D, or $\phi \in \mathscr{C}(D)$.

Intervals appear so frequently in analysis that they are referred to by a special notation. Thus, the *closed* interval $a \leqq x \leqq b$, which is not a domain (why not?) is denoted by $[a, b]$, the *open* interval $a < x < b$ by (a, b), the positive semi-axis $0 \leqq x < +\infty$ by $[0, +\infty)$, and so on. Generally, a round bracket indicates that the endpoint adjacent to it is excluded from the interval, and a square bracket that the adjacent endpoint is included.

Given $F(x)$, the notation $F \in \mathscr{C}^2[1, +\infty)$ thus means that F is twice continuously differentiable in the semi-infinite line $[1, +\infty)$. Considered as a function of the two variables x and y, F is of class $\mathscr{C}^2$ in the *closed* domain including the vertical line $x = 1$ and all points to the right of it in the xy-plane. Likewise, $F \in \mathscr{C}[0, 1]$ means that F is continuous in the vertical strip $0 \leqq x \leqq 1$. Where there is any question of just what domain is referred to below, the domain will be described in words as well as in symbols.

There are a number of obvious facts about the differentiability of solutions of DE's. Such facts about differentiability will be used without special comment where they are irrelevant to the main idea of a proof. For instance, if $g \in \mathscr{C}^n(a, b)$, and $y = f(x)$ is any solution of the DE $y' = g(x)$, then $y \in \mathscr{C}^{n+1}(a, b)$. Again, if $\phi \in \mathscr{C}^n$ and $\psi \in \mathscr{C}^n$ in a domain D, and $F(u, v) \in \mathscr{C}^n$ in the entire uv-plane, then $G(x, y) = F(\phi(x, y), \psi(x, y)) \in \mathscr{C}^n(D)$.

3 Solutions and integrals

According to the definition given in §1, a *solution* of a DE is always a *function*. For example, the solutions of the DE $x + yy' = 0$ in Example 1 are the functions $y = \pm\sqrt{C - x^2}$, whose graphs are semicircles of arbitrary diameter, centered at the origin. The graph of the solution curves are, however, more easily described by the equation $x^2 + y^2 = C$, describing a family of circles centered at the origin. In what sense can such a family of curves be considered as a solution of the DE? To answer this question, we require a new notion.

■ DEFINITION. *An* integral *of* DE (1) *is a function of two variables, $u(x, y)$, which assumes a constant value whenever the variable y is replaced by a solution $y = f(x)$ of the* DE.

In the above example, the function $u(x, y) = x^2 + y^2$ is an integral of the DE $x + yy' = 0$, because, upon replacing the variable y by any function $\pm\sqrt{C - x^2}$, we obtain $u(x, y) = C$.

The second-order DE,

$$\frac{d^2x}{dt^2} = -x, \tag{2'}$$

becomes a first-order DE equivalent to (2) after setting $dx/dt = y$:

$$y\frac{dy}{dx} = -x. \tag{2}$$

As we have seen, the curves $u(x, y) = x^2 + y^2$ are integrals of this DE. When the DE (2′) is interpreted as an equation of motion under Newton's second law (see, for example, the discussion in Chapter 5, §7), the integrals $C = x^2 + y^2$ represent curves of constant energy C. This illustrates an important principle: an integral of a DE representing some kind of motion is a quantity that remains unchanged through the motion.

The relationship between *solutions* and *integrals* of the DE (1) will be made clear by the following theorem.

■ IMPLICIT FUNCTION THEOREM.† *Let $u(x, y)$ be a function of class $\mathscr{C}^n$ in a domain containing the point (x_0, y_0), and let $\partial u(x_0, y_0)/\partial y \neq 0$. Then there exists a unique function $y = f(x, C)$ of class $\mathscr{C}^n$, defined in some open interval (a, b) containing x_0, such that $y_0 = f(x_0, C)$ and $u(x, f(x, C)) = C$ for all x in (a, b) and for all C in an open interval.*

By the Implicit Function Theorem, every integral $u(x, y)$ of class $\mathscr{C}^1$ of the DE (1) defines a family of solutions near any point (x, y) where $\partial u(x, y)/\partial x \neq 0$, obtained by solving the equation $u(x, y) = C$ for the variable y.

