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FRANCES KIRWAN

复代数曲面

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www.wpcbj.com.cn

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CAMBRIDGE
UNIVERSITY PRESS

图书在版编目（CIP）数据

复代数曲面＝Complex Algebraic Curves：英文／（英）柯万（Kirwan,F.）著．—北京：世界图书出版公司北京公司，2008.5
ISBN 978-7-5062-9203-0

Ⅰ.复… Ⅱ.柯… Ⅲ.代数曲面—英文 Ⅳ.O187.1

中国版本图书馆CIP数据核字（2008）第051032号

书　　名:	Complex Algebraic Curves
作　　者:	Frances Kirwan
中 译 名:	复代数曲面
责任编辑:	高蓉 刘慧
出 版 者:	世界图书出版公司北京公司
印 刷 者:	三河国英印务有限公司
发　　行:	世界图书出版公司北京公司（北京朝内大街 137 号　100010）
联系电话:	010-64015659
电子信箱:	kjsk@vip.sina.com
开　　本:	24 开
印　　张:	11.5
版　　次:	2008 年 5月第 1 次印刷
版权登记:	图字:01-2008-1928
书　　号:	978-7-5062-9203-0 / O・594　　定　价: 29.00 元

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
40 West 20th Street, New York, NY 10011-4211 USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1992
Reprinted 1993, 1995

Complex Algebraic Curves, 1st ed. (ISBN978-0-521-42353-3) by Frances Kirwan first published by Cambridge University Press 1992.

This reprint edition for the People's Republic of China is published by arrangement with the Press Syndicate of the University of Cambridge, Cambridge, United Kingdom.

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Preface

This book on complex algebraic curves is intended to be accessible to any third year mathematics undergraduate who has attended courses on algebra, topology and complex analysis. It is an expanded version of notes written to accompany a lecture course given to third year undergraduates at Oxford. It has usually been the case that a number of graduate students have also attended the course, and the lecture notes have been extended somewhat for the sake of others in their position. However this new material is not intended to daunt undergraduates, who can safely ignore it. The original lecture course consisted of Chapters 1 to 5 (except for some of §3.1 including the definition of intersection multiplicities) and part of Chapter 6, although some of the contents of these chapters (particularly the introductory material in Chapter 1) was covered rather briefly.

Each section of each chapter has been arranged as far as possible so that the important ideas and results appear near the start and the more difficult and technical proofs are left to the end. Thus there is no need to finish each section before beginning the next; when the going gets tough the reader can afford to skip to the start of the next section.

The main aim of the course was to show undergraduates in their final year how the basic ideas of pure mathematics they had studied in previous years could be brought together in one of the showpieces of mathematics. In particular it was intended to provide those students not intending to continue mathematics beyond a first degree with a final year course which could be regarded as a culmination of their studies, rather than one consisting of the development of more machinery which they would never have the opportunity to use. As well as being one of the most beautiful areas of mathematics, the study of complex algebraic curves is one in which it is not necessary to develop new machinery before starting - the tools are already available from basic algebra, topology and complex analysis. It was also hoped that the course would give those students who might be tempted to continue mathematics an idea of the flavour, or rather the very varied and exciting array of flavours, of algebraic geometry, illustrating the way it draws on all parts of mathematics while avoiding as much as possible the elaborate and highly developed technical foundations of the subject.

The contents of the book are as follows. Chapter 1 "can be omitted for examination purposes" as the original lecture notes said. This chapter is simply intended to provide some motivation and historical background for the study of complex algebraic curves, and to indicate a few of the numerous reasons why they are of interest to mathematicians working in very different areas. Chapter 2 lays the foundations with the technical definitions and basic results needed to start the subject. Chapter 3 studies algebraic questions about complex algebraic curves, in particular the question of how two curves meet each other. Chapter 4 investigates what complex algebraic curves look like topologically. In Chapters 5 and 6 complex analysis is used to investigate complex algebraic curves from a third point of view. Finally Chapter 7 looks at *singular* complex algebraic curves which are much more complicated objects than nonsingular ones and are mostly ignored in the first six chapters. The three appendices contain results from algebra, complex analysis and topology which are included to make the book as self- contained as possible: they are not intended to be easily readable but simply to be available for those who feel the need to consult them.

