Transport Theory

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Transport Equations

The term "transport theory" is commonly used to refer to the mathematical description of the transport of particles through a host medium. For example, such a theory might be used to describe the diffusion of neutrons through the uranium fuel elements of a nuclear reactor, or the diffusion of light photons through the atmosphere, or perhaps the motion of gas molecules as they stream about, colliding with one another. (Note that in the last example, one cannot really distinguish the "transported" particles from the "host medium.") Transport theory has become an extremely important topic in physics and engineering, since particle transport processes arise in a wide variety of physical phenomena. Much of the early development of this theory was stimulated by astrophysical studies of radiant energy transfer in stellar or planetary atmospheres.[1,2] More recently, the subject of transport theory has been refined to a very high degree for the description of neutron and gamma transport in nuclear systems.[3–6] The mathematical tools used to analyze transport processes also have been applied with some success to problems in rarefied gas dynamics and plasma physics.[7–12] And the list of such applications continues to expand rapidly (as the examples listed in Table 1.1 and Figure 1.1 make apparent).

The transport processes we wish to study can involve a variety of different types of particles such as neutrons, gas molecules, ions, electrons, quanta (photons, phonons), or waves (provided the wavelength is much less than a mean free path), moving through various background media such as the components of a nuclear reactor core, stellar or planetary atmospheres, gases, or plasmas. Transport phenomena range from random walk processes, in which particles stream freely between random interaction events, to highly ordered collective phenomena, in which large numbers of particles interact in a correlated fashion to give rise to wave motion. And yet all these processes can be described by a single unifying theory—indeed, all are governed by the same type of equation. Hence the mathematical tools needed to study these processes are quite similar, although the information desired and the physical interpretation of the solutions differ quite markedly from field to field.

We are concerned with the mathematical description of the transport of particles in matter. Transport theory differs from the usual approaches encountered in classical physics because it is a particle, not a continuum

Table 1.1 □ Applications of Transport Theory

Nuclear reactors
Determination of neutron distributions in reactor cores
Shielding against intense neutron and gamma radiation
Astrophysics
Diffusion of light through stellar atmospheres (radiative transfer)
Penetration of light through planetary atmospheres
Rarefied gas dynamics
Upper atmosphere physics
Sound propagation
Diffusion of molecules in gases
Charged particle transport
Multiple scattering of electrons
Gas discharge physics
Diffusion of holes and electrons in semiconductors
Development of cosmic ray showers
Transport of electromagnetic radiation
Multiple scattering of radar waves in a turbulent atmosphere
Penetration of X-rays through matter
Plasma physics
Microscopic plasma dynamics, microinstabilities
Plasma kinetic theory
Other
Traffic flow (transport of vehicles along highways)
Molecular orientations of macromolecules
The random walk of undergraduates during registration

theory of matter as, for instance, electromagnetism or fluid dynamics. The concept of a continuous field still plays a significant role in transport studies, but it now appears as a probability field, much as one encounters in quantum mechanics.

To be more specific, the usual macroscopic fields encountered in physics involve continuum descriptions. For example, in electromagnetic theory one introduces the electric and magnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ and the charge and current densities $\rho(r,t)$ and $\mathbf{j}(\mathbf{r},t)$. In hydrodynamics the field variables are the mass density $\rho(\mathbf{r},t)$, local flow velocity $\mathbf{u}(\mathbf{r},t)$, and local temperature $T(\mathbf{r},t)$. However in the study of particle transport, the random nature of particle interaction events obliges us to introduce instead a field of probability densities or distribution functions. That is, we cannot predict with certainty the exact number of particles in a certain region at a