The notion of integral has been defined in terms of a solution of a DE. For several classes of DE's, however, it is possible to verify that a function $u(x, y)$ is an integral without first finding any solution. For example, a function $u(x, y)$ of class $\mathscr{C}^1$ is an integral of the *quasilinear* DE

$$\phi(x, y, y') = M(x, y) + N(x, y)y' = 0$$

whenever

$$M(x, y)\frac{\partial u}{\partial y} - N(x, y)\frac{\partial u}{\partial x} = 0,$$

provided that $\partial u/\partial y \neq 0$, as can be verified from the familiar formula $dy/dx = -(\partial u/\partial x)/(\partial u/\partial y)$.

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† Courant, Vol. 2, p. 114; Widder, p. 55. Here and below, page references to authors refer to the books listed in the selected bibliography on pp. 355–357.

A similar method can be applied to DE's of finite *degree* in the derivative y', namely, when the function $\phi(x, y, y')$ is a polynomial in the variable y' (see §8).

Parametric Solutions. A pair of functions $x = g(t)$, $y = h(t)$ of the parameter t describe a *curve* in the xy-plane. Such a curve is a *parametric solution* of DE (1) whenever the function $h(t) = f(g(t))$ for every solution $y = f(x)$ of the DE, as t ranges over the interval of definition of the functions g and h. If $u(g(t), h(t))$ is constant, then the function $u(x, y)$ is an integral of the DE.

Thus we see that there are three different notions of "solution" of a first-order DE (1): a *function* $y = f(x)$, and *integral* $u(x, y) = C$, corresponding to an *implicit solution*, and a parametric *curve* $x = g(t)$, $y = h(t)$.

4 Regular and normal curve families

Let the function $u(x, y)$ be of class $\mathscr{C}^1$ in a region of the xy-plane. The points in the region where both partial derivatives of u vanish are called *critical points.* We shall always assume in this book (unless the contrary is explicitly stated) that the function u has but a finite number of critical points. This implies that the critical points of u are isolated. The reason for this assumption is that it allows us to study the contour lines of u in the neighborhood of a critical point.

For example, the function $u = x^2 + y^2$ has just one critical point in the entire plane; this is at the origin, through which indeed no contour line passes. The contour lines $u = C$ satisfy the following three conditions in any domain D not containing any critical points: (i) one and only one curve of the family passes through each point of D, (ii) each curve of the family has a tangent at every point, and (iii) the tangent direction is a continuous function of position.

We call *any* curve family which satisfies conditions (i)–(iii) a *regular* curve family. The contour lines of any function $u(x, y) \in \mathscr{C}^1$ form a regular curve family in any domain that contains no critical points.

In Example 1, the families of *solution* curves $y = \pm\sqrt{C - x^2}$ are not only a regular curve family in each half-plane but, in addition, the functions $f(x, C) = \sqrt{C - x^2}$ and $g(x, C) = -\sqrt{C - x^2}$ satisfy $\partial f/\partial C = 1/2y > 0$ when $y > 0$, and $\partial g/\partial C = -1/2y > 0$ when $y < 0$, respectively. They constitute a *normal* curve family, as in the following.

■ DEFINITION. *In a domain D of the xy-plane, a* normal *curve family is a family of curves defined explicitly as the graphs of functions $y = f(x, c)$,*

$f \in \mathscr{C}^1(D)$, which depend on a parameter $c = c(x, y)$ which is defined on D and satisfies $\partial c/\partial y \neq 0$ there.

Since, for fixed x, $(\partial f/\partial c)(\partial c/\partial y) = 1$, it is equivalent to require that $\partial f/\partial c \neq 0$.

The relationship of regular and normal curve families to first-order DE's is given by the following result.

■ THEOREM 1. *For curve families in a domain D of the xy plane:*

(i) *every normal curve family is regular;*

(ii) *a regular curve family is normal in any domain where no tangent to any curve of the family is vertical;*

(iii) *the functions $y = f(x, c)$ defining a normal curve family are the solutions of a normal first-order DE;*

(iv) *the functions $u(x, y) = C$ defining any regular curve family are the solutions of a first-order* partial *DE*

$$M(x, y)\frac{\partial u}{\partial y} - N(x, y)\frac{\partial u}{\partial x} = 0;$$

(v) *in any domain where a regular curve family is normal, the curves of the family are the* integrals *of the quasilinear* DE

$$M(x, y) + N(x, y)y' = 0. \tag{6}$$

The proof is essentially contained in the preceding discussion. Statement (i) is an immediate consequence of the definition of a normal curve family. Statement (ii) is a consequence of the Implicit Function Theorem, stated in the preceding Section.

