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Oscar Zariski
Pierre Samuel

Commutative Algebra

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PREFACE

This second volume of our treatise on commutative algebra deals largely with three basic topics, which go beyond the more or less classical material of volume I and are on the whole of a more advanced nature and a more recent vintage. These topics are: (a) valuation theory; (b) theory of polynomial and power series rings (including generalizations to graded rings and modules); (c) local algebra. Because most of these topics have either their source or their best motivation in algebraic geometry, the algebro-geometric connections and applications of the purely algebraic material are constantly stressed and abundantly scattered throughout the exposition. Thus, this volume can be used in part as an introduction to some basic concepts and the arithmetic foundations of algebraic geometry. *The reader who is not immediately concerned with geometric* applications may omit the algebro-geometric material in a first reading (see "Instructions to the reader," page vii), but it is only fair to say that many a reader will find it more instructive to find out immediately what is the geometric motivation behind the purely algebraic material of this volume.

The first 8 sections of Chapter VI (including § 5bis) deal directly with properties of places, rather than with those of the valuation associated with a place. These, therefore, are properties of valuations in which the value group of the valuation is not involved. The very concept of a valua*tion is only introduced for the first time in* § 8, and, from that point on, the more subtle properties of valuations which are related to the value group come to the fore. These are illustrated by numerous examples, taken largely from the theory of algebraic function fields (§§ 14, 15). The last two sections of the chapter contain a general treatment, within the framework of arbitrary commutative integral domains, of two concepts which are of considerable importance in algebraic geometry (the Riemann surface of a field and the notions of normal and derived normal models).

The greater part of Chapter VII is devoted to classical properties of polynomial and power series rings (e.g., dimension theory) and their applications to algebraic geometry. This chapter also includes a treatment of graded rings and modules and such topics as characteristic (Hilbert) functions and chains of syzygies. In the past, these last two topics represented some final words of the algebraic theory, to be followed only by deeper geometric applications. With the modern development of homological methods in commutative algebra, these topics became starting points of extensive, purely algebraic theories, having a much wider range of applications. We could not include, without completely disrupting the balance of this volume, the results which require the use of truly homological methods (e.g., torsion and extension functors, complexes, spectral sequences). However, we have tried to include the results which may be proved by methods which, although inspired by homological algebra, are nevertheless classical in nature. The reader will find these results in Chapter VII, §§ 12 and 13, and in Appendices 6 and 7. No previous knowledge of homological algebra is needed for reading these parts of the volume. The reader who wants to see how truly homological methods may be applied to commutative algebra is referred to the original papers of M. Auslander, D. Buchsbaum, A. Grothendieck, D. Rees, J.-P. Serre, etc., to a forthcoming book of D. C. Northcott, as well, of course, as to the basic treatise of Cartan-Eilenberg.

Chapter VIII deals with the theory of local rings. This theory provides the algebraic basis for the local study of algebraic and analytical varieties. The first six sections are rather elementary and deal with more general rings than local rings. Deeper results are presented in the rest of the chapter, but we have not attempted to give an encyclopedic account of the subject.

While much of the material appears here for the first time in book form, there is also a good deal of material which is new and represents current or unpublished research. The appendices treat special topics of current interest (the first 5 were written by the senior author; the last two by the junior author), except that Appendix 6 gives a smooth treatment of two important theorems proved in the text. Appendices 4 and 5 are of particular interest from an algebro-geometric point of view.

We have not attempted to trace the origin of the various proofs in this volume. Some of these proofs, especially in the appendices, are new. Others are transcriptions or arrangements of proofs taken from original papers.

We wish to acknowledge the assistance which we have received from M. Hironaka, T. Knapp, S. Shatz, and M. Schlesinger in the work of checking parts of the manuscript and of reading the galley proofs. Many improvements have resulted from their assistance.

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INSTRUCTIONS TO THE READER

As this volume contains a number of topics which either are of somewhat specialized nature (but still belong to pure algebra) or belong to algebraic geometry, the reader who wishes first to acquaint himself with the basic algebraic topics before turning his attention to deeper and more specialized results or to geometric applications, may very well skip some parts of this volume during a first reading. The material which may thus be postponed to a second reading is the following:

CHAPTER VI

All of § 3, except for the proof of the first two assertions of Theorem 3 and the definition of the rank of a place; § 5: Theorem 10, the lemma and its corollary; § 5bis (if not immediately interested in geometric applications); § 11: Lemma 4 and pages 57-67 (beginning with part (b) of Theorem 19); § 12; § 14: The last part of the section, beginning with Theorem 34′; § 15 (if not interested in examples); §§ 16, 17, and 18.

CHAPTER VII

§§ 3, 4, 4bis, 5 and 6 (if not immediately interested in geometric applications); all of § 8, except for the statement of Macaulay's theorem and (if it sounds interesting) the proof (another proof, based on local algebra, may be found in Appendix 6); § 9: Theorem 29 and the proof of Theorem 30 (this theorem is contained in Theorem 25); § 11 (the contents of this section are particularly useful in geometric applications).

