Mathematical Thought
from Ancient to Modern T

Mathematical Thought from Ancient to Modern Times

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Preface

> If we wish to foresee the future of mathematics our proper course is to study the history and present condition of the science.
>
> HENRI POINCARÉ

This book treats the major mathematical creations and developments from ancient times through the first few decades of the twentieth century. It aims to present the central ideas, with particular emphasis on those currents of activity that have loomed largest in the main periods of the life of mathematics and have been influential in promoting and shaping subsequent mathematical activity. The very concept of mathematics, the changes in that concept in different periods, and the mathematicians' own understanding of what they were achieving have also been vital concerns.

This work must be regarded as a survey of the history. When one considers that Euler's works fill some seventy volumes, Cauchy's twenty-six volumes, and Gauss's twelve volumes, one can readily appreciate that a one-volume work cannot present a full account. Some chapters of this work present only samples of what has been created in the areas involved, though I trust that these samples are the most representative ones. Moreover, in citing theorems or results, I have often omitted minor conditions required for strict correctness in order to keep the main ideas in focus. Restricted as this work may be, I believe that some perspective on the entire history has been presented.

The book's organization emphasizes the leading mathematical themes rather than the men. Every branch of mathematics bears the stamp of its founders, and great men have played decisive roles in determining the course of mathematics. But it is their ideas that have been featured; biography is entirely subordinate. In this respect, I have followed the advice of Pascal: "When we cite authors we cite their demonstrations, not their names."

To achieve coherence, particularly in the period after 1700, I have treated each development at that stage where it became mature, prominent, and influential in the mathematical realm. Thus non-Euclidean geometry is presented in the nineteenth century even though the history of the efforts to replace or prove the Euclidean parallel axiom date from Euclid's time onward. Of course, many topics recur at various periods.

To keep the material within bounds I have ignored several civilizations such as the Chinese,[1] Japanese, and Mayan because their work had no material impact on the main line of mathematical thought. Also some developments in mathematics, such as the theory of probability and the calculus of finite differences, which are important today, did not play major roles during the period covered and have accordingly received very little attention. The vast expansion of the last few decades has obliged me to include only those creations of the twentieth century that became significant in that period. To continue into the twentieth century the extensions of such subjects as ordinary differential equations or the calculus of variations would call for highly specialized material of interest only to research men in those fields and would have added inordinately to the size of the work. Beyond these considerations, the importance of many of the more recent developments cannot be evaluated objectively at this time. The history of mathematics teaches us that many subjects which aroused tremendous enthusiasm and engaged the attention of the best mathematicians ultimately faded into oblivion. One has but to recall Cayley's dictum that projective geometry is all geometry, and Sylvester's assertion that the theory of algebraic invariants summed up all that is valuable in mathematics. Indeed one of the interesting questions that the history answers is what survives in mathematics. History makes its own and sounder evaluations.

Readers of even a basic account of the dozens of major developments cannot be expected to know the substance of all these developments. Hence except for some very elementary areas the contents of the subjects whose history is being treated are also described, thus fusing exposition with history. These explanations of the various creations may not clarify them completely but should give some idea of their nature. Consequently this book may serve to some extent as a historical introduction to mathematics. This approach is certainly one of the best ways to acquire understanding and appreciation.

I hope that this work will be helpful to professional and prospective mathematicians. The professional man is obliged today to devote so much of his time and energy to his specialty that he has little opportunity to familiarize himself with the history of his subject. Yet this background is important. The roots of the present lie deep in the past and almost nothing in that past is irrelevant to the man who seeks to understand how the present came to be what it is. Moreover, mathematics, despite the proliferation into hundreds of branches, is a unity and has its major problems and goals. Unless the various specialties contribute to the heart of mathematics they are likely to be

1. A fine account of the history of Chinese mathematics is available in Joseph Needham's *Science and Civilization in China*, Cambridge University Press, 1959, Vol. 3, pp. 1–168.

sterile. Perhaps the surest way to combat the dangers which beset our fragmented subject is to acquire some knowledge of the past achievements, traditions, and objectives of mathematics so that one can direct his research into fruitful channels. As Hilbert put it, "Mathematics is an organism for whose vital strength the indissoluble union of the parts is a necessary condition."

