Introduction to Plasma Theory

Dwight R. Nicholson
University of Iowa

John Wiley & Sons
New York • Chichester • Brisbane • Toronto • Singapore

IMPORTANT FORMULAS

(For electrons; note how each formula **scales with mass and charge.**)

Plasma Frequency: $\omega_e(s^{-1}) = \left(\frac{4\pi n e^2}{m_e}\right)^{1/2} = 2\pi\, 9000\sqrt{n(\mathrm{cm}^{-3})}$

Debye Length: $\lambda_e(\mathrm{cm}) = \left(\frac{T_e}{4\pi n e^2}\right)^{1/2} = 740\sqrt{T_e(\mathrm{eV})/n(\mathrm{cm}^{-3})}$

Gyrofrequency: $|\Omega_e|(s^{-1}) = \frac{eB}{m_e c} = 2 \times 10^7\, \mathrm{B(Gauss)}$

Plasma Parameter: $\Lambda = n\lambda_e^3 = 4 \times 10^8\, T_e^{3/2}(\mathrm{eV})/n^{1/2}(\mathrm{cm}^{-3})$

Speed: $v(\mathrm{cm/s}) = (2E/m_e)^{1/2} = 6 \times 10^7\, E^{1/2}(\mathrm{eV})$

Thermal Speed: $v_e(\mathrm{cm/s}) = (T_e/m_e)^{1/2} = 4 \times 10^7\, T_e^{1/2}(\mathrm{eV})$

Gyroradius: $r_g(\mathrm{cm}) = v_\perp/\Omega = 3E_\perp^{1/2}(\mathrm{eV})/\mathrm{B(Gauss)}$

Library of Congress Cataloging in Publication Data:

Nicholson, Dwight R. (Dwight Roy)
Introduction to plasma theory.

(A Wiley series in plasma physics, ISSN 0271-602X)
Includes bibliographies and index.
1. Plasma (Ionized gases) I. Title. II. Series.

QC718.N53 1983 530.4'4 82-13658
ISBN 0-471-09045-X

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

Preface

The purpose of this book is to teach the basic theoretical principles of plasma physics. It is not intended to be an encyclopedia of results and techniques. Nor is it intended to be used primarily as a reference book. It is intended to develop the basic techniques of plasma physics from the beginning, namely, from Maxwell's equations and Newton's law of motion. Absolutely no previous knowledge of plasma physics is assumed. Although the book is primarily intended for a one year course at the first or second year graduate level, it can also be used for a one or two semester course at the junior or senior undergraduate level. Such an undergraduate course would make use of that half of the book which assumes a knowledge only of undergraduate electricity and magnetism. The other half of the book, suitable for the graduate level, requires familiarity with complex variables, Fourier transformation, and the Dirac delta function.

The book is organized in a logical fashion. Although this is not the standard organization of an introductory course in plasma physics, I have found that students at the graduate level respond well to this organization. After the introductory material of Chapters 1 and 2 (single particle motion), the exact theories of Chapters 3 to 5 (Klimontovich and Liouville equations), which are equivalent to Maxwell's equations plus Newton's law of motion, are replaced via approximations by the Vlasov equation of Chapter 6. Further approximations lead to the fluid theory (Chapter 7) and magnetohydrodynamic theory (Chapter 8). The book concludes with two chapters on discrete particle effects (Chapter 9) and weak turbulence theory (Chapter 10). Chapter 6, and Chapters 7 and 8, are meant to be self-contained, so that the book can easily be used by instructors who wish the standard organization. Thus, the introductory material of Chapters 1 and 2 can be immediately followed by Chapters 7 and 8. This would be enough material for a one semester undergraduate course, while the first half of a two semester graduate course could continue with Chapter 6 on Vlasov theory, followed in the second semester by Chapters 3 to 5 on kinetic theory and then by Chapters 9 and 10.