CHAPTER VIII

All of § 5, except for Theorem 13 and its Corollary 2; § 10; § 11: Everything concerning multiplicities; all of § 12, except for Theorem 27 (second proof recommended) and the statement of the theorem of Cohen-Macaulay; § 13.

All appendices may be omitted in a first reading.

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VI. VALUATION THEORY

§ 1. **Introductory remarks.** Homomorphic mappings of rings into fields are very common in commutative algebra and in its applications. We may cite the following examples:

EXAMPLE 1. *The reduction of integers* mod p. More precisely, let p be a prime number; then the canonical mapping of the ring J of integers onto the residue class ring J/Jp maps J *onto* a field with p elements. More generally, we may consider a ring D of algebraic integers (Vol. I, Ch. V, § 4, p. 265), a prime ideal $\mathfrak{p}$ in D, and the mapping of D onto $D/\mathfrak{p}$. These examples are of importance in number theory.

EXAMPLE 2. We now give examples pertaining to algebraic geometry. Let k be a field and K an extension of k. Let $(x_1, \cdots, x_n)$ be a point in the affine n-space A_n^K over K. With every polynomial $F(X_1, \cdots, X_n)$ with coefficients in k we associate its *value* $F(x_1, \cdots, x_n)$ at the given point. This defines a homomorphic mapping of the polynomial ring $k[X_1, \cdots, X_n]$ into K. Now let us say that a point $(x'_1, \cdots, x'_n)$ of A_n^K is a *specialization* of $(x_1, \cdots, x_n)$ over k if every polynomial $F \in k[X_1, \cdots, X_n]$ which vanishes at $(x_1, \cdots, x_n)$ vanishes also at $(x'_1, \cdots, x'_n)$. Then (by taking differences) two polynomials G, H with coefficients in k which take the same value at $(x_1, \cdots, x_n)$ take also the same value at $(x'_1, \cdots, x'_n)$. This defines a mapping of $k[x_1, \cdots, x_n]$ onto $k[x'_1, \cdots, x'_n]$ ($\subset K$), which maps x_i on x'_i for $1 \leq i \leq n$. Such a mapping, and more generally *any homomorphic mapping* φ *of a ring* R *into a field*, such that $\varphi(x) \neq 0$ for some $x \in R$, is called a *specialization* (of $k[x_1, \cdots, x_n]$ into K in our case). Note that this definition implies that $\varphi(1) = 1$ if $1 \in R$. If, as in the above example, the specialization is the identity on some subfield k of the ring, then we shall say that the specialization is *over* k.

EXAMPLE 3. From function theory comes the following example: with any power series in n variables with complex coefficients we associate its constant term, i.e., its value at the origin.

Since any integral domain may be imbedded in its quotient field, a homomorphic mapping of a ring A into a field is the same thing as a homomorphic mapping of A *onto* an integral domain. Thus, by Vol. I, Ch. III, § 8, Theorem 10 a necessary and sufficient condition that a homomorphism f of a ring A map A into a field is that *the kernel of f be a prime ideal.*

From now on we suppose that we are dealing with a ring A which is an *integral domain.* Let K be a field containing A (not necessarily its quotient field), and let f be a specialization of A. An important problem is to investigate whether f may be extended to a specialization defined on as big as possible a subring of K. An answer to this question will be given in § 4. We may notice already that this problem is not at all trivial.

EXAMPLE 4. Consider, in fact, a polynomial ring $k[X, Y]$ in two variables over a field k, and the specialization f of $k[X, Y]$ onto k defined by $f(a)=a$ for a in k, $f(X)=f(Y)=0$ ("the value at the origin"). The value to be given to the rational function X/Y at the origin is not determined by f (since it appears as $0/0$). We have $k[X/Y, Y]\supset k[X, Y]$, and any maximal ideal $\mathfrak{P}$ in $k[X/Y, Y]$ which contains Y contains also X and thus contracts to the maximal ideal (X, Y) in $k[X, Y]$. Since there are infinitely many such maximal ideals $\mathfrak{P}$ (they are the ideals generated by $h(X/Y)$ and Y, where $h(t)$ is any irreducible polynomial in $k[t]$) it follows that f admits infinitely many extensions to the ring $k[X, Y, X/Y]$.

However, there are elements of K to which the given specialization f of A may be extended without further ado and in a unique fashion. Consider, in fact, the elements of K which may be written in the form a/b with a in A, b in A, and $f(b)\neq 0$. These elements constitute the *quotient ring* $A_{\mathfrak{p}}$ where $\mathfrak{p}$ is the kernel of f and is a prime ideal. For such an element a/b let us write $g(a/b)=f(a)/f(b)$. It is readily verified that g is actually a mapping: if $a/b=a'/b'$ with $f(b)\neq 0$ and $f(b')\neq 0$, then $f(a)/f(b)=f(a')/f(b')$ since $ab'=ba'$ and since f is a homomorphism. One sees also in a similar way that g is a homomorphism of $A_{\mathfrak{p}}$ extending f (see Vol. I, Ch. IV, § 9, Theorem 14). Since g takes values in the same field as f does, g is a specialization of $A_{\mathfrak{p}}$. The ring $A_{\mathfrak{p}}$ is sometimes called the *specialization ring* of f; it is a local ring if A is noetherian (Vol. I, Ch. IV, § 11, p. 228).