Statement (iii) is seen to be true as follows. By definition of a normal curve family, we have $\partial f/\partial c \neq 0$; applying the implicit function theorem to solve this equation for the parameter c, we find that there is a function $c = g(x, y)$ of class $\mathscr{C}^1$ for which $y = f(x, g(x, y))$. Differentiating $y = f(x, c)$ relative to x, we find that

$$y' = \partial f(x, g(x, y))/\partial x = F(x, y),$$

as was to be shown.

Statement (iv) follows by setting $M(x, y) = \partial u(x, y)/\partial x$ and $N(x, y) = \partial u(x, y)/\partial y$, and statement (v) follows from the identity

$$\frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y}.$$

EXAMPLE 2. In Example 1, the curves $u(x, y) = x^2 + y^2 = c$ form a regular curve family of integrals of the implicit DE $x = -yy'$.

The curves $y = +\sqrt{c - x^2}$, for $c > 0$, form a *normal* family of curves, solutions of the DE in normal form

$$y' = -x/y$$

in the domain D consisting of the upper half-plane. There is only one critical point, namely the origin.

We have already seen that the solution curves $y = c + f(x)$ for $y' = g(x)$ form a normal curve family in the strip $a \leq x \leq b$ if $g \in \mathscr{C}[a, b]$, with $\partial y/\partial c = 1$. See Figure 1.2, which depicts solution curves for the DE $y' = e^{-x^2}$.

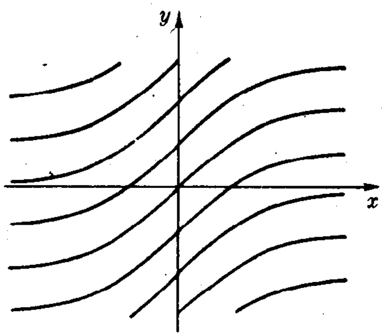

FIGURE 1.2 Solution Curves of $y' = e^{-x^2}$.

It is important to notice that, on the basis of the preceding theorem, we can construct DE's whose solution curves behave in any desired way: we simply draw on a region of the plane a normal family of curves.

EXAMPLE 3. Let $y' = g(x)h(y)$ be any DE with "separable variables" (and g, h continuous). Then the indefinite integrals $\phi(x) = \int g(x)dx$ and $\psi(y) = \int dy/h(y)$ exist in any horizontal strip $y_1 < g(x) < y_2$ between successive zeros of $h(y)$. The implicit equation $u(x, y) = \phi(x) - \psi(y) = c$ defines a regular family of integral curves in each such strip since $\partial u/\partial y = 1/h(y) \neq 0$ has constant sign there.

In (6), the points where $M = N = 0$ are the critical points; they are points where the tangent direction is indeterminate, because (6) reduces to $y' = 0/0$ there. Thus, in Example 1, the origin is such a point, and we see that no integral "curve" passes through it. The behavior of solutions near critical points will be discussed in Chapter 5; it can be very complicated.

Moreover until Chapter 6, §14, we will not even prove the basic fact that, if $F \in \mathscr{C}^1(D)$, the solution curves of $y' = F(x, y)$ form a normal curve family in D. For the present, in order to give precise formulations and proofs of theorems in a reasonably short space, we shall have to content ourselves with much less sweeping conclusions.

5 Exact differentials; integrating factors

A differential $M\,dx + N\,dy$ ($M, N \in \mathscr{C}(D)$) is called *exact* in a domain D when the line integral

$$\int_\tau M(x, y)\,dx + N(x, y)\,dy$$

is the same for all paths of integration τ in D which have the same endpoints. It is shown in the calculus† that $M\,dx + N\,dy$ is exact if and only if there exists a continuously differentiable function $u(x, y)$ such that $M = \partial u/\partial x$ and $N = \partial u/dy$, that is, such that the total differential $du = (\partial u/\partial x)dx + (\partial u/\partial y)dy$ is $M\,dx + N\,dy$. For *continuously differentiable* M, N, a necessary condition for $M\,dx + N\,dy$ to be exact is that M and N satisfy the partial DE $\partial M/\partial y = \partial N/\partial x$; if D is simply connected, this condition is also sufficient.