It is a pleasure to acknowledge the help of many individuals in writing this book. My views on plasma physics have been shaped over the years by dozens of plasma physicists, especially Allan N. Kaufman and Martin V. Goldman. The students in graduate plasma physics courses at the University of Colorado and the University of Iowa have contributed many useful suggestions (Sun Guo-Zheng deserves special mention). The manuscript was professionally typed and edited by Alice Conwell Shank, Gail Maxwell, Susan D. Imhoff, and Janet R. Kephart. The figures were skillfully drafted by John R. Birkbeck, Jr. and Jeana K. Wonderlich. The preparation of this book was supported by the University of Colorado, the University of Iowa, the United States Department of Energy, the United States National Aeronautics and Space Administration, and the United States National Science Foundation.

Dwight R. Nicholson

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CHAPTER

CHAPTER 1

Introduction

1.1 INTRODUCTION

A *plasma* is a gas of charged particles, in which the potential energy of a typical particle due to its nearest neighbor is much smaller than its kinetic energy. The *plasma state* is the fourth state of matter: heating a solid makes a liquid, heating a liquid makes a gas, heating a gas makes a plasma. (Compare the ancient Greeks' earth, water, air, and fire.) The word plasma comes from the Greek plásma, meaning "something formed or molded." It was introduced to describe ionized gases by Tonks and Langmuir [1]. More than 99% of the known universe is in the plasma state. (Note that our definition excludes certain configurations such as the electron gas in a metal and so-called "strongly coupled" plasmas which are found, for example, near the surface of the sun. These need to be treated by techniques other than those found in this book.)

In this book, we shall always consider plasma having roughly equal numbers of singly charged ions ($+e$) and electrons ($-e$), each with average density n_0 (particles per cubic centimeter). In nature many plasmas have more than two species of charged particles, and many ions have more than one electron missing. It is easy to generalize the results of this book to such plasmas.

EXERCISE Name a well-known proposed source of energy that involves plasma with more than one species of ion.

1.2 DEBYE SHIELDING

In a plasma we have many charged particles flying around at high speeds. Consider a special test particle of charge $q_T > 0$ and infinite mass, located at the origin of a three-dimensional coordinate system containing an infinite, uniform plasma. The test charge repels all other ions, and attracts all electrons. Thus, around our test charge the electron density n_e increases and the ion density decreases. The test ion gathers a *shielding cloud* that tends to cancel its own charge (Fig. 1.1).

Consider Poisson's equation relating the electric potential φ to the charge density ρ due to electrons, ions, and test charge,

$$\nabla^2\varphi = -4\pi\rho = 4\pi e(n_e - n_i) - 4\pi q_T\,\delta(\mathbf{r}) \tag{1.1}$$

where $\delta(\mathbf{r}) \equiv \delta(x)\delta(y)\delta(z)$ is the product of three Dirac delta functions. After the introduction of the test charge, we wait for a long enough time that the electrons with temperature T_e have come to thermal equilibrium with themselves, and the ions with temperature T_i have come to thermal equilibrium with themselves, but not so long that the electrons and ions have come to thermal equilibrium with each other at the same temperature (see Section 1.6). Then equilibrium statistical mechanics predicts that

$$n_e = n_0 \exp\left(\frac{e\varphi}{T_e}\right), \qquad n_i = n_0 \exp\left(\frac{-e\varphi}{T_i}\right) \tag{1.2}$$

where each density becomes n_0 at large distances from the test charge where the potential vanishes. Boltzmann's constant is absorbed into the temperatures T_e and T_i, which have units of energy and are measured in units of electron-volts (eV).

Assuming that $e\varphi/T_e \ll 1$ and $e\varphi/T_i \ll 1$, we expand the exponents in (1.2) and write (1.1) away from $\mathbf{r} = 0$ as

$$\nabla^2\varphi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = 4\pi n_0 e^2\left(\frac{1}{T_e} + \frac{1}{T_i}\right)\varphi \tag{1.3}$$

