GREEN'S FUNCTIONS AND BOUNDARY VALUE PROBLEMS

IVAR STAKGOLD
University of Delaware

A WILEY-INTERSCIENCE PUBLICATION

JOHN WILEY & SONS, New York • Chichester • Brisbane • Toronto

Library of Congress Cataloging in Publication Data

Stakgold, Ivar.
Green's functions and boundary value problems.

(Pure and applied mathematics)
"A Wiley-Interscience publication."
Includes index.
1. Boundary value problems. 2. Green's functions.
3. Mathematical physics. I. Title.

QA379.S72 515'.35 78-27259
ISBN 0-471-81967-0

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

Preface

As a result of graduate-level adoptions of my earlier two-volume book, *Boundary Value Problems of Mathematical Physics*, I received many constructive suggestions from users. One frequent recommendation was to consolidate and reorganize the topics into a single volume that could be covered in a one-year course. Another was to place additional emphasis on modeling and to choose examples from a wider variety of physical applications, particularly some emerging ones. In the meantime my own research interests had turned to nonlinear problems, so that, inescapably, some of these would also have to be included in any revision. The only way to incorporate these changes, as well as others, was to write a new book, whose main thrust, however, remains the systematic analysis of boundary value problems. Of course some topics had to be dropped and others curtailed, but I can only hope that your favorite ones are not among them.

My book is aimed at graduate students in the physical sciences, engineering, and applied mathematics who have taken the typical "methods" course that includes vector analysis, elementary complex variables, and an introduction to Fourier series and boundary value problems. Why go beyond this? A glance at modern publications in science and engineering provides the answer. To the lament of some and the delight of others, much of this literature is deeply mathematical. I am referring not only to areas such as mechanics and electromagnetic theory that are traditionally mathematical but also to relative newcomers to mathematization, such as chemical engineering, materials science, soil mechanics, environmental engineering, biomedical engineering, and nuclear engineering. These fields give rise to challenging mathematical problems whose flavor can be sensed from the following short list of examples; integrodifferential equations of neutron transport theory, combined diffusion and reaction in chemical and environmental engineering, phase transitions in metallurgy, free boundary problems for dams in soil mechanics, propagation of impulses along nerves in biology. It would be irresponsible and foolish to claim that readers of

my book will become instantaneous experts in these fields, but they will be prepared to tackle many of the mathematical aspects of the relevant literature.

Next, let me say a few words about the numbering system. The book is divided into ten chapters, and each chapter is divided into sections. Equations do *not* carry a chapter designation. A reference to, say, equation (4.32) is to the thirty-second numbered equation in Section 4 of the chapter you happen to be reading. The same system is used for figures and exercises, the latter being found at the end of sections. The exercises, by the way, are rarely routine and, on occasion, contain substantial extensions of the main text. Examples do not carry any section designation and are numbered consecutively within a section, even though there may be separate clusters of examples within the same section. Some theorems have numbers and others do not; those that do are numbered in a sequence within a section—Theorem 1, Theorem 2, and so on.

A brief description of the book's contents follows. No attempt is made to mention all topics covered; only the general thread of the development is indicated.

Chapter 0 presents background material that consists principally of careful derivations of several of the equations of mathematical physics. Among them are the equations of heat conduction, of neutron transport, and of vibrations of rods. In the last-named derivation an effort is made to show how the usual linear equations for beams and strings can be regarded as first approximations to nonlinear problems. There are also two short sections on modes of convergence and on Lebesgue integration.

Many of the principal ideas related to boundary value problems are introduced on an intuitive level in Chapter 1. A boundary value problem (BVP, for short) consists of a differential equation $Lu=f$ with boundary conditions of the form $Bu=h$. The pair (f,h) is known collectively as the data for the problem, and u is the response to be determined. Green's function is the response when f represents a concentrated unit source and $h=0$. In terms of Green's function, the BVP with arbitrary data can be solved in a form that shows clearly the dependence of the solution on the data. Various examples are given, including some multidimensional ones, some involving interface conditions, and some initial value problems. The useful notion of a well-posed problem is discussed, and a first look is taken at maximum principles for differential equations.

Chapter 2 deals with the theory of distributions, which provides a rigorous mathematical framework for singular sources such as the point charges, dipoles, line charges, and surface layers of electrostatics. The notion of response to such sources is made precise by defining the

distributional solution of a differential equation. The related concepts of weak solution, adjoint, and fundamental solution are also introduced. Fourier series and Fourier transforms are presented in both classical and distributional settings.