There are many excellent books available for those who wish to study the subject further: see for example the books by Arbarello & al, Beardon, Brieskorn and Knörrer, Chern, Clemens, Coolidge, Farkas and Kra, Fulton, Griffiths, Griffiths and Harris, Gunning, Hartshorne, Jones, Kendig, Morrow and Kodaira, Mumford, Reid, Semple and Roth, Shafarevich, Springer, and Walker listed in the bibliography. Many of these references I have used to prepare the lecture course and accompanying notes on which this book was based, as well as the book itself. Indeed, the only reason I had for writing lecture notes and then this book was that each of the books listed either assumes a good deal more background knowledge than undergraduates are likely to have or else takes a very different approach to the subject.

Finally I would like to record my grateful thanks to Graeme Segal, for first suggesting that an undergraduate lecture course on this subject would be worthwhile, to all those students who attended the lecture course and the graduate students who helped run the accompanying classes for their useful comments, to David Tranah of the Cambridge University Press and Elmer Rees for their encouragement and advice on turning the lecture notes into a book, and to Mark Lenssen and Amit Badiani for their great help in producing the final version.

Frances Kirwan
Balliol College, Oxford
August 1991

Contents

Chapter 1

Introduction and background

A complex algebraic curve in $\mathbf{C}^2$ is a subset C of $\mathbf{C}^2 = \mathbf{C} \times \mathbf{C}$ of the form

$$C = \{(x, y) \in \mathbf{C}^2 : P(x, y) = 0\} \tag{1.1}$$

where $P(x, y)$ is a polynomial in two variables with complex coefficients. (See §2.1 for the precise definition). Such objects are called curves by analogy with real algebraic curves or "curved lines" which are subsets of $\mathbf{R}^2$ of the form

$$\{(x, y) \in \mathbf{R}^2 : P(x, y) = 0\} \tag{1.2}$$

where $P(x, y)$ is now a polynomial with real coefficients.

Of course to each real algebraic curve there is associated a complex algebraic curve defined by the same polynomial. Real algebraic curves were studied long before complex numbers were recognised as acceptable mathematical objects, but once complex algebraic curves appeared on the scene it quickly became clear that they have at once simpler and more interesting properties than real algebraic curves. To get some idea why this should be, consider the study of polynomial equations in one variable with real coefficients: it is easier to work with complex numbers, so that the polynomial factorises completely, and then decide which roots are real than not to allow the use of complex numbers at all.

In this book we shall study complex algebraic curves from three different points of view: algebra, topology and complex analysis. An example of the kind of algebraic question we shall ask is

"Do the polynomial equations

$$P(x, y) = 0$$

and

$$Q(x, y) = 0$$

defining two complex algebraic curves have any common solutions $(x, y) \in \mathbf{C}^2$, and if so, how many are there?"

An answer to this question will be given in Chapter 3.

The relationship of the study of complex algebraic curves with complex analysis arises when one attempts to make sense of "multi-valued holomorphic functions" such as

$$z \mapsto z^{\frac{3}{2}}$$

and

$$z \mapsto (z^3 + z^2 + 1)^{\frac{1}{2}}.$$

One ends up looking at the corresponding complex algebraic curves, in these cases

$$y^2 = x^3$$

and

$$y^2 = x^3 + x^2 + 1.$$

Complex analysis will be important in Chapter 5 and Chapter 6 of this book.

We shall also investigate the topology (that is, roughly speaking, the shape) of complex algebraic curves in Chapter 4 and §7.3. It is of course not possible to sketch a complex algebraic curve in $\mathbf{C}^2$ in the same way that we can sketch real algebraic curves in $\mathbf{R}^2$, because $\mathbf{C}^2$ has four real dimensions. None the less, we can draw sketches of complex algebraic curves (with some extra points added "at infinity"), which are accurate *topological* pictures of the curves but which do not reflect the way they sit inside $\mathbf{C}^2$. For some examples, see figure 1.1. It is important to stress the fact that these pictures can only represent the complex curves as topological spaces, and not the way they lie in $\mathbf{C}^2$. For example, the complex curve defined by $xy = 0$ is the union of the two "complex lines" defined by $x = 0$ and $y = 0$ in $\mathbf{C}^2$, which meet at the origin $(0,0)$. Topologically when we add a point at infinity to each complex line it becomes a sphere, and the complex curve becomes the union of two spheres meeting at a point as in figure 1.2. This picture, though topologically correct, represents the two complex lines as tangential to each other at the point of intersection, and this is not the case in $\mathbf{C}^2$. We cannot avoid this problem without making the complex lines look "singular" at the origin as in figure 1.3, which again is not really the case.

1.1 A brief history of algebraic curves.

Real algebraic curves have been studied for more than two thousand years, although it was not until the introduction of the systematic use of coordinates into geometry in the seventeenth century that they could be described in the form (1.2).