In Example 1 this local ring is the set of all fractions m/n whose denominator n is not a multiple of p. In Example 2 it is the set of all rational functions in $X_1, \cdots, X_n$ which are "finite" at the point $(x_1, \cdots, x_n)$ (i.e., whose denominator does not vanish at this point). In Example 3 it is the power series ring itself, as a power series with non-zero constant term is invertible.

On the other hand there are (when the specialization f is not an isomorphic mapping) elements of K to which f cannot be extended by any means. These elements are those which can be written under the form a/b, with a and b in A, with $f(a) \neq 0$ and $f(b) = 0$, for the value $g(a/b)$ of a/b in an extension g of f must satisfy the relation $g(a/b) \cdot f(b) = f(a)$ (since $(a/b) \cdot b = a$), but this is impossible. The elements a/b of the above form are the inverses of the non-zero elements in the maximal ideal of the specialization ring of f.

We are thus led to studying the extreme case in which all elements of K which are not in A are of this latter type. In this case A is identical with the specialization ring of f, and every element of K which is not in A must be of the form $1/x$, where x is an element of A such that $f(x) = 0$.

§ 2. Places

DEFINITION 1. *Let K be an arbitrary field. A place of K is a homomorphic mapping $\mathcal{P}$ of a subring $K_{\mathcal{P}}$ of K into a field Δ, such that the following conditions are satisfied:*

(1) *if $x \in K$ and $x \notin K_{\mathcal{P}}$, then $1/x \in K_{\mathcal{P}}$ and $(1/x)\mathcal{P} = 0$;*

(2) $x\mathcal{P} \neq 0$ *for some x in $K_{\mathcal{P}}$.*

In many applications of ideal theory (and expecially in algebraic geometry) a certain basic field k is given in advance, called *the ground field*, and the above arbitrary field K is restricted to be an extension of k: $k \subset K$. In that case, one may be particularly interested in places $\mathcal{P}$ of K which reduce to the identity on k, i.e., places $\mathcal{P}$ which satisfy the following additional condition:

(3) $c\mathcal{P} = c$ for all c in k (whence k is a subfield of Δ).

Any place $\mathcal{P}$ of K which satisfies (3) is said to be a place of K *over* k, or a place of K/k.

EXAMPLES OF PLACES:

EXAMPLE 1. Let A be a *UFD*, and a an irreducible element in A. The ideal Aa is a prime ideal, whence A/Aa is an integral domain. Denote by Δ its quotient field. The canonical homomorphism of A onto A/Aa is a specialization f of A into Δ. The specialization ring B of f is the set of all fractions x/y, with $x \in A$, $y \in A$, $y \notin Aa$ (i.e., y prime to a). We denote by g the extension of f to B. The homomorphic mapping g is a *place*: in fact, by the unique factorization, any element z of the quotient field K of A which does not belong to B can be written in the form y/x, with $y \in A$, $x \in A$, $y \notin Aa$, $x \in Aa$; then its inverse $1/z = x/y$ belongs to B and satisfies the relation $g(1/z) = 0$.

We call the place g which is thus determined by an irreducible element a of A an a-adic place (of the quotient field of A).

EXAMPLE 2. A similar example may be given if one takes for A a Dedekind domain and if one considers the homomorphic mapping f of A into the quotient field of $A/\mathfrak{p}$ ($\mathfrak{p}$ denoting a prime ideal of A). The extension g of f to the local ring $A_{\mathfrak{p}}$ of f is again a place [notice that $A_{\mathfrak{p}}$ is a *PID* (Vol. I, Ch. V, § 7, Theorem 16), to which the preceding example may be applied]. This place is called the *$\mathfrak{p}$-adic place of A.*

We shall show at once the following property of places: if $\mathscr{P}$ is a place of K, then $\mathscr{P}$ has no proper extensions in K. Or more precisely: if φ is a homomorphic mapping of a subring L of K (into some field), such that $L \supset K_{\mathscr{P}}$ and $\varphi = \mathscr{P}$ on $K_{\mathscr{P}}$, then $L = K_{\mathscr{P}}$. We note first that, by condition (1), the element 1 of K belongs to $K_{\mathscr{P}}$. It follows then from condition (2) that $1\mathscr{P}$ must be the element 1 of Δ. Now, let x be any element of L. We cannot have simultaneously $1/x \in K_{\mathscr{P}}$ and $(1/x)\mathscr{P} = 0$, for then we would have $1 = 1\varphi = (x \cdot 1/x)\varphi = x\varphi \cdot (1/x)\varphi = x\varphi \cdot 0 = 0$, a contradiction. It follows therefore, by condition (1), that $x \in K_{\mathscr{P}}$. Hence $L = K_{\mathscr{P}}$, as asserted.

It will be proved later (§ 4, Theorem 5′, Corollary 4) that the above is a *characteristic property* of places.

