Developments in the theory of turbulence

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Preface

UNTIL about 1960, the attention given to turbulence theory was much less than was justified by its obvious interest and importance. Practising engineers were repelled by its complexity and by its inability to tell them anything that they actually wanted to know. Its relative neglect by the mathematical physicists is harder to understand since its structure is similar to, but simpler than, that of quantum field theories.

In 1958 the theory could give only a few isolated results, and the central problem of closure of the Navier–Stokes equations remained quite untouched. In 1958 and 1959 R. H. Kraichnan published two papers which contain the essence of his Direct Interaction method for effecting this closure. In the years since then, he has devoted much effort to justifying this method and to examining the predictions which it makes of phenomena at high wavenumbers. There is now an increasing body of opinion which holds that, although his work is not conclusive, it does represent a great advance in our understanding, and that much in it is of permanent value.

In 1964, S. F. Edwards offered another method of closing the Navier–Stokes equations which was based on Fokker–Planck principles. Subsequent work by Kraichnan, Edwards, and myself has shown that, although the Fokker–Planck and the Direct Interaction methods are not identical, yet they are very closely related. The same is true of other closure methods which have been devised since.

My aim in writing this book has been to make these developments accessible to a wider circle of people, and to encourage more research workers to take up this difficult but interesting and rewarding subject. I have had in mind readers both with engineering and with mathematical backgrounds. For the sake of the engineers, I have kept the mathematics as simple as is consistent with a proper exposition of the subject and mathematical ideas which such readers may not have met before are explained in the first two Chapters. For mathematical readers, I have tried to explain which aspects of the subject are of particular concern to engineers, and to give a little of the engineering background.

Although the total volume of Kraichnan's published work is very large, his papers are terse in that they omit a good deal of algebraic detail. I have not tried to describe everything that he has done, but I have given more detail than he does of those parts of his work which I do discuss.

As already noted, Chapters 1 and 2 are a mathematical introduction to the subject. Chapter 3 explains the problem of closure, the central difficulty of the theory of turbulence, and also contains further introductory mathematics. The Direct Interaction method is explained in Chapter 4, justified in Chapter 5 and applied to the Navier–Stokes equations in Chapter 6.

Other methods of closure, including the Fokker–Planck method, are described in Chapter 7. Chapter 8 is devoted to the diffusion of a passive scalar by a turbulent velocity field; this chapter includes a number of results which antedate the invention of Direct Interaction. Chapters 9–12 show how Direct Interaction can be combined with a Lagrangian representation of the fluid motion, the need for this change of representation being explained in Chapter 9.

In the last three chapters, an attempt is made to connect the rather abstract theory presented in the rest of the book with the more practical aspects of turbulent flow. Chapter 13 is mainly for the more mathematical reader. It describes the properties of a particular real flow which is used to illustrate the methods. Chapter 14 explains some of the methods which are currently used to make engineering calculations. It lays particular emphasis on a new method due to Hanjalic and Launder which, besides its obvious practical success, seems to be closely related to the more fundamental theory.

Finally, Chapter 15 sets out my own ideas on how Direct Interaction might be used to solve this problem. These ideas have not been fully worked out, and such relatively undeveloped material would not normally be put in a book. I have put it in because it is the only work known to me (apart from a purely formal paper by Kraichnan) which attempts to relate the theory explained in the rest of the book to engineering problems; as such, it will, it is hoped, be of particular interest to engineering readers. Such readers are advised to omit the whole of Chapters 5 and 7, §§ 9.4–9.7 of Chapter 9, the whole of Chapter 10, and §§ 11.4 and 11.5 of Chapter 11 in their first encounter with the book.

A glossary of the notation follows on pp. xv–xix. In addition to a list of symbols, this glossary describes the principles underlying the notation and indicates briefly how it differs from that of other authors.

It is a pleasure to thank the people who have helped me to write this book. My greatest debt is to Dr. S. F. Edwards F.R.S., Plummer Professor of Physics in the University of Cambridge. He introduced me to the subject, and has patiently steered me round some of its many pitfalls. If I persist in falling into them, that is not his fault. I have been much helped by correspondence with, and a visit to, Dr. Robert H. Kraichnan and by discussions with Mr. P. Bradshaw of the Department of Aeronautical Engineering at Imperial College and with Dr. B. E. Launder of the Department of Mechanical Engineering at the same College. These four people have seen the book (or parts of it) in manuscript. Both they and the reviewers of the Clarendon Press have made helpful suggestions and comments, many of which have been incorporated. Naturally, this does not relieve me of sole responsibility for the book's defects.