Chapter 3 returns to a more detailed study of one-dimensional linear boundary value problems. To an equation of order p there are usually associated p independent boundary conditions involving derivatives of order less than p at the endpoints a and b of a bounded interval. If the corresponding BVP with 0 data has only the trivial solution, then the BVP with arbitrary data has one and only one solution which can be expressed in terms of Green's function. If, however, the BVP with 0 data has a nontrivial solution, certain solvability conditions must be satisfied for the BVP with arbitrary data to have a solution. These statements are formulated precisely in an alternative theorem, which recurs throughout the book in various forms. When the BVP with 0 data has a nontrivial solution, Green's function cannot be constructed in the ordinary way, but some of its properties can be salvaged by using a modified Green's function, defined in Section 5.

Chapter 4 begins the study of Hilbert spaces. A Hilbert space is the proper setting for many of the linear problems of applied analysis. Though its elements may be functions or abstract "vectors," a Hilbert space enjoys all the algebraic and geometric properties of ordinary Euclidean space. A Hilbert space is a linear space equipped with an inner product that induces a natural notion of distance between elements, thereby converting it into a metric space which is required to be complete. Some of the important geometric properties of Hilbert spaces are developed, including the projection theorem and the existence of orthonormal bases for separable spaces. Metric spaces can be useful quite apart from any linear structure. A contraction is a transformation on a metric space that uniformly reduces distances between pairs of points. A contraction on a complete metric space has a unique fixed point that can be calculated by iteration from any initial approximation. Examples demonstrate how to use these ideas to prove uniqueness and constructive existence for certain classes of nonlinear differential equations and integral equations.

Chapter 5 examines the theory of linear operators on a separable Hilbert space, particularly integral and differential operators, the latter being unbounded operators. The principal problem of operator theory is the solution of the equation $Au=f$, where A is a linear operator and f an element of the space. A thorough discussion of this problem leads again to adjoint operators, solvability conditions, and alternative theorems. Additional insight is obtained by considering the inversion of the equation

$Au-\lambda u=f$, which leads to the idea of the spectrum, a generalization of the more familiar concept of eigenvalue. For compact operators (which include most integral operators) the inversion problem is essentially solved by the Riesz-Schauder theory of Section 7. Section 8 relates the spectrum of symmetric operators to extremal principles for the Rayleigh quotient. Throughout, the theory is illustrated by specific examples.

In Chapter 6 the general ideas of operator theory are specialized to integral equations. Integral equations are particularly important as alternative formulations of boundary value problems. Special emphasis is given to Fredholm equations with symmetric Hilbert-Schmidt kernels. For the corresponding class of operators, the nonzero eigenvalues and associated eigenfunctions can be characterized through successive extremal principles, and it is then possible to give a complete treatment of the inhomogeneous equation. The last section discusses the Ritz procedure for estimating eigenvalues, as well as other approximation methods for eigenvalues and eigenfunctions. There is also a brief introduction to integrodifferential operators in Exercises 5.3 to 5.8.

Chapter 7 extends the Sturm-Liouville theory of second-order ordinary differential equations to the case of singular endpoints. It is shown, beginning with the regular case, how the necessarily discrete spectrum can be constructed from Green's function. A formal extension of this relationship to the singular case makes it possible to calculate the spectrum, which may now be partly continuous. The transition from regular to singular is analyzed rigorously for equations of the first order, but the Weyl classification for second-order equations is given without proof. The eigenfunction expansion in the singular case can lead to integral transforms such as Fourier, Hankel, Mellin, and Weber. It is shown how to use these transforms and their inversion formulas to solve partial differential equations in particular geometries by separation of variables.

Although partial differential equations have appeared frequently as examples in earlier chapters, they are treated more systematically in Chapter 8. Examination of the Cauchy problem—the appropriate generalization of the initial value problem to higher dimensions—gives rise to a natural classification of partial differential equations into hyperbolic, parabolic, and elliptic types. The theory of characteristics for hyperbolic equations is introduced and applied to simple linear and nonlinear examples. In the second and third sections various methods (Green's functions, Laplace transforms, images, etc.) are used to solve BVPs for the wave equation, the heat equation, and Laplace's equation. The simple and double layers of potential theory make it possible to reduce the Dirichlet problem to an integral equation on the boundary of the domain, thereby providing a rather weak existence proof. In Section 4 a stronger existence

proof is given, using variational principles. Two-sided bounds for some functionals of physical interest, such as capacity and torsional rigidity, are obtained by introducing complementary principles. Another application involving level-line analysis is also given, and there is a very brief treatment of unilateral constraints and variational inequalities.