We introduce the symbol ∞ and we agree to write $x\mathscr{P} = \infty$ if $x \notin K_{\mathscr{P}}$. The following assertions are immediate consequences of conditions (1) and (2) above:

(a) if $x\mathscr{P} = \infty$ and $y\mathscr{P} \neq \infty$, then $(x \pm y)\mathscr{P} = \infty$;
(b) if $x\mathscr{P} = \infty$ and $y\mathscr{P} \neq 0$, then $(xy)\mathscr{P} = \infty$;
(c) if $x \neq 0$, then $x\mathscr{P} = 0$ if and only if $(1/x)\mathscr{P} = \infty$.

If $x \in K_{\mathscr{P}}$ we shall call $x\mathscr{P}$ the *$\mathscr{P}$-value of x*, or *the value of x at the place $\mathscr{P}$*, and we shall say that x is finite at $\mathscr{P}$ or has finite $\mathscr{P}$-value if $x\mathscr{P} \neq \infty$, i.e., if $x \in K_{\mathscr{P}}$. The ring $K_{\mathscr{P}}$ shall be referred to as *the valuation ring of the place $\mathscr{P}$.*

It is clear that the elements $x\mathscr{P}$, $x \in K_{\mathscr{P}}$, form a subring of Δ. It is easily seen that this subring is actually a field, for if $\alpha = x\mathscr{P} \neq 0$, then, by condition (1), also $1/x \in K_{\mathscr{P}}$, and hence $1/\alpha = (1/x)\mathscr{P}$. We call this field the *residue field of $\mathscr{P}$.* The elements of Δ which are not $\mathscr{P}$-values of elements of K do not interest us. Hence we shall assume that the residue field of $\mathscr{P}$ is the field Δ itself.

If K is an extension of a ground field k, if $\mathscr{P}$ is a place of K/k and if s is the transcendence degree of Δ over k (s may be an infinite cardinal), we call *s the dimension of the place $\mathscr{P}$*, over k, or in symbols: $s = \dim \mathscr{P}/k$. If K has transcendence degree r over k, then $0 \leqq s \leqq r$. The place $\mathscr{P}$ of

K/k is *algebraic* (over k) if $s=0$; *rational* if $\Delta=k$. On the other extreme we have the case $s=r$. In this case and under the additional assumption that r is finite, $\mathscr{P}$ is an isomorphism (Vol. I, Ch. II, § 12, Theorem 29), and furthermore it follows at once from condition (1) that $K_{\mathscr{P}}=K$, whence $\mathscr{P}$ is merely a k-isomorphism of K. Places which are isomorphisms of K will be called *trivial* places of K (or trivial places of K/k, if they are k-isomorphisms of K).

It is obvious that the trivial places $\mathscr{P}$ of K are characterized by the condition $K_{\mathscr{P}}=K$. On the other hand, if $\mathscr{P}$ is a place of K and K_1 is a subfield of K, then the restriction $\mathscr{P}_1$ of $\mathscr{P}$ to K_1 is obviously a place of K_1. Therefore, if $K_1 \subset K_{\mathscr{P}}$ then $\mathscr{P}_1$ is a trivial place of K_1. In particular, if K has characteristic $p \neq 0$, then any place $\mathscr{P}$ of K is trivial on the prime subfield of K (for $1 \in K_{\mathscr{P}}$).

From condition (1) of Definition 1 it follows that if an element x of $K_{\mathscr{P}}$ is such that $x\mathscr{P} \neq 0$, then $1/x$ belongs to $K_{\mathscr{P}}$ and hence x is a unit in $K_{\mathscr{P}}$. Hence the kernel of $\mathscr{P}$ consists of all non-units of the ring $K_{\mathscr{P}}$. The kernel of $\mathscr{P}$ is therefore a maximal ideal in $K_{\mathscr{P}}$; in fact it is the only maximal ideal in $K_{\mathscr{P}}$. (However, the valuation ring $K_{\mathscr{P}}$ of a place $\mathscr{P}$ is not necessarily a local ring, since according to our definition, a local ring is noetherian (Vol. I, Ch. IV, § 11, p. 228), while, as we shall see later (§ 10, Theorem 16), a valuation ring need not be noetherian.) The maximal ideal in $K_{\mathscr{P}}$ will be denoted by $\mathfrak{M}_{\mathscr{P}}$ and will be referred to as the prime ideal of the place $\mathscr{P}$. The field $K_{\mathscr{P}}/\mathfrak{M}_{\mathscr{P}}$ and the residue field Δ of $\mathscr{P}$ are isomorphic.

Let L be a subring of K. Our definition of places of K implies that if L is the valuation ring of a place $\mathscr{P}$ of K, then L contains the reciprocal of any element of K which does not belong to L; and, furthermore, L must contain k if L is the valuation ring of a place of K/k. We now prove that also the converse is true:

THEOREM 1. *Let L be a subring of K. If L contains the reciprocal of any element of K which does not belong to L, then there exists a place $\mathscr{P}$ of K such that L is the valuation ring of $\mathscr{P}$. If, furthermore, K contains a ground field k and L contains k, then there also exists a place $\mathscr{P}$ of K/k such that L is the valuation ring of $\mathscr{P}$.*