The first half of this book was written while I was on the staff of the Atomic Energy Establishment, Winfrith, and I am very grateful for the constant support which I received from the Director, Mr. D. W. Fry, and my Head of

Division, Mr. W. S. Eastwood. Finally, I am much indebted to Miss Jennifer Rogers, Mrs. Linda Hawkes, Mrs. Sylvia Monk, and Mrs. Norah Lowe for their devoted typing of a difficult manuscript, and to the authors and editors of journals who have allowed me to reproduce figures from published papers.

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Contents

1. Introductory material

1.1. Historical note

VIEWED mathematically, the central problem of turbulence theory is to obtain statistical solutions of the Navier–Stokes equations, but it has taken a long time for this fact to become apparent. Some remarkable engravings show that Leonardo da Vinci studied the physical appearance of turbulent flow, but the first paper to show any understanding of the phenomenon was published by Osborne Reynolds in 1883. This was followed in 1895 by a second publication in which he introduced the ideas of Reynolds stress and Reynolds number. These papers describe turbulence as 'sinuous motion'; their contents do not suggest that Reynolds was aware that he was dealing with a statistical phenomenon. The statistical nature of turbulence was first realized by Taylor (1921, 1935); this is implicit in his first paper and explicit in the title of his second paper.

There has been a rather agonizing gap between the realization of the nature of the problem and the first steps toward solving it. Although some interesting results were produced between 1930 and 1960, the main development in this period was an increasing realization of the difficulties of turbulence theory; it was found that the normal techniques of fluid mechanics were not powerful enough for the work. The state of knowledge at that time is well summarized in the books of Batchelor (1971), Townsend (1956), and Hinze (1959).

In the late nineteen-fifties and early sixties, first Kraichnan and then Edwards applied the techniques of quantum field theory to this problem and were able to overcome, to some extent at least, the difficulties which had defeated earlier workers. This book is mainly concerned with these later developments.

1.2. Tensor notation

A Cartesian tensor notation is always used in work on turbulence, and no other notation can satisfactorily describe the quantities which arise in this work. Readers who are unfamiliar with the notation should study it in one of the standard texts, of which the book by Jeffreys and Jeffreys (1966) is an attractive example. The general tensor calculus which is used in the theory of relativity is complex, and this complexity has given the subject a bad name. The Cartesian tensor notation, which is all that is needed for this work, is no more complex than vector algebra.

The reader who is unfamiliar with the subject has to master only three basic ideas. The first is the use of a suffix to denote the components of a vector or a tensor; thus the coordinate vector (x, y, z) is written $(x_1, x_2, x_3) = x_i$ and the velocity vector (u, v, w) as $(u_1, u_2, u_3) = u_i$. This vector is a function

of the space coordinates, a fact which is recognized by writing

$$u_i = u_i(\mathbf{x}).$$

The right-hand side should strictly be written $u_i(x_j)$ but this seems unnecessarily pedantic. Similarly the differential vector operator

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

is written $\partial/\partial x_i$.

The second basic idea is that repeated indices are summed from 1 to 3; thus

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

a quantity which is written $\mathbf{a} \cdot \mathbf{b}$ in vector notation. Similarly div $\mathbf{a}$ is written $\partial a_i/\partial x_i$. The tensor notation for curl $\mathbf{a}$ is more awkward, but fortunately we shall have no occasion to use it.†

Lastly, the reader must acquaint himself with the Kronecker symbol

$$\begin{aligned} \delta_{ij} &= 1 \quad \text{if } i = j \\ &= 0 \quad \text{if } i \neq j \end{aligned} \tag{1.1}$$

whose uses will become clear as we go on. Among its more evident properties are

$$\delta_{ii} = 3 \qquad \text{and} \qquad \delta_{ij} a_j = a_i.$$

1.3. The equations of motion

Throughout this book, we confine ourselves to the motion of incompressible Newtonian fluids with constant physical properties, since it seems sensible to restrict ourselves to the simplest possible problem until we start to get some real results. There is, of course, no implication that the effects of compressibility, non-Newtonian behaviour, or varying physical properties are uninteresting or unimportant.

