Graduate Texts in Mathematics

Raoul Bott
Loring W. Tu

Differential Forms in Algebraic Topology

代数拓扑中微分形式

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(continued after index)

For
Phyllis Bott
and
Lichu and Tsuchih Tu

Preface

The guiding principle in this book is to use differential forms as an aid in exploring some of the less digestible aspects of algebraic topology. Accordingly, we move primarily in the realm of smooth manifolds and use the de Rham theory as a prototype of all of cohomology. For applications to homotopy theory we also discuss by way of analogy cohomology with arbitrary coefficients.

Although we have in mind an audience with prior exposure to algebraic or differential topology, for the most part a good knowledge of linear algebra, advanced calculus, and point-set topology should suffice. Some acquaintance with manifolds, simplicial complexes, singular homology and cohomology, and homotopy groups is helpful, but not really necessary. Within the text itself we have stated with care the more advanced results that are needed, so that a mathematically mature reader who accepts these background materials on faith should be able to read the entire book with the minimal prerequisites.

There are more materials here than can be reasonably covered in a one-semester course. Certain sections may be omitted at first reading without loss of continuity. We have indicated these in the schematic diagram that follows.

This book is not intended to be foundational; rather, it is only meant to open some of the doors to the formidable edifice of modern algebraic topology. We offer it in the hope that such an informal account of the subject at a semi-introductory level fills a gap in the literature.

It would be impossible to mention all the friends, colleagues, and students whose ideas have contributed to this book. But the senior author would like on this occasion to express his deep gratitude, first of all to his primary topology teachers E. Specker, N. Steenrod, and

K. Reidemeister of thirty years ago, and secondly to H. Samelson, A. Shapiro, I. Singer, J.-P. Serre, F. Hirzebruch, A. Borel, J. Milnor, M. Atiyah, S.-s. Chern, J. Mather, P. Baum, D. Sullivan, A. Haefliger, and Graeme Segal, who, mostly in collaboration, have continued this word of mouth education to the present; the junior author is indebted to Allen Hatcher for having initiated him into algebraic topology. The reader will find their influence if not in all, then certainly in the more laudable aspects of this book. We also owe thanks to the many other people who have helped with our project: to Ron Donagi, Zbig Fiedorowicz, Dan Freed, Nancy Hingston, and Deane Yang for their reading of various portions of the manuscript and for their critical comments, to Ruby Aguirre, Lu Ann Custer, Barbara Moody, and Caroline Underwood for typing services, and to the staff of Springer-Verlag for its patience, dedication, and skill.

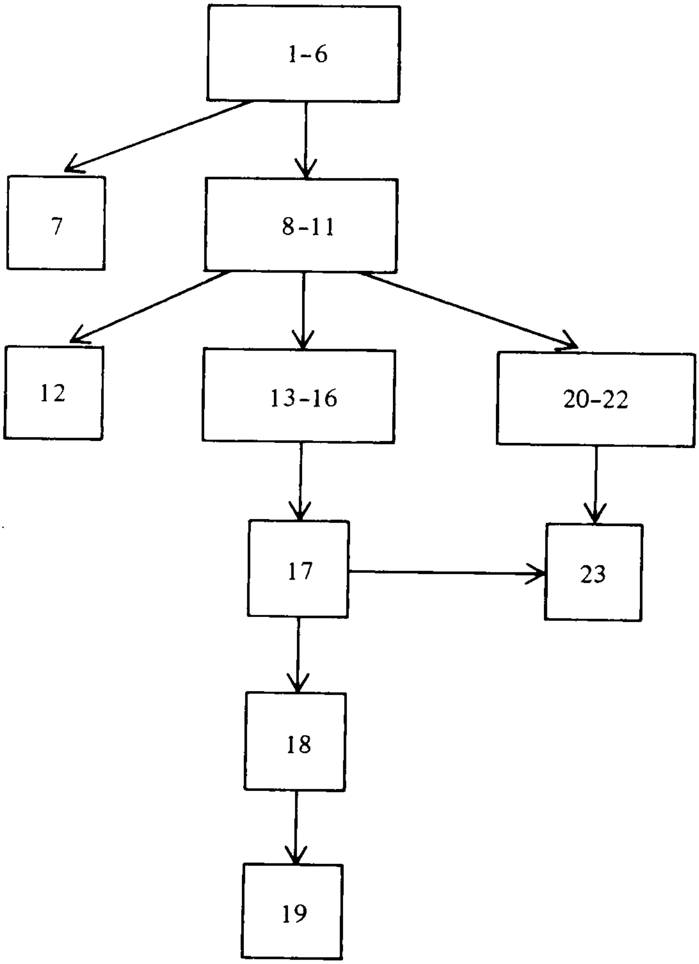
For the Revised Third Printing

While keeping the text essentially the same as in previous printings, we have made numerous local changes throughout. The more significant revisions concern the computation of the Euler class in Example 6.44.1 (pp. 75–76), the proof of Proposition 7.5 (p. 85), the treatment of constant and locally constant presheaves (p. 109 and p. 143), the proof of Proposition 11.2 (p. 115), a local finite hypothesis on the generalized Mayer–Vietoris sequence for compact supports (p. 139), transgressive elements (Prop. 18.13, p. 248), and the discussion of classifying spaces for vector bundles (pp. 297–300).