For students of mathematics this work may have other values. The usual courses present segments of mathematics that seem to have little relationship to each other. The history may give perspective on the entire subject and relate the subject matter of the courses not only to each other but also to the main body of mathematical thought.

The usual courses in mathematics are also deceptive in a basic respect. They give an organized logical presentation which leaves the impression that mathematicians go from theorem to theorem almost naturally, that mathematicians can master any difficulty, and that the subjects are completely thrashed out and settled. The succession of theorems overwhelms the student, especially if he is just learning the subject.

The history, by contrast, teaches us that the development of a subject is made bit by bit with results coming from various directions. We learn, too, that often decades and even hundreds of years of effort were required before significant steps could be made. In place of the impression that the subjects are completely thrashed out one finds that what is attained is often but a start, that many gaps have to be filled, or that the really important extensions remain to be created.

The polished presentations in the courses fail to show the struggles of the creative process, the frustrations, and the long arduous road mathematicians must travel to attain a sizable structure. Once aware of this, the student will not only gain insight but derive courage to pursue tenaciously his own problems and not be dismayed by the incompleteness or deficiencies in his own work. Indeed the account of how mathematicians stumbled, groped their way through obscurities, and arrived piecemeal at their results should give heart to any tyro in research.

To cover the large area which this work comprises I have tried to select the most reliable sources. In the pre-calculus period these sources, such as T. L. Heath's *A History of Greek Mathematics*, are admittedly secondary, though I have not relied on just one such source. For the subsequent development it has usually been possible to go directly to the original papers, which fortunately can be found in the journals or in the collected works of the prominent mathematicians. I have also been aided by numerous accounts and surveys of research, some in fact to be found in the collected works. I have tried to give references for all of the major results; but to do so for all assertions would have meant a mass of references and the consumption of space that is better devoted to the account itself.

The sources have been indicated in the bibliographies of the various chapters. The interested reader can obtain much more information from these sources than I have extracted. These bibliographies also contain many references which should not and did not serve as sources. However, they have been included either because they offer additional information, because the level of presentation may be helpful to some readers, or because they may be more accessible than the original sources.

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Contents

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I

Mathematics in Mesopotamia

> Logic can be patient for it is eternal.
>
> OLIVER HEAVISIDE

1. *Where Did Mathematics Begin?*

Mathematics as an organized, independent, and reasoned discipline did not exist before the classical Greeks of the period from 600 to 300 B.C. entered upon the scene. There were, however, prior civilizations in which the beginnings or rudiments of mathematics were created. Many of these primitive civilizations did not get beyond distinguishing among one, two, and many; others possessed, and were able to operate with, large whole numbers. Still others achieved the recognition of numbers as abstract concepts, the adoption of special words for the individual numbers, the introduction of symbols for numbers, and even the use of a base such as ten, twenty, or five to denote a larger unit of quantity. One also finds the four operations of arithmetic, but confined to small numbers, and the concept of a fraction, limited, however, to 1/2, 1/3, and the like, and expressed in words. In addition, the simplest geometric notions, line, circle, and angle, were recognized. It is perhaps of interest that the concept of angle must have arisen from observation of the angle formed by man's thigh and lower leg or his forearm and upper arm because in most languages the word for the side of an angle is either the word for leg or the word for arm. In English, for example, we speak of the arms of a right triangle. The uses of mathematics in these primitive civilizations were limited to simple trading, the crude calculation of areas of fields, geometric decoration on pottery, patterns woven into cloth, and the recording of time.

Until we reach the mathematics of the Babylonians and the Egyptians of about 3000 B.C. we do not find more advanced steps in mathematics. Since primitive peoples settled down in one area, built homes, and relied upon agriculture and animal husbandry as far back as 10,000 B.C., we see how slowly the most elementary mathematics made its first steps; moreover, the existence of vast numbers of civilizations with no mathematics to speak of shows how sparsely this science was cultivated.