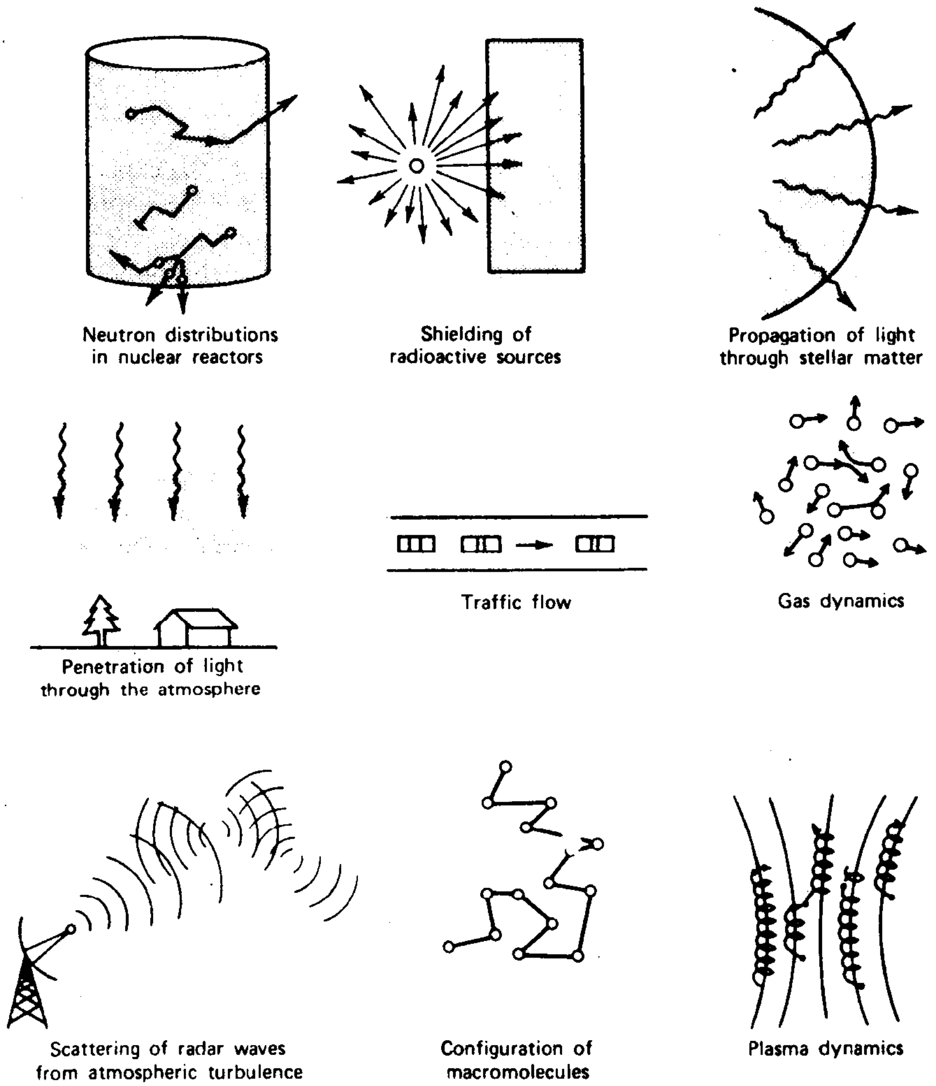

Fig. 1.1 □ Examples of transport processes.

given time, but only the expected particle density $N(\mathbf{r}, t)$ defined by

$$N(\mathbf{r}, t)d^3r = \text{expected number of particles in } d^3r \text{ about } \mathbf{r} \text{ at time } t$$

This density would then be described by an appropriate partial differential equation such as the diffusion equation:

$$\frac{\partial N}{\partial t} - \nabla \cdot D \nabla N(\mathbf{r},t) = S(\mathbf{r},t)$$

It is important to stress here that $N(\mathbf{r},t)$ characterizes only the expected or average particle density at $\mathbf{r}$ and t. Our mathematical description of particle transport processes involves such a statistical approach. We return to consider this feature and provide a more precise definition of this statistical average in the last section of this chapter.

One can distinguish between two classes of problems that arise in transport theory. First are the *direct problems* in which one is given the composition and geometry of the host medium and the location and magnitude of any sources of particles and asked to determine the distribution of particles in the medium. This is the most common class of transport problems. It arises in a host of applications, including nuclear reactor theory, radiation transport, plasma physics, and gas dynamics. The second and far more difficult class involves *inverse problems* in which one is given the distribution and asked to determine characteristics of the medium through which the particles have propagated or the sources that have generated the particles. Such problems are encountered in fields such as astrophysics in which one measures the intensity and spectral distribution of light in order to infer properties of stars, and in nuclear medicine where radioisotopes are injected into patients, and the radiation emitted by such sources is used in diagnosis—for example, to determine whether a tumor is present.