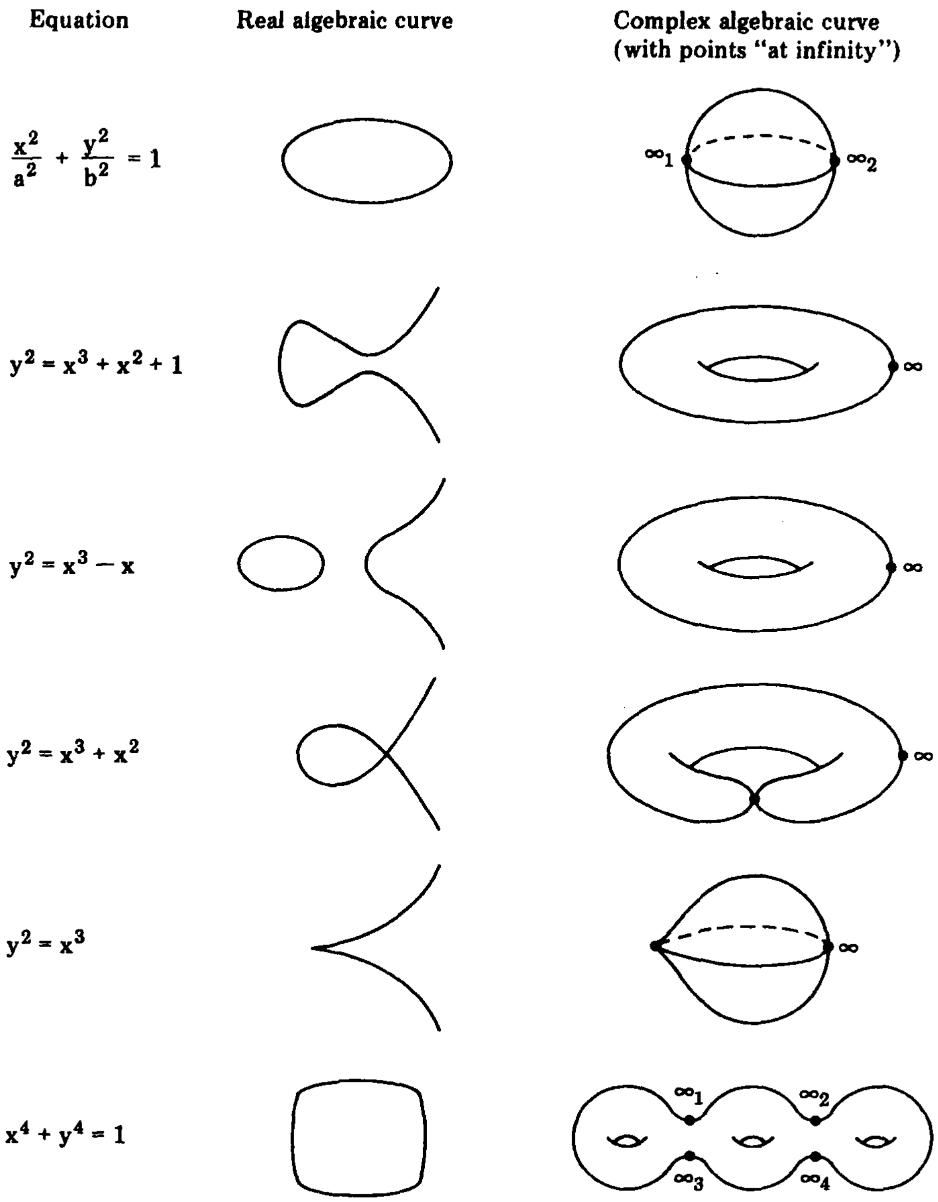

Figure 1.1: Some algebraic curves

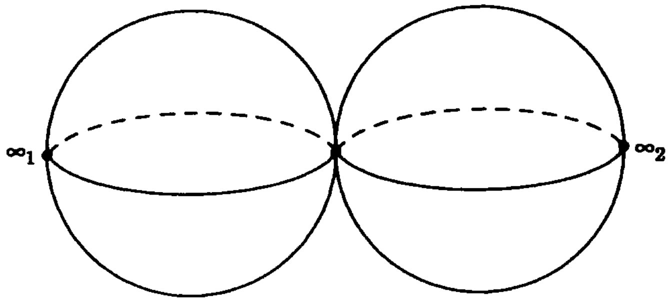

Figure 1.2: The complex curve $xy = 0$

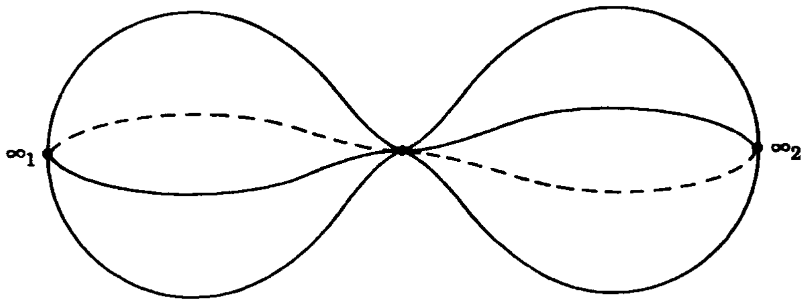

Figure 1.3: Another view of the complex curve $xy = 0$

The story starts with the Greeks, who had very sophisticated geometrical methods but a relatively primitive understanding of algebra. To them a circle was not defined by an equation

$$(x-a)^2 + (y-b)^2 = r^2$$

but was instead the locus of all points having equal distance r from a fixed point $P = (a, b)$. Similarly a parabola to the Greeks was the locus of all points having equal distance from a given point P and a given line L, while an ellipse (hyperbola) was the locus of all points for which the sum (difference) of the distances from two given points P and Q had a fixed value.

Lines and circles can of course be drawn with a ruler and compasses, and the Greeks devised more complicated mechanisms to construct parabolas, ellipses and hyperbolas. With these they were able to solve some famous problems such as "duplicating the cube"; in other words constructing a cube whose volume is twice the volume of a given cube (this was called the Delian problem. This comes down to constructing a line segment of length $2^{1/3}$ times the length of a given unit segment. The Greeks realised that this could be done by constructing the points of intersection of the parabolas

$$y^2 = 2x$$

and

$$x^2 = y.$$

They tried very hard to do this and other constructions (such as trisecting an arbitrary angle and drawing regular polygons) using ruler and compasses alone. They failed, and in fact it can be shown using Galois theory (see for example [Stewart 73] pp.57-67) that these constructions are impossible with ruler and compasses.

Besides lines and circles, ellipses, parabolas and hyperbolas the Greeks knew constructions for many other curves, for example the epicyclic curves used to describe the paths of planets before the discovery of Kepler's laws. (An epicyclic curve is the path of a point on a circle which rolls without slipping on the exterior of a