Finally, in Chapter 9, a number of methods applicable to nonlinear problems are developed. Section 1 points out some of the features that distinguish nonlinear problems from linear ones and illustrates these differences through some simple examples. In Section 2 the principal qualitative results of branching theory (also known as bifurcation theory) are presented. The phenomenon of bifurcation is understood most easily in terms of the buckling of a rod under compressive thrust. As the thrust is increased beyond a certain critical value, the state of simple compression gives way to the buckled state with its appreciable transverse deflection. Section 3 shows how a variety of linear problems can be handled by perturbation theory (inhomogeneous problems, eigenvalue problems, change in boundary conditions, domain perturbations). These techniques, as well as monotone methods, are then adapted to the solution of nonlinear BVPs. The concluding section discusses the possible loss of stability of the basic steady state when an underlying parameter is allowed to vary.

I have already acknowledged my debt to the students and teachers who were kind enough to comment on my earlier book. There are, however, two colleagues to whom I am particularly grateful: Stuart Antman, who generously contributed the ideas underlying the derivation of the equations for rods in Chapter 0, and W. Edward Olmstead, who suggested some of the examples on contractions in Chapter 4 and on branching in Chapter 9.

IVAR STAKGOLD

Newark, Delaware
April 1979

Contents

0

Preliminaries

As its name and number indicate, this chapter contains background material that has no precise place in the systematic development beginning with Chapter 1. Many readers may prefer to look over this preliminary material rather casually and then refer to it again as need arises.

The principal purpose here is to give fairly careful derivations of some of the equations of mathematical physics that will be studied extensively in the rest of the book. The attention paid to modeling in the present chapter could, regrettably, not be sustained in subsequent ones. Readers who are particularly interested in this aspect of applied mathematics are encouraged to consult the books by Lin and Segel, by Aris (1978), and by Segel.

Two other isolated sections of a mathematical nature complete the chapter. One section reviews some fundamental ideas on convergence. The other presents a short treatment of Lebesgue integration. Although only a few essential properties of this integral are needed in the book, it seemed worthwhile to take a few pages to explain its construction. These limited goals made it convenient to use Tonelli's approach as presented in Silverman's fine translation of Shilov's book.

A few words about terminology are in order. R_n stands for n-dimensional Euclidean space. The definitions below are given for R_3 but are easily modified for R_n. A point in R_3 is identified by its position vector $\mathbf{x}=(x_1,x_2,x_3)$, where x_1,x_2,x_3 are *Cartesian* coordinates; $|\mathbf{x}|=(x_1^2+x_2^2+x_3^2)^{1/2}$, where the nonnegative square root is understood; $d\mathbf{x}$ stands for a volume element $dx_1\,dx_2\,dx_3$. In later chapters the distinguishing notation for vectors is dropped.

An *open ball* of radius a, centered at the origin, is the set of points $\mathbf{x}$ such that $|\mathbf{x}|<a$. The set $|\mathbf{x}|\leq a$ is a *closed ball*, and the set $|\mathbf{x}|=a$ is a *sphere*. In R_2 the words *disk* and *circle* are often substituted for ball and sphere, respectively. An *open set* Ω has the property that, whenever $x\in\Omega$, so does some sufficiently small ball with center at $\mathbf{x}$. A point $\mathbf{x}$ belongs to the

boundary Γ of an open set Ω if $\mathbf{x}$ is not in Ω but if every open ball centered at $\mathbf{x}$ contains a point of Ω. The *closure* $\bar{\Omega}$ of Ω is the union of Ω and Γ. These ideas are best illustrated by an egg with a very thin shell. The interior of the egg is an open set Ω, the shell is Γ, and the egg with shell is $\bar{\Omega}$. An open set Ω is *connected* if each pair of points in Ω can be connected by a curve lying entirely in Ω. A *domain* is an open connected set. Thus an open ball is a domain, but the union of two disjoint open balls is not.