We now consider first-order DE's of the form

$$\phi(x, y, y') = M(x, y) + N(x, y)y' = 0 \tag{7}$$

with $M, N \in \mathscr{C}$ in a domain D. Dividing by N, we obtain the algebraically equivalent normal form

$$y' = F(x, y) = -M(x, y)/N(x, y) \tag{7'}$$

except on the closed set where N vanishes, i.e., where $\partial\phi/\partial y' = 0$. This set (the x-axis in Example 1) divides the domain D into a number of subdomains, in each of which the normal form (7′) is equivalent to (7).

It is often stated that the solution curves of the DE (7) are the contour lines $u(x, y) = C$, whenever $M\,dx + N\,dy = du$ is an exact differential. But this is not true, as Example 1 shows. A correct statement of the relation in question is the following.

■ THEOREM 2. *If $M(x, y)\,dx + N(x, y)\,dy$ is an exact differential du in a domain D, then the contour lines $u(x, y) = C$ are integral curves of (7) for any constant C. These contour lines form a regular curve family in the domain D^*, consisting of D with the critical points where $M = N = 0$ (that is, where grad $u = 0$) deleted.*

The proof is immediate, for any continuous M, N. Along any solution curve $y = f(x)$, we have

$$du/dx = \partial u/\partial x + y'\,\partial u/\partial y = M(x, y) + N(x, y)y' = 0.$$

† Courant, Vol. 2, p. 352; Widder, p. 251.

The regularity of the family of contour lines follows directly from the implicit function theorem. They form a normal family in any subdomain where N does not vanish.

When the differential $M\,dx + N\,dy$ is not exact, one can often find a function $\mu(x, y)$ such that the product

$$(\mu M)\,dx + (\mu N)\,dy = du$$

is an exact differential. The contour lines $u(x, y) = C$ will then again be integral curves of the DE $M(x, y) + N(x, y)y' = 0$ because $du/dx = \mu(M + Ny') = 0$; and segments of these contour lines between points of vertical tangency will be solution curves. Such a function μ is called an integrating factor.

■ DEFINITION. *An* integrating factor *for a differential* $M(x, y)\,dx + N(x, y)\,dy$ *is a nonvanishing function* $\mu(x, y)$ *such that the product* $(\mu M)\,dx + (\mu N)\,dy$ *is an exact differential.*

For example, consider the DE $xy' = y$. The differential $x\,dy - y\,dx$ which is associated with it is not exact, but it has the integrating factor $1/(x^2 + y^2)$ in the right half-plane $x > 0$. In fact, the function $\theta(x, y)$ defined by the line integral

$$\theta(x, y) = \int_{(1,0)}^{(x,y)} (x\,dy - y\,dx)/(x^2 + y^2)$$

is the angle made with the positive x-axis by the vector (x, y). That is, it is just the polar angle θ when the point (x, y) is expressed in polar coordinates. Thus, the integral curves of $xy' = y$ in the domain $x > 0$ are the radii $\theta = C$, where $-\pi/2 < \theta < \pi/2$; the solution curves are the same.

Note that the differential $(x\,dy - y\,dx)/(x^2 + y^2)$ is not exact in the punctured plane, consisting of the xy-plane with the origin deleted. For θ changes by 2π in going around the origin. This is possible, even though $\partial[x/(x^2 + y^2)]/\partial x = \partial[-y/(x^2 + y^2)]/\partial y$, because the punctured plane is not a simply connected domain.

EXERCISES A

1. Plot the integral curves of the DE $y' = y^2/x^2$. In which regions of the plane do they form a regular curve family? A normal curve family?
2. Find equations describing all solutions of $y' = (x + y)^2$.
3. Find equations describing all solutions of $y' = (2x + y)^{-1}$.
4. Show that if a normal curve family is invariant under horizontal translation, then the curves of the family are the solution curves of a DE of the form $y' = f(y)$.
5. Find all functions $f(x)$ whose definite integral between 0 and any x equals the reciprocal of $f(x)$. [HINT: DE is $y' = -y^3$.]
6. For what pairs of positive integers n, r is the function $|x|^n$ of class $\mathscr{C}^r$?