PROOF. Assume that L contains the reciprocal of any element of K which does not belong to L. Then it follows in the first place that $1 \in L$. We next show that the non-units of L form an ideal. For this it is only necessary to show that if x and y are non-units of L, then also $x+y$ is a non-unit, and in the proof we may assume that both x and y are different from zero. By assumption, either y/x or x/y belongs to L. Let, say, $y/x \in L$. Then $x+y=x(1+y/x)$, and since $1+y/x \in L$ and x

is a non-unit in L, we conclude that $x+y$ is a non-unit in L, as asserted. Let, then, $\mathfrak{M}$ be the ideal of non-units of L, and let $\mathscr{P}$ be the canonical homomorphism of L onto the field $L/\mathfrak{M}$. Then condition (1) of Definition 1 is satisfied, with $K_{\mathscr{P}}=L$ (while Δ is now the field $L/\mathfrak{M}$), for if $x \in K$ and $x \notin L$, then $1/x \in L$, whence $1/x \in \mathfrak{M}$ and therefore $(1/x)\mathscr{P}=0$. It is obvious that also condition (2) is satisfied, since $L/\mathfrak{M}$ is a field and since $\mathscr{P}$ maps L onto $L/\mathfrak{M}$.

Assume now that the additional condition $k \subset L$ is also satisfied. Then the field $L/\mathfrak{M}$ contains the *isomorphic* image $k\mathscr{P}$ of k. We may therefore identify each element c of k with its image $c\mathscr{P}$, and then also condition (3) is satisfied. Q.E.D.

An important property of the valuation ring $K_{\mathscr{P}}$ of a place $\mathscr{P}$ is that *it is integrally closed in* K. For let x be any element of K which is integrally dependent on $K_{\mathscr{P}}$: $x^n+a_1x^{n-1}+ \cdots +a_n=0$, $a_i \in K_{\mathscr{P}}$. Dividing by x^n we find $1=-a_1(1/x)-a_2(1/x)^2- \cdots -a_n(1/x)^n$. If $x \notin K_{\mathscr{P}}$, then $1/x \in K_{\mathscr{P}}$, $(1/x)\mathscr{P}=0$, and hence equating the $\mathscr{P}$-values of both sides of the above relation we get $1=0$, a contradiction. Hence $x \in K_{\mathscr{P}}$, and $K_{\mathscr{P}}$ is integrally closed in K, as asserted.

DEFINITION 2. *If $\mathscr{P}$ and $\mathscr{P}'$ are places of K (or of K/k), with residue fields Δ and Δ' respectively, then $\mathscr{P}$ and $\mathscr{P}'$ are said to be isomorphic places (or k-isomorphic places) if there exists an isomorphism ψ (or a k-isomorphism ψ) of Δ onto Δ' such that $\mathscr{P}'=\mathscr{P}\psi$.*

A necessary and sufficient condition that two places $\mathscr{P}$ and $\mathscr{P}'$ of K (or of K/k) be isomorphic (or k-isomorphic) is that their valuation rings $K_{\mathscr{P}}$ and $K_{\mathscr{P}'}$ coincide. It is obvious that the condition is necessary. Assume now that the condition is satisfied, and let φ be the canonical homomorphism of $K_{\mathscr{P}}$ onto $K_{\mathscr{P}}/\mathfrak{M}_{\mathscr{P}}$. Then $\mathscr{P}^{-1}\varphi$ is an isomorphism of Δ onto $K_{\mathscr{P}}/\mathfrak{M}_{\mathscr{P}}$, and similarly $\mathscr{P}'^{-1}\varphi$ is an isomorphism of Δ' onto $K_{\mathscr{P}}/\mathfrak{M}_{\mathscr{P}}$. Hence $\mathscr{P}^{-1}\mathscr{P}'(=\mathscr{P}^{-1}\varphi \cdot \varphi^{-1}\mathscr{P}')$ is an isomorphism ψ of Δ onto Δ', showing that $\mathscr{P}$ and $\mathscr{P}'$ are isomorphic places. If, moreover, $\mathscr{P}$ and $\mathscr{P}'$ are places of K/k, then ψ is a k-isomorphism of Δ onto Δ', whence $\mathscr{P}$ and $\mathscr{P}'$ are k-isomorphic places.

It is clear that k-isomorphic places of K/k have the same dimension over k.

Isomorphic *algebraic* places of K/k will be referred to as *conjugate places* (over k) if their residue fields are subfields of one and the same algebraic closure $\bar{k}$ of k. In that case, these residue fields are conjugate subfields of $\bar{k}/k$.

If $\mathscr{P}$ is a place of K/k, where k is a ground field, then K and the residue field Δ of $\mathscr{P}$ have the same characteristic (since $k \subset \Delta$). Conversely, assume that $\mathscr{P}$ is a place of K such that K and Δ have the same characteristic p. (Note that this assumption is satisfied for any place $\mathscr{P}$ of K if K has characteristic $\neq 0$, for in that case the restriction of $\mathscr{P}$ to the prime subfield of K is an isomorphism.) Let Γ denote the prime subfield of K. We know that if $p \neq 0$ then the restriction of $\mathscr{P}$ to Γ is an isomorphism. If $p=0$ and if J denotes the ring of integers in Γ, then $J \subset K_{\mathscr{P}}$ (since $1 \in K_{\mathscr{P}}$) and the restriction of $\mathscr{P}$ to J must be an isomorphism (for otherwise Δ would be of characteristic $\neq 0$). Hence again the restriction of $\mathscr{P}$ to Γ is an isomorphism (and we have $\Gamma \subset K_{\mathscr{P}}$). *It follows at once* (as in the proof of the last part of Theorem 1) *that $\mathscr{P}$ is isomorphic to a place of K/Γ.* We thus see that the theory of places over ground fields is essentially as general as the theory of arbitrary places $\mathscr{P}$ *in the equal characteristic case* (i.e., in the case in which K and Δ have the same characteristic).

