Noncommutative Geometry

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Alain Connes

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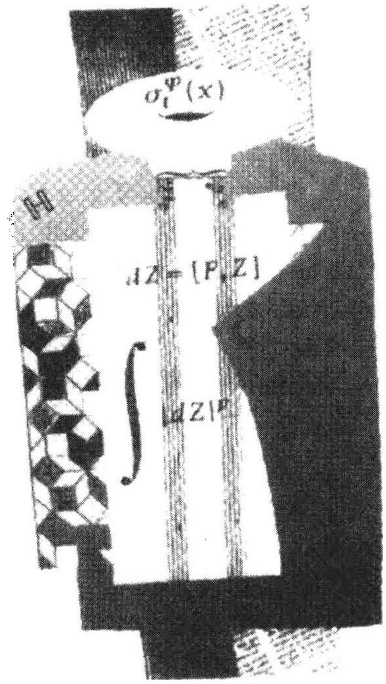
PREFACE



This book is the English version of the French "Géométrie non commutative" published by InterEditions Paris (1990). After the excellent initial translation by S.K. Berberian, a considerable amount of rewriting was done and many additions made, multiplying by 3.8 the size of the original manuscript. In particular the present text contains several unpublished results.

My thanks go first of all to Cécile whose patience and care for the manuscript have been essential to its completion. This second version of the book greatly benefited from the modifications suggested by many people: foremost was Marc Rieffel, but important contributions were made by D. Sullivan, J.-L. Loday, J. Lott, J. Bellissard, P. B. Cohen, R. Coquereaux, J. Dixmier, M. Karoubi, P. Krée, H. Bacry, P. de la Harpe, A. Hof, G. Kasparov, J. Cuntz, D. Testard, D. Kastler, T. Loring, J. Pradines, V. Nistor, R. Plymen, R. Brown, C. Kassel, and M. Gerstenhaber, with several of whom I have shared the pleasure of collaboration.

Patrick Ion and Arthur Greenspoon played a decisive rôle in the finalisation of the book, clearing up many mathematical imprecisions and considerably smoothing the initial manuscript. I wish to express my deep gratitude for their generous help and their insight.

Finally, my thanks go to Marie Claude for her help in creating the picture on the cover of the book, to Gilles who took the photograph, and to Bonnie Ion and Françoise for their help with the bibliography. Many thanks go also to Peter Renz who orchestrated the whole thing.

Alain Connes
30 June 1994
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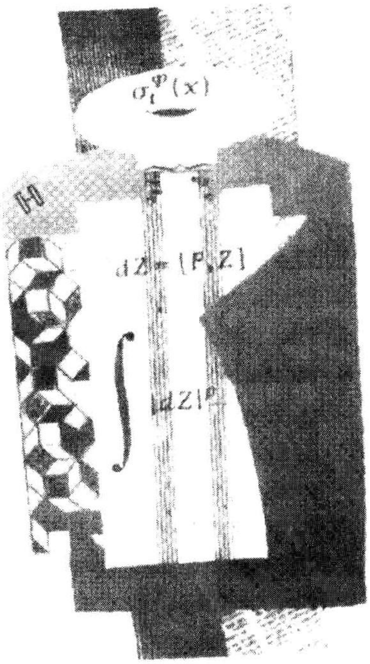

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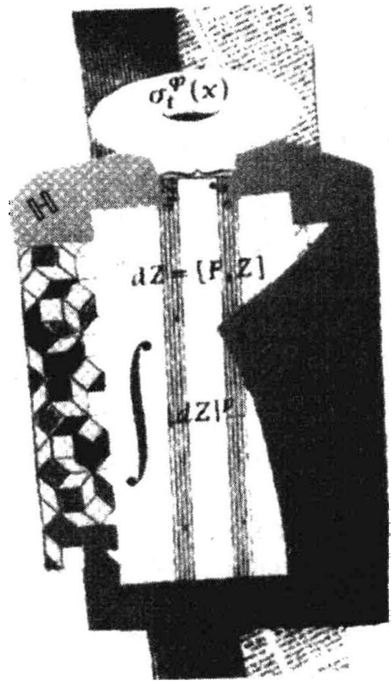
INTRODUCTION



The correspondence between geometric spaces and commutative algebras is a familiar and basic idea of algebraic geometry. The purpose of this book is to extend this correspondence to the noncommutative case in the framework of real analysis. The theory, called noncommutative geometry, rests on two essential points:

1. The existence of many natural spaces for which the classical set-theoretic tools of analysis, such as measure theory, topology, calculus, and metric ideas lose their pertinence, but which correspond very naturally to a noncommutative algebra. Such spaces arise both in mathematics and in quantum physics and we shall discuss them in more detail below; examples include:

a) The space of Penrose tilings

b) The space of leaves of a foliation

c) The space of irreducible unitary representations of a discrete group

d) The phase space in quantum mechanics

e) The Brillouin zone in the quantum Hall effect

f) Space-time.

Moreover, even for classical spaces, which correspond to commutative algebras, the new point of view will give new tools and results, for instance for the Julia sets of iteration theory.