In tensor notation the incompressibility condition reads

$$\partial u_i/\partial x_i = 0, \tag{1.2}$$

while the Navier–Stokes equations take the form

$$\frac{\partial u_i}{\partial t} + u_m \frac{\partial u_i}{\partial x_m} - \nu \nabla^2 u_i = -\frac{\partial p}{\partial x_i}. \tag{1.3}$$

Here ν is the kinematic viscosity, which has dimensions $L^2 T^{-1}$ p is the normal pressure divided by the (constant) density; it is sometimes called the kinematic pressure. Note that we do not hesitate to write ∇^2 instead of

$$\frac{\partial^2}{\partial x_j \partial x_j}$$

† The scalar quantity A_{ii} is called the trace of the tensor A_{ij}, and the operation $j = i$ is known as contraction.

which would again be rather pedantic. Also, the 'inertial term' is easily written in tensor notation. The various vector notations for this term are all rather strained and artificial.

We shall also consider the convection of a passive scalar by this velocity field. This is governed by the equation

$$\frac{\partial \psi}{\partial t} + u_m \frac{\partial \psi}{\partial x_m} - \kappa \nabla^2 \psi = 0, \tag{1.4}$$

κ being the appropriate diffusivity. For example, ψ might be the temperature, in which case κ would be the thermal diffusivity $k/\rho C_p$, k, ρ, and C_p being the thermal conductivity, the density, and the specific heat respectively. In this case, it is implied that the temperature differences are too small to affect the physical properties of the fluid. Alternatively, ψ might be the concentration of a second substance, and κ would then be the molecular diffusivity. Again, the concentration of this second substance would have to be small enough not to affect the physical properties of the first substance.

There are, of course, many useful and interesting extensions of eqns (1.3) and (1.4). One of the simplest is obtained by making appropriate allowance for the thermal expansion of the fluid by adding the 'Boussinesq term' $\alpha g \psi \delta_{i3}$ to the right-hand side of (1.3), where α is the coefficient of thermal expansion of the fluid, g is the acceleration due to gravity, $\psi = T - T_0$, T being a general temperature and T_0 a reference temperature, and x_3 points vertically upwards. The resulting equations describe Boussinesq turbulence, which occurs in a fluid heated from below and which is driven by thermal expansion. Similarly, if we add an electromagnetic force term to eqn (1.3) and replace (1.4) by the equations of the electromagnetic field, we can study magnetohydrodynamic turbulence. All such possibilities will be ignored, tempting though they are, since they can only aggravate the central problem of solving eqns (1.3) and (1.4).

It is sometimes asked whether the Navier–Stokes equations (1.3) really do represent turbulence. They are exact for a continuous fluid which shows Newtonian behaviour under all conditions, and it is perhaps more appropriate to ask whether this is an adequate approximation for real turbulent fluids. We shall see in § 2.8 that there is a fairly definite minimum to the size of eddy which is found in a turbulent field, and that, for flow at a Reynolds number of 10^6 in a pipe 10 mm in diameter (these being very extreme conditions), this minimum eddy is more than 10^{-4} mm in diameter.

The continuum approximation will be valid as long as the minimum eddy size is substantially larger than the mean free path. This condition is well satisfied in liquids, in which the mean free path is comparable with atomic dimensions (10^{-7} mm). It is also satisfied, though not by such a comfortable margin, by any gas dense enough to produce a Reynolds number of 10^6 in a passage only 10 mm in diameter.

Velocity and temperature gradients can be high in turbulent fields and one must also ask whether Newton's approximation may be inadequate, even though there is no departure from continuum behaviour. This can be investigated by using the exact statistical-mechanical equations of motion. The Navier–Stokes equations are the first approximation to these exact equations when the velocity and temperature gradients are assumed to be small and when the properties of the individual molecules are not too bizarre. This last proviso excludes aggregation and molecular chains which can tie themselves in knots. The next approximation can be evaluated roughly in some simple cases and it seems that, while not utterly negligible, it is not large enough to cause concern.

There is another line of argument which, though rather pragmatic, weighs heavily with those who are working on this problem. If the Navier–Stokes equations are in fact inadequate, the proper equations must be more complex and more difficult to solve. It seems sensible to stick to the Navier–Stokes equations until we have obtained some proper solutions, which should give real evidence as to their adequacy.