§ 3. Specialization of places. Let $\mathscr{P}$ and $\mathscr{P}'$ be places of K. We say that *$\mathscr{P}'$ is a specialization of $\mathscr{P}$* and we write $\mathscr{P} \to \mathscr{P}'$, if the valuation ring $K_{\mathscr{P}'}$ of $\mathscr{P}'$ is contained in the valuation ring $K_{\mathscr{P}}$ of $\mathscr{P}$, and we say that *$\mathscr{P}'$ is a proper specialization of $\mathscr{P}$* if $K_{\mathscr{P}'}$ is a proper subring of $K_{\mathscr{P}}$. If both $\mathscr{P}$ and $\mathscr{P}'$ are places of K/k and $\mathscr{P}'$ is a specialization of $\mathscr{P}$, then we shall write $\mathscr{P} \overset{k}{\to} \mathscr{P}'$.

It is clear that $\mathscr{P} \to \mathscr{P}'$ if and only if either one of the following conditions is satisfied: (a) $x\mathscr{P}' \neq \infty$ implies $x\mathscr{P} \neq \infty$; (b) $x\mathscr{P}=0$ implies $x\mathscr{P}'=0$ (for, $x\mathscr{P}=0$ implies $(1/x)\mathscr{P}=\infty$, whence $(1/x)\mathscr{P}'=\infty$, or $x\mathscr{P}'=0$). Hence we have, in view of (b):

$$(1) \qquad \mathscr{P} \to \mathscr{P}' \Leftrightarrow K_{\mathscr{P}} \supset K_{\mathscr{P}'} \quad \textit{and} \quad \mathfrak{M}_{\mathscr{P}} \subset \mathfrak{M}_{\mathscr{P}'}.$$

In particular, if both $\mathscr{P}$ and $\mathscr{P}'$ are places of K/k and $\mathscr{P} \overset{k}{\to} \mathscr{P}'$, then we conclude at once with the following result: *If $x_1, x_2, \cdots, x_n$ are any elements of K which are finite at $\mathscr{P}'$ (and therefore also at $\mathscr{P}$), then any algebraic relation, over k, between the $\mathscr{P}$-values of the x_i is also satisfied by the $\mathscr{P}'$-values of the x_i.* Thus, our definition of specialization of places is a natural extension of the notion of specialization used in algebraic geometry.

Every place of K is a specialization of any trivial place of K. Furthermore, isomorphic places are specializations of each other. Conversely, if two places $\mathscr{P}$ and $\mathscr{P}'$ are such that each is a specialization of the other, then they are isomorphic places. As a generalization of the last statement, we have the following theorem:

THEOREM 2. *Let $\mathscr{P}$ and $\mathscr{P}'$ be places of K, with residue fields Δ and Δ' respectively. Then $\mathscr{P} \to \mathscr{P}'$ if and only if there exists a place $\mathscr{Q}$ of Δ such that $\mathscr{P}'=\mathscr{P}\mathscr{Q}$ on $K_{\mathscr{P}'}$.*

PROOF. Assume that $\mathscr{P} \to \mathscr{P}'$. We set $\Delta_2 = K_{\mathscr{P}'}\mathscr{P}$ and we observe that since $K_{\mathscr{P}'} \subset K_{\mathscr{P}}$, Δ_2 is a subring of Δ. On the other hand, we have, by (1), that $\mathfrak{M}_{\mathscr{P}}$ is a prime ideal in $K_{\mathscr{P}'}$. Let now φ and φ' denote the canonical homomorphisms of $K_{\mathscr{P}'}$ onto $K_{\mathscr{P}'}/\mathfrak{M}_{\mathscr{P}}$ and $K_{\mathscr{P}'}/\mathfrak{M}_{\mathscr{P}'}$ respectively, and let $\mathscr{P}_1$ be the restriction of $\mathscr{P}$ to $K_{\mathscr{P}'}$. Since $\mathfrak{M}_{\mathscr{P}}$ is the kernel of $\mathscr{P}_1$, the product $\mathscr{P}_1{}^{-1}\varphi$ is an isomorphism of Δ_2 onto $K_{\mathscr{P}'}/\mathfrak{M}_{\mathscr{P}}$. Similarly $\varphi'^{-1}\mathscr{P}'$ is an isomorphism of $K_{\mathscr{P}'}/\mathfrak{M}_{\mathscr{P}'}$ onto Δ'. Since $\mathfrak{M}_{\mathscr{P}} \subset \mathfrak{M}_{\mathscr{P}'}$, $\varphi^{-1}\varphi'$ is a homomorphism of $K_{\mathscr{P}'}/\mathfrak{M}_{\mathscr{P}}$ onto $K_{\mathscr{P}'}/\mathfrak{M}_{\mathscr{P}'}$. We set $\mathscr{Q} = \mathscr{P}_1{}^{-1}\varphi \cdot \varphi^{-1}\varphi' \cdot \varphi'^{-1}\mathscr{P}' = \mathscr{P}_1{}^{-1}\mathscr{P}'$. Then $\mathscr{Q}$ is a homomorphism of Δ_2 onto Δ'. If ξ is an element of Δ which is not in Δ_2 and x is some fixed element of $K_{\mathscr{P}}$ such that $x\mathscr{P} = \xi$, then $x \notin K_{\mathscr{P}'}$, $(1/x)\mathscr{P}' = 0$, and hence $(1/\xi)\mathscr{Q} = 0$. We have thus proved that $\mathscr{Q}$ is a place of Δ, with residue field Δ', and that $\mathscr{P}_1\mathscr{Q} = \mathscr{P}'$. Hence $\mathscr{P}'$ and $\mathscr{P}\mathscr{Q}$ coincide on $K_{\mathscr{P}'}$.