We would like to thank Robert Lyons, Jonathan Dorfman, Peter Law, Peter Landweber, and Michael Maltenfort, whose lists of corrections have been incorporated into the second and third printings.

Raoul Bott
Loring Tu

Interdependence of the Sections



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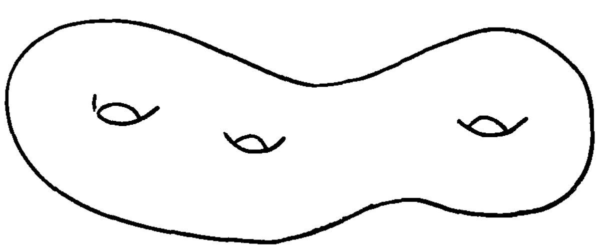
Introduction

The most intuitively evident topological invariant of a space is the number of connected pieces into which it falls. Over the past one hundred years or so we have come to realize that this primitive notion admits in some sense two higher-dimensional analogues. These are the *homotopy* and *cohomology groups* of the space in question.

The evolution of the higher homotopy groups from the component concept is deceptively simple and essentially unique. To describe it, let $\pi_0(X)$ denote the set of path components of X and if p is a point of X, let $\pi_0(X, p)$ denote the set $\pi_0(X)$ with the path component of p singled out. Also, corresponding to such a point p, let $\Omega_p X$ denote the space of maps (continuous functions) of the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ which send 1 to p, made into a topological space via the compact open topology. The path components of this so-called *loop space* $\Omega_p X$ are now taken to be the elements of $\pi_1(X, p)$:

$$\pi_1(X, p) = \pi_0(\Omega_p X, \bar{p}).$$

The composition of loops induces a group structure on $\pi_1(X, p)$ in which the constant map $\bar{p}$ of the circle to p plays the role of the identity; so endowed, $\pi_1(X, p)$ is called the *fundamental group* or the *first homotopy group* of X at p. It is in general not Abelian. For instance, for a Riemann surface of genus 3, as indicated in the figure below:



$\pi_1(X, p)$ is generated by six elements $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ subject to the single relation

$$\prod_{i=1}^{3} [x_i, y_i] = 1,$$

where $[x_i, y_i]$ denotes the commutator $x_i y_i x_i^{-1} y_i^{-1}$ and 1 the identity. The fundamental group is in fact sufficient to classify the closed oriented 2-dimensional surfaces, but is insufficient in higher dimensions.

To return to the general case, all the higher homotopy groups $\pi_k(X, p)$ for $k \geq 2$ can now be defined through the inductive formula:

$$\pi_{k+1}(X, p) = \pi_k(\Omega_p X, \tilde{p}).$$

By the way, if p and p' are two points in X in the same path component, then

$$\pi_k(X, p) \simeq \pi_k(X, p'),$$

but the correspondence is not necessarily unique. For the Riemann surfaces such as discussed above, the higher π_k's for $k \geq 2$ are all trivial, and it is in part for this reason that π_1 is sufficient to classify them. The groups π_k for $k \geq 2$ turn out to be Abelian and therefore do not seem to have been taken seriously until the 1930's when W. Hurewicz defined them (in the manner above, among others) and showed that, far from being trivial, they constituted the basic ingredients needed to describe the homotopy-theoretic properties of a space.

The great drawback of these easily defined invariants of a space is that they are very difficult to compute. To this day not all the homotopy groups of say the 2-sphere, i.e., the space $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$, have been computed! Nonetheless, by now much is known concerning the general properties of the homotopy groups, largely due to the formidable algebraic techniques to which the "cohomological extension" of the component concept lends itself, and the relations between homotopy and cohomology which have been discovered over the years.

This cohomological extension starts with the dual point of view in which a component is characterized by the property that on it *every locally constant function is globally constant*. Such a component is sometimes called a connected component, to distinguish it from a *path component*. Thus, if we define $H^0(X)$ to be the vector space of real-valued *locally constant* functions on X, then dim $H^0(X)$ tells us the number of connected components of X. Note that on reasonable spaces where path components and connected components agree, we therefore have the formula

$$\text{cardinality } \pi_0(X) = \dim H^0(X).$$

Still the two concepts are dual to each other, the first using maps of the unit interval into X to test for connectedness and the second using maps of X

into $\mathbb{R}$ for the same purpose. One further difference is that the cohomology group $H^0(X)$ has, by fiat, a natural $\mathbb{R}$-module structure.