We end this section with two simple transformations which will be useful later on. Using the incompressibility condition (1.2), the Navier–Stokes equations (1.3) may be rewritten

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_m}(u_i u_m) - \nu \nabla^2 u_i = -\frac{\partial p}{\partial x_i}. \tag{1.5}$$

Taking the divergence of this equation and using (1.2) again, we have

$$\nabla^2 p = -\frac{\partial^2}{\partial x_i \partial x_m}(u_i u_m) \tag{1.6}$$

and we write the solution of this as

$$p = -\frac{1}{\nabla^2}\frac{\partial^2}{\partial x_i \partial x_m}(u_i u_m). \tag{1.7}$$

At this stage, the symbol $1/\nabla^2$ should not be thought of as implying any more than that p is the solution of equation (1.6). It will be given a more definite meaning in § 13.5.

We can now eliminate the pressure from the Navier–Stokes equations, which may be rewritten

$$\frac{\partial u_i}{\partial t} - \nu \nabla^2 u_i = -\tfrac{1}{2} P_{ijm}(\nabla)(u_j u_m), \tag{1.8}$$

where

$$P_{ijm}(\nabla) = \frac{\partial}{\partial x_m} P_{ij}(\nabla) + \frac{\partial}{\partial x_j} P_{im}(\nabla) \tag{1.9}$$

and

$$P_{ij}(\nabla) = \delta_{ij} - \frac{1}{\nabla^2}\frac{\partial^2}{\partial x_i \partial x_j}, \tag{1.10}$$

δ_{ij} being defined by eqn (1.1.).

1.4. Survey of statistical questions

Fig. 1.1 summarizes the difference between laminar conditions and turbulence. In steady laminar flow the velocity does not change with time while in turbulent flow it fluctuates in a fairly unpredictable way, although obviously one can say that it is unlikely either to jump outside certain limits or to change very quickly. These restrictions are perfectly compatible with the notion of randomness, which may be thought of as unpredictability.

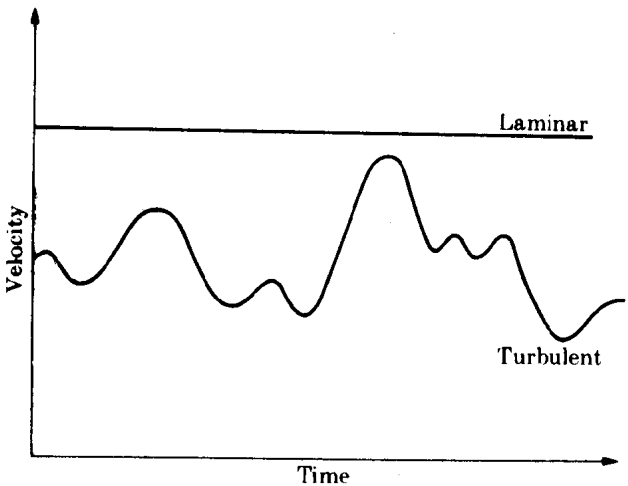

FIG. 1.1. Time dependence of the velocity in laminar and in turbulent flow.

It is often asked how a motion which is described by the Navier–Stokes equations can be random, since a given set of initial conditions determines the motion for the rest of time. This question is still being debated, consequently the answer given below may meet with some dissent. It can be demonstrated both experimentally and theoretically that the Navier–Stokes equations have enormous amplifying power under the right conditions. Even though two sets of initial conditions may be very similar, the resulting flows can diverge greatly as time goes on. This means that the initial conditions must be specified with unrealistic and indeed submolecular accuracy in order to determine the flow uniquely and, if one considers only the molecular level, the motion is random. One therefore either accepts that a proper specification of the initial conditions is unattainable and is content to regard the flow as random, or one asserts that the visible

randomness is, in the end, due to amplification of molecular randomness. The second point of view is probably the less unsatisfying but, as far as practical calculations are concerned, either will serve.