Conversely, if we have $\mathscr{P}' = \mathscr{P}\mathscr{Q}$ on $K_{\mathscr{P}'}$, where $\mathscr{Q}$ is a place of Δ, then it is clear that $x\mathscr{P}' \neq \infty$ implies $x\mathscr{P} \neq \infty$, whence $K_{\mathscr{P}'} \subset K_{\mathscr{P}}$, and $\mathscr{P}'$ is a specialization of $\mathscr{P}$. This completes the proof.

We note that $\mathscr{P}'$ and $\mathscr{P}\mathscr{Q}$ coincide not only on $K_{\mathscr{P}'}$ but also on $K_{\mathscr{P}}$, in the following sense: *if $x \in K_{\mathscr{P}}$ and $x \notin K_{\mathscr{P}'}$ (whence $x\mathscr{P} \in \Delta$ and $x\mathscr{P}' = \infty$), then* $(x\mathscr{P})\mathscr{Q} = \infty$. For, if $x \notin K_{\mathscr{P}'}$, then $(1/x)\mathscr{P}' = 0$, and hence $(1/x)\mathscr{P}\mathscr{Q} = 0$ (since $\mathscr{P}' = \mathscr{P}\mathscr{Q}$ on $K_{\mathscr{P}'}$), i.e., $(1/x\mathscr{P})\mathscr{Q} = 0$ and $(x\mathscr{P})\mathscr{Q} = \infty$, as asserted.

We note also that in the special case of isomorphic places $\mathscr{P}$, $\mathscr{P}'$, $\mathscr{Q}$ is an isomorphism of Δ, i.e., $\mathscr{Q}$ is a trivial place of Δ.

It is clear that the place $\mathscr{Q}$ whose existence is asserted in Theorem 2 is uniquely determined by $\mathscr{P}$ and $\mathscr{P}'$ and that if both $\mathscr{P}$ and $\mathscr{P}'$ are places *over* k, then also $\mathscr{Q}$ is a place over k (i.e., a place of Δ/k).

COROLLARY. *If $\mathscr{P}$ and $\mathscr{P}'$ are places of K/k and $\mathscr{P} \xrightarrow{k} \mathscr{P}'$, then* $\dim \mathscr{P}'/k \leq \dim \mathscr{P}/k$. *Furthermore, if the residue field Δ of $\mathscr{P}$ has finite transcendence degree over k and $\mathscr{P}'$ is a specialization of $\mathscr{P}$ over k, then* $\dim \mathscr{P}'/k = \dim \mathscr{P}/k$ *if and only if $\mathscr{P}$ and $\mathscr{P}'$ are k-isomorphic places.*

We shall now investigate the following question: given a place $\mathscr{P}$ of K, find all the places of K of which $\mathscr{P}$ is a specialization. From Theorem 1 (§ 2) it follows at once that any ring (in K) which contains the valuation ring of a place of K is itself a valuation ring of a place of K. Hence our question is equivalent to the following: find all the subrings of K which contain $K_{\mathscr{P}}$. The answer to this equation is given by the following theorem:

THEOREM 3. *Any subring of K which contains $K_{\mathscr{P}}$ is necessarily the quotient ring of $K_{\mathscr{P}}$ with respect to some prime ideal of $K_{\mathscr{P}}$. If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are ideals in $K_{\mathscr{P}}$, then either $\mathfrak{M}_1$ contains $\mathfrak{M}_2$ or $\mathfrak{M}_2$ contains $\mathfrak{M}_1$ (and hence* *the set of rings between $K_{\mathscr{P}}$ and K is totally ordered by set-theoretic inclusion* $\subset$). *If $\mathscr{P}$ is a place of K/k and if tr.d. $K/k=r\neq\infty$, then $K_{\mathscr{P}}$ has only a finite number of prime ideals, and the number of prime ideals of $K_{\mathscr{P}}$ (other than $K_{\mathscr{P}}$ itself) is at most equal to $r-s$, where $s=\dim \mathscr{P}/k$.*