Although transport theory arises in a wide variety of disciplines, within each field it has become a very specialized subject, almost an art, dealing with the solution of a very particular type of equation. Furthermore, most of the applications of transport theory have developed almost totally independent of one another. For example, the essential physics of transport processes was already highly developed in the kinetic theory of gases developed by Boltzmann more than a century ago. The mathematical methods used to solve transport equations were developed to analyze problems in radiative transfer during the 1930s. Despite this heritage, the field of neutron transport theory has developed almost independently of kinetic theory or radiative transfer, partly because of the highly specialized nature of neutron transport problems in nuclear systems, but also partly because of the enormous emphasis placed on this discipline in the atomic energy program. Particular emphasis was directed toward the development of accurate computational (computer-based) methods, most of which are quite unfamiliar to physicists in other fields.

Hence there is a very strong incentive to unify the various approaches used to analyze and solve transport problems in different fields. The task of drawing together these applications and presenting a general, unified theory of particle transport processes is one of the primary motivations for writing this book.

1.1 □ PARTICLE DISTRIBUTION FUNCTIONS □ The ultimate goal of transport theory is to determine the distribution of particles in a medium, taking account of the motion of the particles and their interactions with the host medium. Although knowledge of the particle density $N(\mathbf{r},t)$ would be sufficient for most applications, unfortunately there is no equation that adequately describes this quantity in most physical situations. Therefore we must generalize the concept of the particle density somewhat to account for more of the independent variables that characterize particle motion.

The state of a classical point particle can be characterized by specifying the particle position $\mathbf{r}$ and velocity $\mathbf{v}$. This level of description is usually sufficient for describing the transport of more complicated particles (neutrons, photons, molecules, automobiles), since internal variables such as spin, polarization, or structure usually do not influence the motion of the particles as they stream freely between interactions—although such internal variables certainly influence the interactions between the particles and the host medium. (Exceptions to this include the transport of polarized light through an atmosphere[13] and the transport of a polarized neutron beam through a magnetic field.[14] We indicate later how the theory can be generalized to account for spin or polarization effects.) Therefore it suffices to define a particle *phase space density* function $n(\mathbf{r},\mathbf{v},t)$ that depends only on the particle position and velocity:

$n(\mathbf{r},\mathbf{v},t)d^3r\,d^3v =$ expected number of particles in d^3r about $\mathbf{r}$ with velocity in d^3v about $\mathbf{v}$ at time t

This function contains all the information that is usually required for the description of transport processes. For example, we can integrate $n(\mathbf{r},\mathbf{v},t)$ over velocity to obtain the particle density

$$N(\mathbf{r},t) = \int d^3v\, n(\mathbf{r},\mathbf{v},t)$$

In certain [illegible] cases $n(\mathbf{r},\mathbf{v},t)$ may be rather easy to calculate. For example, if the particles are in thermal equilibrium at a temperature T, then $n(\mathbf{r},\mathbf{v},t)$ becomes just the familiar Maxwell-Boltzmann distribution function

$$n(\mathbf{r},\mathbf{v},t)\to n_0 M(\mathbf{v}) = n_0\left(\frac{m}{2\pi kT}\right)^{3/2}\exp\left(\frac{-mv^2}{2kT}\right)$$

where n_0 is the average number density of the particles. More generally we are faced with solving a special type of equation for $n(\mathbf{r},\mathbf{v},t)$ known as a "transport" or "kinetic" equation. However it is usually possible to derive such an equation to describe $n(\mathbf{r},\mathbf{v},t)$ to a rather high degree of accuracy.