Having accepted that the flow is random, one then enquires what information one can reasonably ask for. The concept of the mean velocity is almost implicit in Fig. 1.1, and one can also see how it is to be measured. One will measure the velocity as a function of time, and will then average this measurement:

$$\bar{u} = (1/T)\int_0^T u(t)\,\mathrm{d}t. \tag{1.11}$$

This is an integral average, which is the limit of averaging measurements taken at successive moments of time. This average will itself fluctuate, depending on the starting-point and duration of the averaging process, but we can see intuitively that the process will eventually yield a definite quantity. Mathematically we define $\bar{u}$ by

$$\bar{u} = \lim_{T\to\infty}(1/T)\int_0^T u(t)\,\mathrm{d}t \tag{1.12}$$

while experimentally we continue the averaging process until the fluctuations in $\bar{u}$ are acceptably small (and we have to say what we will regard as acceptable).

This seems quite straightforward, but one encounters logical and mathematical difficulties if the $\bar{u}$ defined by eqn (1.12) depends on the starting time of the integration. For this reason statisticians prefer to *define* the averaging process in a different way. If one wants to know the velocity at a particular point in a pipe at a particular time, one *imagines* making the experiment on a great number of pipes at the same time. If the pipes are labelled $1, \ldots, n, \ldots N$ and the corresponding measurements are

$$u^{(1)}(t), \ldots, u^{(n)}(t), \ldots, u^{(N)}(t)$$

(t being the time at which all the imaginary measurements are made), the average velocity at time t is now redefined as

$$\langle u(t)\rangle = \lim_{N\to\infty}(1/N)\sum_{n=1}^{N} u^{(n)}(t). \tag{1.13}$$

This is known as 'averaging over repeated realizations' and is the fundamental definition as used by turbulence theorists. $\langle\ \rangle$ always means a realization average, while $\bar{\ }$ is an average over time; $\langle\ \rangle$ is also called an 'ensemble average'.

It may be shown that, if $\langle u(t)\rangle$ as defined by eqn (1.13) is in fact independent of t, then $\bar{u}$ as defined by eqn (1.12) is

1. independent of the time at which the integration is started and
2. equal to $\langle u\rangle$.

This is known as the *ergodic theorem*. If $\langle u(t)\rangle$ is independent of t, the velocity field is said to be 'stationary', a stationary turbulent field being the equivalent of a steady laminar field. (We shall see later that stationarity implies more than this.) This theorem assures us that we are theorizing about the same quantity that the experimenter is measuring.

Averaging as defined by eqn (1.13) is a linear operation; this has a number of useful consequences. For example

$$\langle Au(t)\rangle = A\langle u(t)\rangle,$$

where A is a constant,

$$\langle u_1(t) + u_2(t)\rangle = \langle u_1(t)\rangle + \langle u_2(t)\rangle$$

and

$$\left\langle \frac{\partial u(t)}{\partial s} \right\rangle = \frac{\partial}{\partial s}\langle u(t)\rangle, \tag{1.14}$$

where s is any variable on which u might depend (time, space, temperature and so on). However,

$$\langle u_1(t)\, u_2(t)\rangle$$

is not in general equal to

$$\langle u_1(t)\rangle \langle u_2(t)\rangle.$$

From now on, it is convenient to write

$$u(t) = U + \tilde{u}(t) \tag{1.15}$$

where the mean velocity $U = \langle u(t)\rangle$ does not fluctuate, while $\tilde{u}$ is a fluctuating quantity with zero mean. The quantity

$$\langle \tilde{u}(t)\tilde{u}(t+\tau)\rangle \tag{1.16}$$

is a typical correlation which tends to zero as the time interval τ tends to infinity. If the velocity field is stationary, the correlation is independent of t and depends only on τ. (1.16) must be positive when $\tau = 0$, since then it is just $\langle \tilde{u}^2(t)\rangle$, but it can be negative for some values of τ; this happens quite commonly in turbulence fields.

In the previous paragraph we treated u as though it were a scalar function of t only. In fact it is a vector function of $\mathbf{x}$ and t, and should be written $u_i(\mathbf{x}, t)$. Eqn (1.15) should now be rewritten

$$u_i(\mathbf{x}, t) = U_i(\mathbf{x}) + \tilde{u}_i(\mathbf{x}, t) \tag{1.17}$$

since although the mean velocity U_i is independent of t, it can, and usually does, vary with $\mathbf{x}$. A quantity of great importance in turbulence theory is the spatial correlation

$$\langle \tilde{u}_i(\mathbf{x}, t)\tilde{u}_j(\mathbf{x}', t')\rangle. \tag{1.18}$$