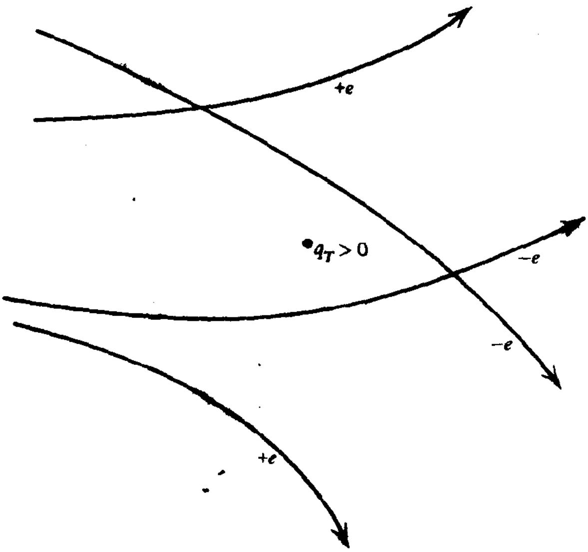

Fig. 1.1 A test charge in a plasma attracts particles of opposite sign and repels particles of like sign, thus forming a shielding cloud that tends to cancel its charge.

If we define the electron and ion *Debye lengths*

$$\lambda_{e,i} \equiv \left(\frac{T_{e,i}}{4\pi n_0 e^2}\right)^{1/2} \tag{1.4}$$

and the *total Debye length*

$$\lambda_D^{-2} = \lambda_e^{-2} + \lambda_i^{-2} \tag{1.5}$$

Eq. (1.3) then becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = \lambda_D^{-2}\varphi \tag{1.6}$$

Trying a solution of the form $\varphi = \tilde{\varphi}/r$, we find

$$\frac{d^2\tilde{\varphi}}{dr^2} = \lambda_D^{-2}\tilde{\varphi} \tag{1.7}$$

The solution that falls off properly at large distances is $\tilde{\varphi} \propto \exp(-r/\lambda_D)$. From elementary electricity and magnetism we know that the solution to (1.1) at locations very close to $\mathbf{r} = 0$ is $\varphi = q_T/r$; thus, the desired solution to (1.1) at all distances is

$$\varphi = \frac{q_T}{r}\exp\left(\frac{-r}{\lambda_D}\right) \tag{1.8}$$

The potential due to a test charge in a plasma falls off much faster than in vacuum. This phenomenon is known as *Debye shielding*, and is our first example of plasma *collective behavior*. For distances $r >>$ the Debye length λ_D, the shielding cloud effectively cancels the test charge q_T. Numerically, the Debye length of species s with temperature T_s is roughly $\lambda_s \approx 740[T_s(\text{eV})/n(\text{cm}^{-3})]^{1/2}$ in units of cm.

EXERCISE Prove that the net charge in the shielding cloud exactly cancels the test charge q_T.

It is not necessary that q_T be a special particle. In fact, each particle in a plasma tries to gather its own shielding cloud. However, since the particles are moving, they are not completely successful. In an equal temperature plasma ($T_e = T_i$), a typical slowly moving ion has the full electron component of its shielding cloud and a part of the ion component, while a typical rapidly moving electron has a part of the electron component of its shielding cloud and almost none of the ion component.

1.3 PLASMA PARAMETER

In a plasma where each species has density n_0, the distance between a particle and its nearest neighbor is roughly $n_0^{-1/3}$. The average potential energy Φ of a particle due to its nearest neighbor is, in absolute value,

$$|\Phi| \sim \frac{e^2}{r} \sim n_0^{1/3}e^2 \tag{1.9}$$

Our definition of a plasma requires that this potential energy be much less than the typical particle's kinetic energy

$$\frac{1}{2} m_s\langle v^2\rangle = \frac{3}{2} T_s = \frac{3}{2} m_s v_s^2 \tag{1.10}$$

where m_s is the mass of species s, $\langle\ \rangle$ means an average over all particle velocities at a given point in space, and we have defined the *thermal speed* v_s of species s by

$$v_s \equiv \left(\frac{T_s}{m_s}\right)^{1/2} \tag{1.11}$$

For electrons, $v_e \approx 4 \times 10^7\ T_e^{1/2}$ (eV) in units of cm/s. Our definition of a plasma requires

$$n_0^{1/3} e^2 << T_s \tag{1.12}$$

or

$$n_0^{2/3} \left(\frac{T_s}{n_0\, e^2}\right) >> 1 \tag{1.13}$$

Raising each side of (1.13) to the 3/2 power, and recalling the definition (1.4) of the Debye length, we have (dropping factors of 4π, etc.)