The symbol $\doteq$ means "set equal to." It is occasionally used to define a new expression. For instance, in writing $D \doteq dS/dx$ we are defining D as dS/dx which, in turn, is presumably known from earlier discussion.

The terms

$$\inf_{\mathbf{x}\in\Omega} f(\mathbf{x}), \quad \sup_{\mathbf{x}\in\Omega} f(\mathbf{x})$$

stand for the infimum (greatest lower bound) and supremum (least upper bound) of the real-valued function f on Ω. For instance, if Ω is the open ball in R_n with radius a and center at the origin, and $f(\mathbf{x}) = |\mathbf{x}|$, then

$$\inf_{\mathbf{x}\in\Omega} f(\mathbf{x}) = 0, \quad \sup_{\mathbf{x}\in\Omega} f(\mathbf{x}) = a,$$

even though the supremum is not attained for any element $\mathbf{x}$ in Ω.

1. HEAT CONDUCTION

We shall consider the flow of heat in an inhomogeneous medium occupying the three-dimensional domain Ω with boundary Γ. The temperature $u(\mathbf{x}, t)$ is a scalar function defined for $\mathbf{x}$ in Ω and a time interval $t_1 < t < t_2$. In the general situation there will exist within Ω certain sources of heat, known as *body sources*, whose nature will be specified more precisely later. The discussion of boundary and initial conditions may safely be postponed since they do not affect internal heat balances.

Let R be an *arbitrary* portion of Ω with boundary B. It is vital that R be allowed to range through a variety of subdomains of Ω, so that we can obtain sufficient information for our purposes. A heat balance over R for the time interval $(t, t + dt)$ gives

(1.1) heat produced by body sources

= rise in heat content + outflow of heat through B.

We now make two physical assumptions that have successfully weathered the passage of time:

1. If a material element of volume $d\mathbf{x}$ is raised from the temperature u to the temperature $u + du$, its heat content is raised by $C du\, d\mathbf{x}$, where C is

the *specific heat* (which may depend on u) in calories per degree per cubic centimeter.

2. Fourier's law: Consider an element of surface with normal $\mathbf{n}$ and area dS. The amount of heat flowing in time dt across this element is

$$dt\,dS\,\mathbf{n}\cdot(-k\,\mathrm{grad}\,u) = -dt\,dS\,k\frac{\partial u}{\partial n},$$

where k is the thermal conductivity (which may depend on u) measured in calories per second per centimeter per degree. The vector $-k\,\mathrm{grad}\,u$ is known as the *heat flow vector*, the minus sign being consistent with the fact that heat flows in the direction of decreasing temperature.

Since our medium may be inhomogeneous, both C and k may depend on $\mathbf{x}$ as well as on u. If we let $du = du(\mathbf{x}, t)$ stand for the rise in temperature at $\mathbf{x}$ in the time from t to $t+dt$, the change in heat content of R in that time is given by

$$\int_R C\,du\,d\mathbf{x}, \tag{1.2}$$

and the heat flowing outward through B by

$$-dt\int_B k\frac{\partial u}{\partial n}\,dS, \tag{1.3}$$

where n is the outward normal to B.

In the absence of sources that liberate heat instantaneously in time, we can write the heat produced by body sources as $F_R(t)\,dt$, where $F_R(t)$ is the *rate*, in calories per second, at which body sources generate heat in the whole of R. After division by dt and passage to the limit, (1.1) takes the form

$$F_R(t) = \int_R C(\mathbf{x}, u)\frac{\partial u}{\partial t}\,d\mathbf{x} - \int_B k(\mathbf{x}, u)\frac{\partial u}{\partial n}\,dS, \tag{1.4}$$

which we regard as the *primary equation* describing heat conduction. From (1.4) we can specialize in different directions, one of which leads to the familiar but more restrictive partial differential equation of heat conduction.