In kinetic theory[9] it is common to normalize $n(\mathbf{r},\mathbf{v},t)$ by dividing through by the particle density $N(\mathbf{r},t)$

$$f(\mathbf{r},\mathbf{v},t) = \frac{n(\mathbf{r},\mathbf{v},t)}{N(\mathbf{r},t)}$$

This terminology is useful because $f(\mathbf{r},\mathbf{v},t)$ can then be identified as a probability distribution or density function with a unit normalization:

$$\int d^3v f(\mathbf{r},\mathbf{v},t) = 1$$

Both $n(\mathbf{r},\mathbf{v},t)$ and $f(\mathbf{r},\mathbf{v},t)$ contain information only about the expected number of particles in a differential volume element of phase space, $d^3r\,d^3v$. Neither function provides any information about higher order statistical correlations such as the "doublet" distribution $f(\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2;t)$ characterizing the probability that two particles will be found simultaneously with coordinates in $d^3r_1\,d^3v_1\,d^3r_2\,d^3v_2$. Actually there is little interest in such higher order correlations or fluctuations from $n(\mathbf{r},\mathbf{v},t)$ for random walk processes in which the particles of interest do not interact and therefore can be correlated only by special types of source conditions (e.g., the simultaneous emission of two or more neutrons in a fission reaction[6]). However such higher order phase space densities or distribution functions are of major interest in collective processes that are dominated by interactions (hence correlations) among particles.

It is sometimes convenient to decompose the particle velocity vector $\mathbf{v}$ into two components, one variable characterizing the particle speed, and a second corresponding to the direction of motion. The particle kinetic energy, $E=\frac{1}{2}mv^2$ is used frequently instead of the speed v. To specify the direction of particle motion, we introduce a unit vector $\hat{\boldsymbol{\Omega}}$ in the direction of the velocity vector $\mathbf{v}$ (see Figure 1.2)

$$\hat{\boldsymbol{\Omega}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \hat{\mathbf{e}}_x \sin\theta\cos\phi + \hat{\mathbf{e}}_y \sin\theta\sin\phi + \hat{\mathbf{e}}_z\cos\theta$$

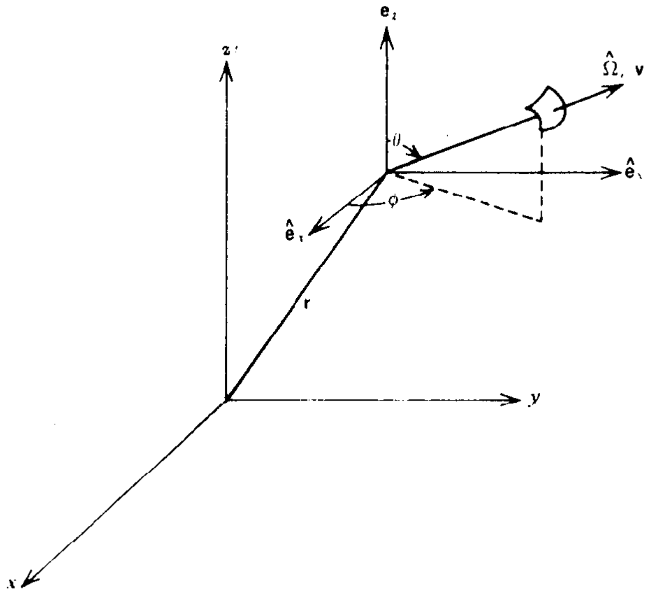

Fig. 1.2 □ The position and direction variables characterizing the state of a particle.

where we have chosen to represent this direction unit vector in spherical velocity-space coordinates (θ, ϕ). The particle phase space density can then be defined in terms of these new variables as

$n(\mathbf{r}, E, \hat{\mathbf{\Omega}}, t)\, d^3r\, dE\, d\hat{\mathbf{\Omega}} =$ expected number of particles in d^3r about $\mathbf{r}$ with kinetic energy E in dE moving in direction $\hat{\mathbf{\Omega}}$ in solid angle $d\hat{\mathbf{\Omega}}$

Integration over these velocity space variables would then take the form

$$\int d^3v\, n(\mathbf{r}, \mathbf{v}, t) = \int_0^\infty dv\, v^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta\, n(\mathbf{r}, v, \hat{\mathbf{\Omega}}, t)$$

$$= \int_0^\infty dE \int d\hat{\mathbf{\Omega}}\, n(\mathbf{r}, E, \hat{\mathbf{\Omega}}, t)$$

where we have identified the differential solid angle $d\hat{\mathbf{\Omega}} = \sin\theta\, d\theta\, d\phi$. One can easily transform back and forth between various sets of variables by noting:

$$n(\mathbf{r}, E, \hat{\Omega}, t) = \left(\frac{v}{m}\right) n(\mathbf{r}, \mathbf{v}, t)$$

$$n(\mathbf{r}, v, \hat{\Omega}, t) = v^2 n(\mathbf{r}, \mathbf{v}, t)$$

$$n(\mathbf{r}, E, \hat{\Omega}, t) = \left(\frac{1}{mv}\right) n(\mathbf{r}, v, \hat{\Omega}, t)$$

where $E = \frac{1}{2} mv^2$ and $\hat{\Omega} = \mathbf{v}/|\mathbf{v}|$.