$$\boxed{\Lambda_s \equiv n_0 \lambda_s^3 >> 1} \tag{1.14}$$

where Λ_s is called the *plasma parameter of species s*. (*Note:* Some authors call Λ_s^{-1} the plasma parameter.) The plasma parameter is just the number of particles of species s in a box each side of which has length the Debye length (a Debye cube). Equation (1.14) tells us that, by definition, a plasma is an ionized gas that has many particles in a Debye cube. Numerically, $\Lambda_s \approx 4 \times 10^8\ T_s^{3/2}(\text{eV})/n_0^{1/2}(\text{cm}^{-3})$. We will often substitute the total Debye length λ_D in (1.14), and define the result $\Lambda \equiv n_0\, \lambda_D^3$ to be the *plasma parameter.*

EXERCISE Evaluate the electron thermal speed, electron Debye length, and electron plasma parameter for the following plasmas.

(a) A tokamak or mirror machine with $T_e \approx 1$ keV, $n_0 \approx 10^{13}$ cm^{-3}.
(b) The solar wind near the earth with $T_e \approx 10$ eV, $n_0 \approx 10$ cm^{-3}.
(c) The ionosphere at 300 km above the earth's surface with $T_e \approx 0.1$ eV, $n_0 \approx 10^6$ cm^{-3}.
(d) A laser fusion, electron beam fusion, or ion beam fusion plasma with $T_e \approx 1$ keV, $n_0 \approx 10^{20}$ cm^{-3}.
(e) The sun's center with $T_e \approx 1$ keV, $n_0 \approx 10^{23}$ cm^{-3}.

It is fairly easy to see why many ionized gases found in nature are indeed plasmas. If the potential energy of a particle due to its nearest neighbor were greater than its kinetic energy, then there would be a strong tendency for electrons and ions to bind together into atoms, thus destroying the plasma. The need to keep ions and electrons from forming bound states means that most plasmas have temperatures in excess of one electron-volt.

EXERCISE The temperature of intergalactic plasma is currently unknown, but it could well be much lower than 1 eV. How could the plasma maintain itself at such a low temperature? (*Hint:* $n_0 \approx 10^{-5}$ cm^{-3}).

Of course, it is possible to find situations where a plasma exists jointly with another state. For example, in the lower ionosphere there are regions where 99% of the atoms are neutral and only 1% are ionized. In this *partially ionized plasma*, the ionized component can be a legitimate plasma according to (1.14), where Λ_s should be calculated using only the parameters of the ionized component. Typically, there will be a continuous exchange of particles between the unionized gas and the ionized plasma, through the processes of atomic recombination and ionization.

We can now evaluate the validity of the assumption made before (1.3), that $e\varphi/T_s << 1$. This assumption is most severe for the nearest neighbor to the test charge (which we now take to have charge $q_T = +e$). Using the unshielded form of the potential, we require

$$\frac{e}{T_s}\left(\frac{e}{r}\right) \approx \frac{e}{T_s}\left(\frac{e}{n_0^{-1/3}}\right) << 1 \tag{1.15}$$

or

$$n_0^{1/3} e^2 << T_s \tag{1.16}$$

which is just the condition (1.12) required by the definition of a plasma. Thus, our derivation of Debye shielding is correct for any ionized gas that is indeed a plasma.

1.4 PLASMA FREQUENCY

Consider a hypothetical slab of plasma of thickness L, where for the present we consider the ions to have infinite mass, but equal density n_0 and opposite charge to the electrons while the electrons are held rigidly in place with respect to each other, but can move freely through the ions. Suppose the electron slab is displaced a distance δ to the right of the ion slab and then allowed to move freely (Fig. 1.2). What happens?

An electric field will be set up, causing the electron slab to be pulled back toward the ions. When the electrons exactly overlap the ions, the net force is zero, but the electron slab has substantial speed to the left. Thus, the electron slab overshoots, and the net result is harmonic oscillation. The frequency of the oscillation is called the *electron plasma frequency*. It depends only on the electron density, the electron charge, and the electron mass. Let's calculate it.