By excluding spatially "singular" sources (those that are concentrated at points, on curves, or on surfaces), we can write

$$F_R(t) = \int_\Omega f(\mathbf{x}, t)\,d\mathbf{x}, \tag{1.5}$$

where $f(\mathbf{x},t)$, measured in calories per second per cubic centimeter, is the volume rate at which body sources produce heat at $(\mathbf{x},t)$. The corresponding temperature $u(\mathbf{x},t)$ will then be smooth enough so that the divergence theorem can be used on the surface integral in (1.4) to give

$$\int_R \left[C\frac{\partial u}{\partial t} - \operatorname{div}(k \operatorname{grad} u) - f(\mathbf{x},t) \right] d\mathbf{x} = 0, \tag{1.6}$$

which holds for every portion R of Ω. The integrand is a function of $\mathbf{x}$ and t defined for $\mathbf{x}\in\Omega$ and $t_1<t<t_2$. Assuming that this function is continuous, we claim it must vanish identically. Indeed, if the function differed from 0 at a point $\mathbf{x}$ at time t, there would exist a neighborhood R of that point such that the function would be of one sign throughout R at time t. The integral in (1.6) would then fail to vanish for that particular R and t. We therefore conclude that

$$C\frac{\partial u}{\partial t} - \operatorname{div}(k \operatorname{grad} u) = f, \qquad \mathbf{x}\in\Omega, \quad t_1<t<t_2, \tag{1.7}$$

which is the usual equation of heat conduction. If C and k are *constants*, the equation reduces to

$$\frac{\partial u}{\partial t} - a\Delta u = \frac{f}{C}, \tag{1.8}$$

where $a = k/C$ is the *thermal diffusivity* in square centimeters per second, and $\Delta \doteq \operatorname{div}\operatorname{grad}$ is the Laplacian operator, which has the familiar form

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

in Cartesian coordinates.

Remark. In the simplest case the production term f is prescribed as a function of $\mathbf{x}$ and t. In that case (1.8) is a linear inhomogeneous equation. There are many problems, however, in which the production term also depends on the unknown temperature u. If, for instance, a *chemical reaction* takes place which liberates heat, it is reasonable to assume that the rate at which heat is released is given by the Arrhenius law:

$$Ae^{-B/u}, \tag{1.9}$$

where A and B are known positive constants. We must therefore substitute

$$f(\mathbf{x},t)=Ae^{-B/u(\mathbf{x},t)}$$

on the right sides of (1.7) and (1.8), so that these become nonlinear partial differential equations.

In *steady-state heat conduction* the temperature $u(\mathbf{x})$ is independent of time, and (1.4) reduces to

$$-\int_B k\frac{\partial u}{\partial n}\,dS=F_R, \tag{1.10}$$

where F_R is the steady heat input per unit time over the domain R bounded by B. If these body sources stem from a volume density $f(\mathbf{x})$, we can use reasoning similar to that yielding (1.7) to conclude that

$$-\operatorname{div}(k\operatorname{grad}u)=f(\mathbf{x}), \qquad \mathbf{x}\in\Omega. \tag{1.11}$$

To show that (1.10) is indeed more general than (1.11), consider the case of a single source concentrated at $\mathbf{x}=\boldsymbol{\xi}$ and generating p calories per second. Then (1.10) tells us that

$$-\int_B k\frac{\partial u}{\partial n}\,dS=\begin{cases}0 & \text{if } \boldsymbol{\xi} \text{ is not in } R,\\ p & \text{if } \boldsymbol{\xi} \text{ is in } R.\end{cases} \tag{1.12}$$

By a now familiar argument, the first line shows that $-\operatorname{div}(k\operatorname{grad}u)=0$ for $\mathbf{x}\neq\boldsymbol{\xi}$. By specializing the second line to a small sphere centered at $\boldsymbol{\xi}$, it is possible to extract precise information on the nature of the singularity in u at $\boldsymbol{\xi}$ (see Section 4, Chapter 1, for instance).

Whenever there is possible ambiguity in the interpretation of (1.11), it is wise to return to the integral formulation (1.10) for guidance. As another example, suppose that Ω consists of two media separated by an interface σ. The thermal conductivity k is continuous with the possible exception of a jump discontinuity across σ. Assuming that there are no prescribed sources or any heat losses (caused by films or imperfect fitting) on the interface, we can apply (1.10) to a thin pillbox straddling the interface with its bases parallel to σ. It is permissible to neglect the contribution from the lateral surface of the pillbox to obtain

$$k_+\frac{\partial u}{\partial n_+}=k_-\frac{\partial u}{\partial n_-}, \tag{1.13}$$

where the subscripts + and − denote the two sides of σ. Equation (1.13) and the continuity of u comprise the interface conditions (see also Section 4, Chapter 1).