When the particle phase space density is written in terms of the variables $(\mathbf{r}, E, \hat{\Omega}, t)$, it is sometimes referred to as the *angular* density (since it depends on the angles θ and ϕ characterizing the direction of particle motion) to distinguish it from the *total* particle density $N(\mathbf{r}, t)$.

A closely related concept is the *phase space current density* function or *angular current density* $\mathbf{j}(\mathbf{r}, \mathbf{v}, t)$, which is defined by

$\mathbf{j}(\mathbf{r}, \mathbf{v}, t) \cdot d\mathbf{S}\, d^3v = \mathbf{v} n(\mathbf{r}, \mathbf{v}, t) \cdot d\mathbf{S}\, d^3v =$ expected number of particles that cross an area dS per second with velocity $\mathbf{v}$ in d^3v at time t (see Figure 1.3)

If this quantity is integrated over particle velocities, one arrives at a definition of the particle *current density* $\mathbf{J}(\mathbf{r}, t)$

$$\mathbf{J}(\mathbf{r}, t) = \int d^3v\, \mathbf{j}(\mathbf{r}, \mathbf{v}, t)$$

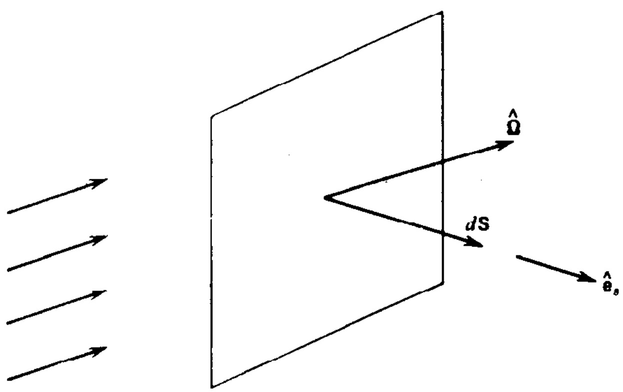

Fig. 1.3 ☐ Particles incident on a surface element $d\mathbf{S}$.

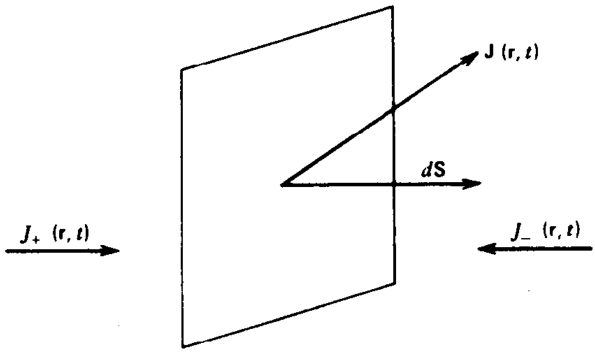

Fig. 1.4 □ Partial and total current densities.

Here, of course, $\mathbf{J}(\mathbf{r},t)\cdot d\mathbf{S}$ would be interpreted as the rate at which particles pass through a differential surface area dS.

A similar concept is the *partial current density* $J_{\pm}(\mathbf{r},t)$, which characterizes the rate at which particles flow through an area in a given direction. That is, we define

$$J_{\pm}(\mathbf{r},t)=\pm\int_{\pm} d^3v\, \hat{\mathbf{e}}_s\cdot\mathbf{j}(\mathbf{r},\mathbf{v},t)$$

where $\hat{\mathbf{e}}_s$ is the unit normal to the surface, and the velocity space integration is taken over only those particle directions in the positive or negative direction (see Figure 1.4). From this definition it is apparent that

$$\hat{\mathbf{e}}_s\cdot\mathbf{J}(\mathbf{r},t)=J_{+}(\mathbf{r},t)-J_{-}(\mathbf{r},t)$$

In this sense, $\mathbf{J}(\mathbf{r},t)$ might be referred to as the "net" current density.