Poisson's equation in one dimension is ($\partial_x \equiv \partial/\partial x$)

$$\partial_x E = 4\pi\rho \tag{1.17}$$

where E is the electric field. Referring to Fig. 1.3, we take the boundary condition $E(x = 0) = 0$, and assume throughout that $\delta << L$. From (1.17) the electric field over most of the slab is $4\pi n_0 e\delta$, and the force per unit area on the electron slab is (electric field) $\times$ (charge per unit area) or $-4\pi n_0^2 e^2 \delta L$. Newton's second law is

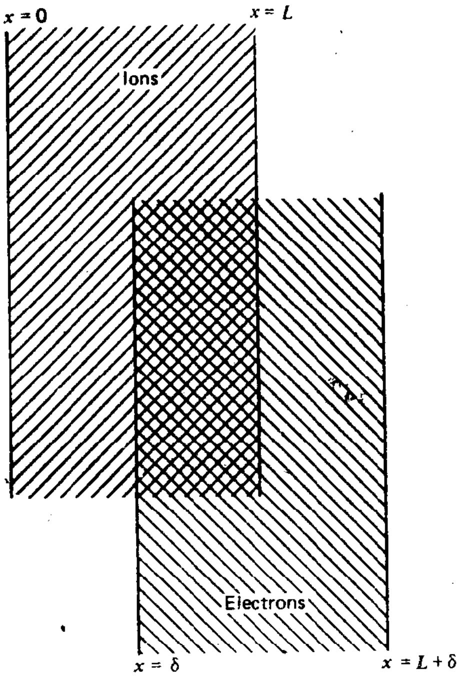

Fig. 1.2 Plasma slab model used to calculate the plasma frequency.

(force per unit area) = (mass per unit area) × (acceleration), or

$$(-4\pi n_0^2 e^2 \delta L) = (n_0 m_e L)(\ddot{\delta}) \tag{1.18}$$

where an overdot is a time derivative. Equation (1.18) is in the standard form of a harmonic oscillator equation,

$$\ddot{\delta} + \left(\frac{4\pi n_0 e^2}{m_e}\right) \delta = 0 \tag{1.19}$$

with characteristic frequency

$$\boxed{\omega_e \equiv \left(\frac{4\pi n_0 e^2}{m_e}\right)^{1/2}} \tag{1.20}$$

which is called the *electron plasma frequency*. Numerically, $\omega_e = 2\pi \times 9000\, n_e^{1/2}$ (cm^{-3}) in units of s^{-1}.

EXERCISE Calculate the electron plasma frequency ω_e and $\omega_e/2\pi$ (e.g., in MHz and kHz) for the five plasmas in the exercise below (1.14).

By analogy with the electron plasma frequency (1.20) we define the ion plasma frequency ω_i for a general ion species with density n_i and ion charge Ze as

$$\omega_i \equiv \left(\frac{4\pi n_i Z^2 e^2}{m_i}\right)^{1/2} \tag{1.21}$$

The total plasma frequency ω_p for a two-component plasma is defined as

$$\omega_p^2 \equiv \omega_e^2 + \omega_i^2 \tag{1.22}$$

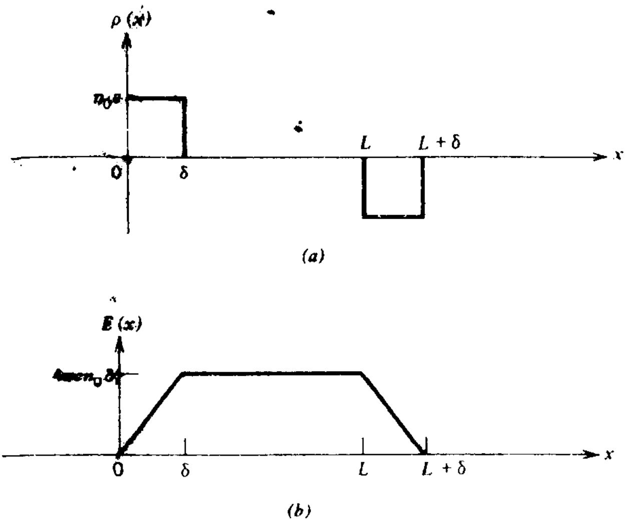

Fig. 1.3 Calculation of the electron plasma frequency. (*a*) Charge density. (*b*) Electric field.