fixed circle: see figure 1.4). Greek mathematics was almost forgotten in Western Europe for many centuries after the end of the Roman Empire, but in the late Middle Ages and Renaissance period it was gradually rediscovered through contact with Arab mathematicians. It was during the Renaissance that new algebraic curves were discovered by artists such as Leonardo da Vinci who were interested in drawing outlines of three-dimensional shapes in perspective.

As well as reintroducing Greek mathematics the Arabs introduced to Europe a much more sophisticated understanding of algebra and a good algebraic

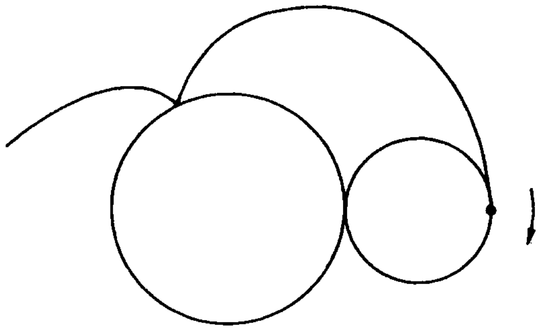

Figure 1.4: An epicyclic curve

notation. It can be difficult for us to realise how important good notation is in the solution of a mathematical problem. For example the simple argument

$$x^2 + 3 = 5x \Rightarrow (x - 5/2)^2 = 13/4 \Rightarrow x = (5 \pm \sqrt{13})/2$$

becomes much harder to express and to follow using words alone.

By the end of the seventeenth century mathematicians were familiar with the idea pioneered by Descartes and Fermat of describing a locus of points in the plane by one or more equations in two variables x and y. The methods of the differential calculus were gradually being understood and applied to curves. It was known that many real algebraic curves turned up in problems in applied mathematics (one example being the nephroid or kidney curve which is seen when light is reflected from a mirror whose cross-section is part of a circle).

Around 1700 Newton made a detailed study of cubic curves (that is, curves defined by polynomials of degree three) and described seventy two different cases. He investigated the *singularities* of a curve C defined by a polynomial $P(x, y)$, i.e. the points $(x, y) \in C$ satisfying

$$\frac{\partial P}{\partial x}(x, y) = 0 = \frac{\partial P}{\partial y}(x, y).$$

These are points where the curve does not look "smooth", such as the origin in the cubic curves defined by $y^2 = x^3 + x^2$ and $y^2 = x^3$ (cf. figure 1.1). We shall investigate the singularities of curves in greater detail in Chapter 7.