We have employed a consistent notation in which quantities that are dependent on phase space or angle are denoted by lowercase symbols (e.g., n or $\mathbf{j}$) and configuration space- or angle-integrated quantities are denoted by uppercase symbols (N or $\mathbf{J}$).

1.2 □ DERIVATION OF A GENERIC FORM OF THE TRANSPORT EQUATION □ We now derive an exact (albeit formal) equation for the phase space density $n(\mathbf{r},\mathbf{v},t)$ characterizing a transport process by simply balancing the various mechanisms by which particles can be gained or lost from a volume of material. That is, we begin by considering an arbitrary volume V and attempt to calculate the time rate of change of the number of particles in this volume that have velocities $\mathbf{v}$ in d^3v (see Figure 1.5). If

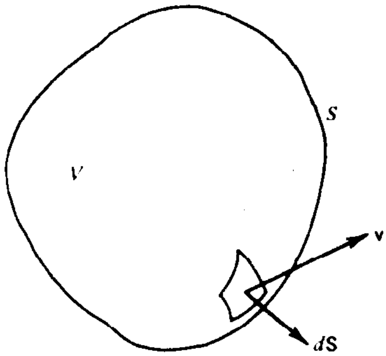

Fig. 1.5 □ An arbitrary volume V with surface area S.

we ignore for the moment macroscopic forces that might change the trajectories of the particles in V, it is apparent that the only mechanisms that can change the particle number are leakage through the surface of V, collision events that change the particle velocities, or sources in V:

$$\begin{bmatrix}\text{time rate}\\ \text{of change}\\ \text{of } n\end{bmatrix} = \begin{bmatrix}\text{change due}\\ \text{to leakage}\\ \text{through } S\end{bmatrix} + \begin{bmatrix}\text{change due}\\ \text{to}\\ \text{collisions}\end{bmatrix} + \begin{bmatrix}\\ \text{sources}\\ \\ \end{bmatrix}$$

We can express this balance condition mathematically as follows:

$$\frac{\partial}{\partial t}\int_V d^3r\, n(\mathbf{r},\mathbf{v},t)d^3v = -\int_S d\mathbf{S}\cdot\mathbf{j}(\mathbf{r},\mathbf{v},t)d^3v + \int_V d^3r\left(\frac{\partial n}{\partial t}\right)_{\text{coll}} d^3v$$
$$+\int_V d^3r\, s(\mathbf{r},\mathbf{v},t)d^3v$$

where we have defined the source density function $s(\mathbf{r},\mathbf{v},t)$ and the time rate of change due to collisions $(\partial n/\partial t)_{\text{coll}}$. If our choice of the arbitrary volume does not depend on time, we can bring $\partial/\partial t$ inside the integral over V. Furthermore we can use Gauss's law to rewrite the surface integral for the leakage contribution as a volume integral

$$\int_S d\mathbf{S}\cdot\mathbf{j}(\mathbf{r},\mathbf{v},t) = \int_V d^3r\,\nabla\cdot\mathbf{j}(\mathbf{r},\mathbf{v},t) = \int_V d^3r\,\nabla\cdot\mathbf{v}n(\mathbf{r},\mathbf{v},t) = \int_V d^3r\,\mathbf{v}\cdot\nabla n(\mathbf{r},\mathbf{v},t)$$

where we have noted that $\nabla\cdot\mathbf{v}n(\mathbf{r},\mathbf{v},t) = \mathbf{v}\cdot\nabla n(\mathbf{r},\mathbf{v},t)$, since $\mathbf{r}$ and $\mathbf{v}$ are independent variables. Thus our balance condition can be rewritten as follows:

$$\int d^3r\,d^3v\left\{\frac{\partial n}{\partial t} + \mathbf{v}\cdot\nabla n - \left(\frac{\partial n}{\partial t}\right)_{\text{coll}} - s\right\} = 0 \qquad (1.1)$$

But since V is arbitrary, Eq. 1.1 can be satisfied for all V only if the integrand itself is identically zero:

$$\frac{\partial n}{\partial t}+\mathbf{v}\cdot\nabla n(\mathbf{r},\mathbf{v},t)=\left(\frac{\partial n}{\partial t}\right)_{\text{coll}}+s(\mathbf{r},\mathbf{v},t) \qquad (1.2)$$

Hence we have arrived at an equation for the phase space density $n(\mathbf{r},\mathbf{v},t)$. This is the general form taken by the *transport equations* that characterize particle transport processes in an enormous variety of applications.