(See Problem 1.3.) For most plasmas in nature $\omega_e >> \omega_i$, so $\omega_p^2 \approx \omega_e^2$. We will see in a later chapter that the general response of an unmagnetized plasma to a perturbation in the electron density is a set of oscillations with frequencies very close to the electron plasma frequency ω_e.

The relation among the Debye length λ_s, the plasma frequency ω_s, and the thermal speed v_s, for the species s, is

$$\boxed{\lambda_s = v_s/\omega_s} \tag{1.23}$$

EXERCISE Demonstrate (1.23).

1.5 OTHER PARAMETERS

Many of the plasmas in nature and in the laboratory occur in the presence of magnetic fields. Thus, it is important to consider the motion of an individual charged particle in a magnetic field. The Lorentz force equation for a particle of charge q_s and mass m_s moving in a constant magnetic field $\mathbf{B} = B_0\hat{z}$ is

$$m_s\ddot{\mathbf{r}} = \frac{q_s}{c}(\dot{\mathbf{r}} \times B_0\hat{z}) \tag{1.24}$$

For initial conditions $\mathbf{r}(t = 0) = (x_0, y_0, z_0)$ and $\dot{\mathbf{r}}(t = 0) = (0, v_\perp, v_z)$ the solution of (1.24) is

$$x(t) = x_0 + \frac{v_\perp}{\Omega_s}(1 - \cos \Omega_s t)$$

$$y(t) = y_0 + \frac{v_\perp}{\Omega_s}\sin \Omega_s t$$

$$z(t) = z_0 + v_z t \tag{1.25}$$

where we have defined the *gyrofrequency*

$$\boxed{\Omega_s \equiv \frac{q_s B_0}{m_s c}} \tag{1.26}$$

EXERCISE Verify that (1.25) is the solution of (1.24) with the desired initial conditions.

Numerically, $\Omega_e = -2 \times 10^7 B_0$ (gauss, abbreviated G) in units of s^{-1}, and $\Omega_i \doteq 10^4 B_0$ (gauss) in units of s^{-1} if the ions are protons.

The nature of the motion (1.25) is a constant velocity in the $\hat{z}$-direction, and a circular gyration in the x-y plane with angular frequency $|\Omega_s|$ and center at the *guiding center* position $\mathbf{r}_{gc}$ given by

$$\mathbf{r}_{gc} = (x_0 + v_\perp/\Omega_s, y_0, z_0 + v_z t) \tag{1.27}$$

The radius of the circle in the x-y plane is the *gyroradius* $v_\perp/|\Omega_s|$. The *mean gyroradius* r_s of species s is defined by setting $v_\perp$ equal to the thermal speed, so

$$\boxed{r_s \equiv v_s/|\Omega_s|} \tag{1.28}$$

EXERCISE In the exercise below (1.14), calculate and order the frequencies ω_e, ω_i, $|\Omega_e|$, Ω_i; also calculate the gyroradii r_e and r_i; take $T_i = T_e$ and use the following parameters.
(a) Protons, $B_0 = 10$ kG.
(b) Protons, $B_0 = 10^{-5}$ G.
(c) O^+ ions, $B_0 = 0.5$ G.
(d) Deuterons, $B_0 = 0$ and $B_0 = 10^6$ G.
(e) Protons, $B_0 = 100$ G.

At this point, let us briefly mention relativistic and quantum effects. For simplicity, we shall always treat nonrelativistic plasmas. In principle, there is no difficulty in generalizing any of the results of this course to include special relativistic effects; these are discussed at length in the book by Clemmow and Dougherty [2].

EXERCISE To what regime of electron temperature are we limited by the nonrelativistic assumption? How about ion temperature if the ions are protons?

There are, of course, many plasmas in which special relativistic effects do become important. For example, cosmic rays may be thought of as a component of the interstellar and intergalactic plasma with relativistic temperature.

We shall also neglect quantum mechanical effects. For most of the laboratory and astrophysical plasmas in which we might be interested, this is a good assumption. There are, of course, plasmas in which quantum effects are very important. An example would be solid state plasmas. As a rough criterion for the neglect of quantum effects, one might require that the typical de Broglie length $h/m_s v_s$ be much less than the average distance between particles $n_0^{-1/3}$.