Despite their usefulness (1.10) and (1.4) are not general enough to treat certain important idealized singularities. For instance, a dipole located at ξ would apparently go undetected in (1.10) since $F_R = 0$ whether ξ is in R or outside R. We shall see in Chapter 2 that such problems are best handled within the theory of distributions.

Boundary and Initial Conditions

Equation (1.7) alone does not determine the temperature $u(\mathbf{x}, t)$. We must in addition give the initial temperature $u(\mathbf{x}, t_1)$ and a boundary condition on Γ for $t_1 < t < t_2$. This boundary condition is usually of one of the following three types:

$$(1.14)\quad \begin{cases} \text{(i)} & \text{temperature } u \text{ prescribed on } \Omega \text{ for } t_1 < t < t_2, \\ \text{(ii)} & \text{heat flow } -k\dfrac{\partial u}{\partial n} \text{ prescribed on } \Gamma \text{ for } t_1 < t < t_2, \\ \text{(iii)} & -k\dfrac{\partial u}{\partial n} = \alpha u \text{ on } \Gamma \text{ for } t_1 < t < t_2. \end{cases}$$

For $\alpha > 0$ the last condition is Newton's law of cooling, which characterizes radiation into a surrounding medium at uniform temperature [which is then taken to be the datum of temperature in (1.7)]. Thus heat is lost from the surface of the body at a rate proportional to the difference between the surface temperature and the surrounding temperature. If $\alpha < 0$, condition (iii) states that the larger the boundary temperature the more heat enters the body, a circumstance which obviously tends to increase the internal temperature. Newton's law of cooling can be regarded as an approximation to Stefan's radiation law, which has a term βu^4 on the right side of (iii). It is of course possible to have an inhomogeneous version of (iii) if the surface is simultaneously heated by prescribed heat flow.

One-Dimensional Problems

Equation (1.7) can sometimes be reduced to an equation involving derivatives in only one space direction. Let (x_1, x_2, x_3) be Cartesian coordinates; we want to describe classes of problems in which u depends only on x_1 and t.

1. Suppose Ω is the slab $0 < x_1 < a$, $-\infty < x_2, x_3 < \infty$. Assume that the source term depends only on x_1 and t, that the boundary conditions on the

faces of the slab depend only on t, that the initial temperature depends only on x_1, and that k and C depend only on x_1 and u. On physical grounds it is clear that heat flows only in the x_1 direction, so that u will be a function of x_1 and t alone. The differential equation (1.7) then becomes

$$C\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_1}\left(k\frac{\partial u}{\partial x_1}\right) = f(x_1,t), \qquad 0<x_1<a, \quad t_1<t<t_2. \tag{1.15}$$

The solution $u(x_1,t)$ of the boundary-initial value problem associated with (1.15) obviously also satisfies the related boundary-initial value problem associated with (1.7). If we can prove uniqueness in the latter case, we will have shown that $u(x_1,t)$ is in fact the desired solution of (1.7).

2. Let Ω be a cylindrical rod (of arbitrary cross section) whose axis coincides with the segment $0<x_1<a$. In addition to the assumptions in part 1, let us suppose that the *lateral surface of the rod is insulated*. We may imagine the rod as having been punched out of the slab of part 1. Since the temperature in the slab is independent of x_2 and x_3, it must satisfy the condition $\partial u/\partial n=0$ on the lateral surface of the cylinder, which is the criterion of insulation. Thus the flow in the rod is one-dimensional, and (1.15) holds as before. In particular, the steady-state equation is

$$-\frac{d}{dx_1}\left(k\frac{du}{dx_1}\right) = \tilde{f}(x_1), \qquad 0<x_1<a, \tag{1.16}$$

where $\tilde{f}(x_1)$ is the *volume density* of sources. If A is the cross-sectional area of the rod, we can write $\tilde{f}(x_1)=f(x_1)/A$, where $f(x_1)$ is the source density per unit length of the rod. Equation (1.16) becomes

$$-\frac{d}{dx_1}\left(k\frac{du}{dx_1}\right) = \frac{f(x_1)}{A}, \qquad 0<x_1<a. \tag{1.17}$$

2. DIFFUSION

With a different interpretation of the terms, (1.4) and (1.7) also govern the *concentration* $c(\mathbf{x},t)$ of a substance diffusing through some medium. The energy balance (1.1) is replaced by a mass balance of the substance in question over R for the time interval $(t,t+dt)$:

(2.1) mass created by body sources

= increase of mass + outflow of mass through B.