We can give a somewhat shorter derivation of this equation (and relax the assumption concerning macroscopic forces on the particles) by simply equating the substantial derivative[15] describing the time rate of change of the local particle density along the particle trajectory to the change in the local density due to collisions and sources

$$\frac{Dn}{Dt}=\left(\frac{\partial n}{\partial t}\right)_{\text{coll}}+s$$

We can calculate Dn/Dt explicitly as

$$\frac{Dn}{Dt}=\frac{\partial n}{\partial t}+\frac{\partial \mathbf{r}}{\partial t}\cdot\frac{\partial n}{\partial \mathbf{r}}+\frac{\partial \mathbf{v}}{\partial t}\cdot\frac{\partial n}{\partial \mathbf{v}}=\frac{\partial n}{\partial t}+\mathbf{v}\cdot\frac{\partial n}{\partial \mathbf{r}}+\frac{\mathbf{F}}{m}\cdot\frac{\partial n}{\partial \mathbf{v}}$$

where we have introduced the obvious notation for the vector differentiation operations: $\partial/\partial\mathbf{r}\equiv\nabla$ (e.g., $\text{grad}\, n=\nabla n=\partial n/\partial\mathbf{r}$). Therefore we find that the transport equation takes the form

$$\frac{\partial n}{\partial t}+\mathbf{v}\cdot\frac{\partial n}{\partial \mathbf{r}}+\frac{\mathbf{F}}{m}\cdot\frac{\partial n}{\partial \mathbf{v}}=\left(\frac{\partial n}{\partial t}\right)_{\text{coll}}+s$$

The source term $s(\mathbf{r},\mathbf{v},t)$ in this equation is usually assumed to be specified in advance; therefore it is independent of the solution $n(\mathbf{r},\mathbf{v},t)$. However in certain situations (e.g., neutrons generated in a fission reaction) it may be convenient to allow the source to contain an "intrinsic" component dependent on the phase space density $n(\mathbf{r},\mathbf{v},t)$ itself.

To proceed further, we must be a bit more specific about the collision term $(\partial n/\partial t)_{\text{coll}}$, and this calls for a few more definitions, to enable us to adequately describe collision processes. For the present we assume that such collisions or interactions with the background medium occur instantaneously at a point in space. That is, we assume that particles stream along until they suffer a collision, at which point they are instantaneously absorbed or scattered to a new velocity. It should be apparent that such an assumption would not be valid for processes in which the ranges of the interaction forces are large, or in which the particle is absorbed, then reemitted some time later. We patch up these deficiencies later.

We now introduce the concept of a *mean free path* (mfp) to characterize such "local" interaction events:

$(mfp)^{-1} \equiv \Sigma(\mathbf{r}, \mathbf{v}) \equiv$ probability of particle interaction per unit distance traveled by particle of velocity $\mathbf{v}$ at position $\mathbf{r}$

We follow the customary terminology of radiation transport by referring to the inverse *mfp* $\Sigma(\mathbf{r}, \mathbf{v})$ as the *macroscopic cross section* characterizing the interaction. This latter quantity can be related to the more familiar concept of a microscopic interaction cross section σ by noting

$$\Sigma(\mathbf{r}, \mathbf{v}) = N_B(\mathbf{r})\sigma(\mathbf{v})$$

where $N_B(\mathbf{r})$ is the number density of the background medium.

We must generalize this concept a bit to describe scattering processes or interaction processes in which the incident particle is absorbed in the collision event and several secondary particles are then emitted (e.g., nuclear fission events or the stimulated emission of light). Indeed, since transport theory is essentially just a mathematical description of "multiple scattering" processes in which the particles of interest wander through a medium, making numerous collisions as they go,[6] it is important to introduce the concept of a *scattering probability function* $f(\mathbf{v}' \to \mathbf{v})$ defined by

$f(\mathbf{v}' \to \mathbf{v})d^3v \equiv$ probability that any secondary particles induced by an incident particle with velocity $\mathbf{v}'$ will be emitted with velocity $\mathbf{v}$ in d^3v

Note that $f(\mathbf{v}' \to \mathbf{v})$ is essentially just a transition probability characterizing a change of state of the particle from $\mathbf{v}'$ to $\mathbf{v}$.