PROOF. Let L be a ring between $K_{\mathscr{P}}$ and K: $K_{\mathscr{P}}<L<K$. Then L is the valuation ring $K_{\mathscr{Q}}$ of a place $\mathscr{Q}$ of which $\mathscr{P}$ is a specialization and hence the prime ideal $\mathfrak{M}_{\mathscr{Q}}$ of $\mathscr{Q}$ is also a prime ideal in $K_{\mathscr{P}}$. Any element of $K_{\mathscr{P}}$ which is not in $\mathfrak{M}_{\mathscr{Q}}$ is a unit in $K_{\mathscr{Q}}$ (since $\mathfrak{M}_{\mathscr{Q}}$ is the ideal of non-units of $K_{\mathscr{Q}}$ and since $K_{\mathscr{P}}\subset K_{\mathscr{Q}}$). Hence the quotient ring of $K_{\mathscr{P}}$ with respect to the prime ideal $\mathfrak{M}_{\mathscr{Q}}$ (i.e., the set of all quotients a/b, where $a, b\in K_{\mathscr{P}}$ and $b\notin\mathfrak{M}_{\mathscr{Q}}$) is contained in $K_{\mathscr{Q}}$. On the other hand, we now show that any element x of $K_{\mathscr{Q}}$ belongs to the above quotient ring. This is obvious if $x\in K_{\mathscr{P}}$. Assume that $x\notin K_{\mathscr{P}}$. If we set $y=1/x$, then $y\in K_{\mathscr{P}}$ (since $K_{\mathscr{P}}$ is a valuation ring). Furthermore, $x\notin\mathfrak{M}_{\mathscr{Q}}$ (since $\mathfrak{M}_{\mathscr{Q}}\subset K_{\mathscr{P}}$), and hence x is a unit in $K_{\mathscr{Q}}$. Therefore also y is a unit in $K_{\mathscr{Q}}$, and so $y\notin\mathfrak{M}_{\mathscr{Q}}$. It follows that $x(=1/y)$ belongs to the quotient ring of $K_{\mathscr{P}}$ with respect to $\mathfrak{M}_{\mathscr{Q}}$. This proves the first part of the theorem.

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be any two proper ideals in $K_{\mathscr{P}}$ (not necessarily prime ideals) and assume that $\mathfrak{M}_1\not\subset\mathfrak{M}_2$. Let x be an element of $\mathfrak{M}_1$, not in $\mathfrak{M}_2$, and let y be any element of $\mathfrak{M}_2$, $y\neq 0$. Then $x/y\notin K_{\mathscr{P}}$, and hence $y/x\in K_{\mathscr{P}}$, $y\in\mathfrak{M}_1$ (since $\mathfrak{M}_1$ is an ideal and $x\in\mathfrak{M}_1$). Hence $\mathfrak{M}_2\subset\mathfrak{M}_1$.

Assume now that $\mathscr{P}$ is a place of K/k and that tr.d. $K/k=r\neq\infty$. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two prime ideals in $K_{\mathscr{P}}$ and let us assume that, say, $\mathfrak{M}_1>\mathfrak{M}_2$. Let L_i, $i=1, 2$, be the quotient ring of $K_{\mathscr{P}}$ with respect to $\mathfrak{M}_i$, and let $\mathscr{P}_i$ be a place of K whose valuation ring is L_i. We have $L_2>L_1$, and hence $\mathscr{P}_1$ is a *proper* specialization of $\mathscr{P}_2$. On the other hand, $\mathscr{P}$ is a specialization of $\mathscr{P}_1$. It follows by Theorem 2, Corollary, that $\dim \mathscr{P}/k\leqq\dim \mathscr{P}_1/k<\dim \mathscr{P}_2/k\leqq r$. This shows that the number of prime ideals of $K_{\mathscr{P}}$ is finite and that the number of prime ideals in $K_{\mathscr{P}}$, other than $K_{\mathscr{P}}$ itself, is at most $r-s$. This completes the proof of the theorem.

DEFINITION 1. *The ordinal type*† *of the totally ordered set of proper prime ideals $\mathfrak{q}$ of $K_{\mathscr{P}}$ ($\mathfrak{q}\neq(0)$, $\mathfrak{q}\neq K_{\mathscr{P}}$; $\mathfrak{q}_1$ precedes $\mathfrak{q}_2$ if $\mathfrak{q}_1>\mathfrak{q}_2$) is called the rank of the place $\mathscr{P}$.*

† In most axiomatic systems of set theory it is possible to attach to every totally ordered set E a well-defined object $o(E)$ in such a way that we have $o(E)=o(F)$ if and only if E and F are isomorphic ordered sets (i.e., if there exists a one-to-one mapping f of E onto F such that the relations $x\leqq y$ and $f(x)\leqq f(y)$ are equivalent). The object $o(E)$ is called the *ordinal type* of E. Furthermore, if E is isomorphic to the set $\{1, 2, \cdots, n\}$ (i.e., if E is a finite, totally ordered set with n elements), we shall identify its ordinal type with its cardinal number n.