Once the use of complex numbers was understood in the nineteenth century it was realised that very often it is easier and more profitable to study the complex solutions to a polynomial equation $P(x, y) = 0$ instead of just the real solutions. For example, if we allow complex projective changes of coordinates

$$(x,y) \mapsto \left(\frac{ax+by+c}{hx+jy+k}, \frac{dx+ey+f}{hx+jy+k} \right)$$

where

$$\begin{pmatrix} a & b & c \\ d & e & f \\ h & j & k \end{pmatrix}$$

is a nonsingular matrix (see Chapter 2 for more details), then many of Newton's seventy two different cubics become equivalent to one another. In fact any complex curve defined by an irreducible cubic polynomial can be put into one of the forms

$$\begin{array}{rcll} y^2 & = & x(x-1)(x-\lambda) & \text{with } \lambda \neq 0,1 \text{ (nonsingular cubic)} \\ y^2 & = & x^2(x+1) & \text{(nodal cubic)} \\ y^2 & = & x^3 & \text{(cuspidal cubic)} \end{array}$$

(see corollary 3.34 and exercise 3.9).

Another example of the "better" behaviour of complex curves than real ones is the fact that a real algebraic curve can be so degenerate it doesn't look like a curve at all. For example, the subset

$$\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 = 0\}$$

of $\mathbf{R}^2$ is the single point $(0,0)$ and

$$\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 = -1\}$$

is the empty set. But if $P(x,y)$ is any nonconstant polynomial with complex coefficients then the subset of the *complex* space $\mathbf{C}^2$ given by

$$\{(x,y) \in \mathbf{C}^2 : P(x,y) = 0\}$$

is nonempty and "has complex dimension one" in a reasonable sense.

In the nineteenth century it was realised that if suitable "points at infinity" are added to a complex algebraic curve it becomes a compact topological space, just as the Riemann sphere is made by adding an extra point ∞ to $\mathbf{C}$. Moreover one can make sense of the concepts of holomorphic and meromorphic functions on this topological space, and much of the theory of complex analysis on $\mathbf{C}$ can be applied. This leads to the theory of Riemann surfaces[1],

[1]There is an unfortunate inconsistency of terminology in the theory of complex algebraic curves and Riemann surfaces. A complex algebraic curve is called a curve because its *complex* dimension is one, but its *real* dimension is two so it can also be called a surface.

called after Bernhard Riemann (1826-1866). Riemann was extremely influential in developing the idea that geometry should deal not only with ordinary Euclidean space but also with much more general and abstract spaces.

At the same time as Riemann and his followers were investigating complex algebraic curves using complex analysis and topology, other mathematicians began to use purely algebraic methods to obtain the same results. In 1882 Dedekind and Weber showed that much of the theory of algebraic curves remained valid when the field of complex numbers was replaced by another (preferably algebraically closed) field K. Instead of studying the curve C defined by an irreducible polynomial $P(x, y)$ directly, they studied the "field of rational functions on C" which consists of all functions $f : C \to K \cup \{\infty\}$ of the form

$$f(x, y) = Q(x, y)/R(x, y)$$

where $Q(x, y)$ and $R(x, y)$ are polynomials with coefficients in K such that $R(x, y)$ is not divisible by $P(x, y)$ (i.e. such that $R(x, y)$ does not vanish identically on C).

It is often useful to study curves defined over fields other than the fields of real and complex numbers. For example number theorists interested in the integer solutions to a diophantine equation

$$P(x, y) = 0,$$

where $P(x, y)$ is a polynomial with integer coefficients, often first regard the equation as a congruence modulo a prime number p. Such a congruence can be thought of as defining an algebraic curve over the finite field

$$\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$$

consisting of the integers modulo p, or over its algebraic closure.

By the end of the nineteenth century mathematicians had begun to make progress in studying the solutions of systems of more than one polynomial equation in more than two variables. During the twentieth century many more ideas and results have been developed in this area of mathematics (known as algebraic geometry). Algebraic curves and surfaces are now reasonably well understood, but the theory of algebraic varieties of dimension greater than two remains very incomplete. (An algebraic variety is, roughly speaking, the set of solutions to finitely many polynomial equations in finitely many variables over a field K).

The study of algebraic curves and Riemann surfaces, involving as it does a rich interplay between algebra, analysis, topology and geometry, with applications in many different areas of mathematics, has been the subject of active