As a further characterization of such processes, we define the mean number of secondary particles emitted per collision event, $c(\mathbf{r}, \mathbf{v})$, by

$c(\mathbf{r}, \mathbf{v}) \equiv$ mean number of secondary particles emitted in a collision event experienced by an incident particle with velocity $\mathbf{v}$ at position $\mathbf{r}$

It is also useful to define the *collision kernel* $\Sigma(\mathbf{v}' \to \mathbf{v})$ characterizing such processes by

$$\Sigma(\mathbf{r}, \mathbf{v}' \to \mathbf{v}) = \Sigma(\mathbf{r}, \mathbf{v}')c(\mathbf{r}, \mathbf{v}')f(\mathbf{r}, \mathbf{v}' \to \mathbf{v})$$

This describes the probability per unit distance traveled that an incident particle of velocity $\mathbf{v}'$ will suffer a collision in which a secondary particle of velocity $\mathbf{v}$ is produced (which may be the original particle but with a new velocity in a simple scattering event). Note that by definition

$$\Sigma(\mathbf{r},\mathbf{v}) = \int d^3v' \Sigma(\mathbf{r},\mathbf{v}\rightarrow\mathbf{v}')$$

Again we stress that these definitions are useful only if the collision events are localized and uncorrelated. For example, if the particles are wavelike (e.g., photons or quantum mechanical particles), the interaction events would have to be sufficiently well separated to ensure the loss of phase information from one event to another—that is, mean free paths must be much larger than the particle wavelengths. In a similar sense, the mean free path must be much larger than the range of the interaction forces characterizing the collision events.

These concepts can now be used to obtain an explicit form for the collision term $(\partial n/\partial t)_{\text{coll}}$ appearing in the transport equation. First we note that the frequency of collision events experienced by a particle of velocity $\mathbf{v}$ is given by

$$v\Sigma(\mathbf{r},\mathbf{v}) \equiv \text{collision frequency}$$

Hence the rate at which such reactions will occur in a unit volume can be written as

$$v\Sigma(\mathbf{r},\mathbf{v})n(\mathbf{r},\mathbf{v},t) \equiv \text{reaction rate density}$$

If we now note that the rate at which particles of velocity $\mathbf{v}$ suffer interactions that change their velocity or perhaps destroy the particle is $v\Sigma(\mathbf{r},\mathbf{v})\ n(\mathbf{r},\mathbf{v},t)$, while the rate at which particles of different velocities $\mathbf{v}'$ induce the production of secondary particles of velocity $\mathbf{v}$ is $v'\Sigma(\mathbf{r},\mathbf{v}'\rightarrow\mathbf{v})\ n(\mathbf{r},\mathbf{v}',t)d^3v'$, we can immediately identify the collision term in the transport equation as

$$\left(\frac{\partial n}{\partial t}\right)_{\text{coll}} = \int d^3v'\, v'\Sigma(\mathbf{r},\mathbf{v}'\rightarrow\mathbf{v})n(\mathbf{r},\mathbf{v}',t) - v\Sigma(\mathbf{r},\mathbf{v})n(\mathbf{r},\mathbf{v},t)$$

(If we recall the identification of $\Sigma(\mathbf{r},\mathbf{v}'\rightarrow\mathbf{v})$ as essentially a transition kernel, it is apparent that the collision term assumes a form reminiscent of the master equation characterizing Markov stochastic processes.) We can now write the general form of the transport equation as

$$\frac{\partial n}{\partial t} + \mathbf{v}\cdot\frac{\partial n}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m}\cdot\frac{\partial n}{\partial \mathbf{v}} + v\Sigma n = \int d^3v'\, v'\Sigma(\mathbf{v}'\rightarrow\mathbf{v})n(\mathbf{r},\mathbf{v}',t) + s \qquad (1.3)$$