2. The extension of the classical tools, such as measure theory, topology, differential calculus and Riemannian geometry, to the noncommutative situation. This extension involves, of course, an algebraic reformulation of the above tools, but passing from the commutative to the noncommutative case is never straightforward. On the one hand, completely new phenomena arise in the noncommutative case, such as the existence of a canonical time evolution for a

noncommutative measure space. On the other hand, the constraint of developing the theory in the noncommutative framework leads to a new point of view and new tools even in the commutative case, such as cyclic cohomology and the quantized differential calculus which, unlike the theory of distributions, is perfectly adapted to products and gives meaning and uses expressions like $\int f(Z)|dZ|^p$ where Z is not differentiable (and p not necessarily an integer).

Let us now discuss in more detail the extension of the classical tools of analysis to the noncommutative case.

A. Measure theory (Chapters I and V)

It has long been known to operator algebraists that the theory of von Neumann algebras and weights constitutes a far reaching generalization of classical measure theory. Given a countably generated measure space X, the linear space of square-integrable (classes of) measurable functions on X forms a Hilbert space. It is one of the great virtues of the Lebesgue theory that every element of the latter Hilbert space is represented by a measurable function, a fact which easily implies the Radon-Nikodým theorem, for instance. There is, up to isomorphism, only one Hilbert space with a countable basis, and in the above construction the original measure space is encoded by the representation (by multiplication operators) of its algebra of bounded measurable functions. This algebra turns out to be the prototype of a *commutative* von Neumann algebra, which is dual to an (essentially unique) measure space X.

In general a construction of a Hilbert space with a countable basis provides one with specific automorphisms (unitary operators) of that space. The algebra of operators in the Hilbert space which commute with these particular automorphisms is a *von Neumann algebra*, and all von Neumann algebras are obtained in that manner. The theory of not necessarily commutative von Neumann algebras was initiated by Murray and von Neumann and is considerably more difficult than the commutative case.

The center of a von Neumann algebra is a commutative von Neumann algebra, and, as such, dual to an essentially unique measure space. The general case thus decomposes over the center as a direct integral of so-called *factors*, i.e. von Neumann algebras with trivial center.

In increasing degree of complexity the factors were initially classified by Murray and von Neumann into three types, I, II, and III.

The type I factors and more generally the type I von Neumann algebras, (i.e. direct integrals of type I factors) are isomorphic to commutants of *commutative* von Neumann algebras. Thus, up to the notion of multiplicity they correspond to classical measure theory.

The type II factors exhibit a completely new phenomenon, that of *continuous dimension*. Thus, whereas a type I factor corresponds to the geometry of lines, planes, ..., k-dimensional complex subspaces of a given Hilbert space, the subspaces that belong to a type II factor are no longer classified by a dimension which is an integer but by a dimension which is a positive *real number* and will span a continuum of values (an interval). Moreover, crucial properties such as the equality

$$\dim(E \wedge F) + \dim(E \vee F) = \dim(E) + \dim(F)$$

remain true in this continuous geometry ($E \wedge F$ is the intersection of the subspaces and $E \vee F$ the closure of the linear span of E and F).

The type III factors are those which remain after the type I and type II cases have been considered. They appear at first sight to be singular and intractable. Relying on Tomita's theory of modular Hilbert algebras and on the earlier work of Powers, Araki, Woods and Krieger, I showed in my thesis that type III is subdivided into types III_λ, $\lambda \in [0,1]$ and that a factor of type III_λ, $\lambda \neq 1$, can be reconstructed uniquely as a crossed product of a type II von Neumann algebra by an automorphism contracting the trace. This result was then extended by M. Takesaki to cover the III_1 case as well, using a one-parameter group of automorphisms instead of a single automorphism.

These results thus reduce the understanding of type III factors to that of type II factors and their automorphisms, a task which was completed in the hyperfinite case and culminates in the complete classification of hyperfinite von Neumann algebras presented briefly in Chapter I Section 3 and in great detail in Chapter V.

The reduction from type III to type II has some resemblance to the reduction of arbitrary locally compact groups to unimodular ones by a semidirect product construction. There is one essential difference, however, which is that the range of the module, which is a closed subgroup of $\mathbb{R}_+^*$ in the locally compact group case, has to be replaced for type III_0 factors by an ergodic action of $\mathbb{R}_+^*$: the flow of weights of the type III factor. This flow is an invariant of the factor and can, by Krieger's theorem (Chapter V) be any ergodic flow, thus exhibiting an intrinsic relation between type III factors and ergodic theory and lending support to the ideas of G. Mackey on virtual subgroups. Indeed, in Mackey's terminology, a virtual subgroup of $\mathbb{R}_+^*$ corresponds exactly to an ergodic action of $\mathbb{R}_+^*$.

Since general von Neumann algebras have such an unexpected and powerful structure theory it is natural to look for them in more common parts of mathematics and to start using them as tools. After some earlier work by Singer, Coburn, Douglas, and Schaeffer, and by Shubin (whose work is the first application of type II techniques to the spectral theory of operators), a decisive step in this direction was taken up by M.F. Atiyah and I. M. Singer. They showed that the type II von Neumann algebra generated by the regular representation of a discrete group (already considered by Murray and von Neumann) provides, thanks to the *continuous dimension*, the necessary tool to measure the multiplicity of the kernel of an invariant elliptic differential operator on a Galois covering space. Moreover, they showed that the type II index on the

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