Now what should the proper higher-dimensional analogue of $H^0(X)$ be? Unfortunately there is no decisive answer here. Many plausible definitions of $H^k(X)$ for $k > 0$ have been proposed, all with slightly different properties but all isomorphic on "reasonable spaces". Furthermore, in the realm of differentiable manifolds, all these theories coincide with the *de Rham theory* which makes its appearance there and constitutes in some sense the most perfect example of a cohomology theory. The de Rham theory is also unique in that it stands at the crossroads of topology, analysis, and physics, enriching all three disciplines.

The gist of the "de Rham extension" is comprehended most easily when M is assumed to be an open set in some Euclidean space $\mathbb{R}^n$, with coordinates $x_1, \ldots, x_n$. Then amongst the C^∞ functions on M the locally constant ones are precisely those whose gradient

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

vanishes identically. Thus here $H^0(M)$ appears as the space of solutions of the differential equation $df = 0$. This suggests that $H^1(M)$ should also appear as the space of solutions of some natural differential equations on the manifold M. Now consider a 1-form on M:

$$\theta = \sum a_i \, dx_i,$$

where the a_i's are C^∞ functions on M. Such an expression can be integrated along a smooth path γ, so that we may think of θ as a function on paths γ:

$$\gamma \mapsto \int_\gamma \theta.$$

It then suggests itself to seek those θ which give rise to *locally constant* functions of γ, i.e., for which the integral $\int_\gamma \theta$ is left unaltered under small variations of γ—but keeping the endpoints fixed! (Otherwise, only the zero 1-form would be locally constant.) Stokes' theorem teaches us that these line integrals are characterized by the differential equations:

$$\frac{\partial a_i}{\partial x_j} - \frac{\partial a_j}{\partial x_i} = 0 \qquad \text{(written } d\theta = 0\text{)}.$$

On the other hand, the fundamental theorem of calculus implies that $\int_\gamma df = f(Q) - f(P)$, where P and Q are the endpoints of γ, so that *the gradients are trivally locally constant.*

One is here irresistibly led to the definition of $H^1(M)$ as the vector space of *locally constant line integrals modulo the trivially constant ones.* Similarly the higher cohomology groups $H^k(M)$ are defined by simply replacing line integrals with their higher-dimensional analogues, the *k-volume integrals.*

The Grassmann calculus of exterior differential forms facilitates these extensions quite magically. Moreover, the differential equations characterizing the locally constant k-integrals are seen to be C^∞ invariants and so extend naturally to the class of C^∞ manifolds.

Chapter I starts with a rapid account of this whole development, assuming little more than the standard notions of advanced calculus, linear algebra and general topology. A nodding acquaintance with singular homology or cohomology helps, but is not necessary. No real familiarity with differential geometry or manifold theory is required. After all, the concept of a manifold is really a very natural and simple extension of the calculus of several variables, as our fathers well knew. Thus for us a manifold is essentially a space constructed from open sets in $\mathbb{R}^n$ by patching them together in a smooth way. This point of view goes hand in hand with the "computability" of the de Rham theory. Indeed, the decisive difference between the π_k's and the H^k's in this regard is that if a manifold X is the union of two open submanifolds U and V:

$$X = U \cup V,$$

then the cohomology groups of U, V, $U \cap V$, and X are linked by a much stronger relation than the homotopy groups are. The linkage is expressed by the exactness of the following sequence of linear maps, *the Mayer–Vietoris sequence*:

$$0 \to H^0(X) \to \cdots \to H^{k-1}(U \cap V) \xrightarrow{d^*} H^k(X) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \xrightarrow{d^*} H^{k+1}(X) \to \cdots$$

starting with $k = 0$ and extending up indefinitely. In this sequence every arrow stands for a linear map of the vector spaces and exactness asserts that the kernel of each map is precisely the image of the preceding one. The horizontal arrows in our diagram are the more or less obvious ones induced by restriction of functions, but the coboundary operator d^* is more subtle and uses the existence of a *partition of unity* subordinate to the cover $\{U, V\}$ of X, that is, smooth functions ρ_U and ρ_V such that the first has support in U, the second has support in V, and $\rho_U + \rho_V \equiv 1$ on X. The simplest relation imaginable between the H^k's of U, V, and $U \cup V$ would of course be that H^k behaves additively; the Mayer–Vietoris sequence teaches us that this is indeed the case if U and V are disjoint. Otherwise, there is a geometric feedback from $H^k(U \cap V)$ described by d^*, and one of the hallmarks of a topologist is a sound intuition for this d^*.

The exactness of the Mayer–Vietoris sequence is our first goal once the basics of the de Rham theory are developed. Thereafter we establish the