Topological Theory on Graphs

图的拓扑理论

刘彦佩 著

中国科学技术大学出版社

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内容提要

本书不在于图的拓扑性质本身，而是着意以图为代表的一些组合构形为出发点，揭示与拓扑学中一些典型对象，如多面形、曲面、嵌入、纽结等的联系，特别是显示了定理有效化的途径对于以拓扑学为代表的基础数学的作用。同时，也提出了一些新的曲面模型，为超大规模集成电路的布线尝试构建多方面的理论基础。

本书可作为基础数学、应用数学、系统科学、计算机科学等专业高年级本科生和研究生的补充教材，也可供相关专业的教师和科研工作者参考。

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Preface

The subject of this book reflects new developments mainly by the author himself in company with cooperators most of them his former and present graduate students on the foundation established in Liu, Y.P.[33–34]. The central idea is to extract suitable parts of a topological object such as a graph not necessary to be with symmetry, as linear spaces which are all with symmetry for exploiting global properties in construction of the object. This is a way of combinatorizations and further algebraications of an object via relationship among their subspaces.

Graphs are dealt with three vector spaces over GF(2), the finite field of order 2, generated by 0(dimensional)-cells, 1(dimensional)-cells and 2(dimensional)-cells. The first two spaces were known from, *e.g.*, Lefschetz, S.[2] by taking 0-cells and 1-cells as, respectively, vertices and edges. Of course, a graph is only a 1-complex without two cells.

Since the fifties of last century, in Wu, W.J.[1] and Tutte, W.T.[4, 16], the chain groups generated by 0-cells and 1-cells over, respectively, GF(2) and the real field were independently used for describing a graph. And they both then after ten years adopted nonadjacent pair of edges as a 2-cell for which the cohomology on a graph was allowed to be established.

Their results especially in Wu, W.J.[1–6] enabled the present author to create a number of types of planarity auxiliary graphs induced from the graph considered for the study of the efficiency of theorems in Liu, Y.P.[1,2,19,22,42] as one approach. Another approach can be seen in Liu, Y.P.[23–25,43].

More interestingly, two decades later than Liu, Y.P.[1], in Archdeacon, D. and J. Siran[1] a theta graph(network) was used for characterizing the planarity of a given graph. The theta graph can be seen to be a type of planarity auxiliary graph(network) because our planarity auxiliary graphs are subgraphs of the theta graph. However, in virtue of the order of theta network upper bounded by an exponential function of the size of given graph and that of planarity auxiliary network by a quadratic polynomial of the size of given graph, theorems deduced from a theta network are all without efficiency while those from a planarity auxiliary network are all with efficiency.

The effects of planarity auxiliary graphs are reflected in Chapters 8, 10, 11, 12 and 13 with a number of extensions.

On the other hand, in Liu, Y.P.[31] a graph was dealt with a set of polyhedra via double covering the edge set by travels under certain condition so that travels were treated as 2-cells. These enable us to discover homology and another type of cohomology for showing the sufficiency of Eulerian necessary condition in this circumstance. Further, all the results for the planarity of a graph in Whitney, H.[7] on the duality, MacLane, S.[1–2] on a circuit basis and Lefschetz, S.[1] on a circuit double covering have a universal view in this way. In fact, our polyhedra are all on such surfaces, *i.e.*, 2-dimensional compact manifolds without boundary. If a boundary is allowed on a surface, the Eulerian necessary condition is not always sufficient in general. Some person used to have missing the boundary condition in Abrams, L. and C.D. Slilaty[1].

The effects of this theoretical thinking are reflected in Chapters 4,5,7 and 14.

Because of the clarification of the joint tree model of a polyhedron in Liu, Y.P.[35–36] by the present author recently on the basis of Liu, Y.P.[8–9], we are allowed to write a chapter for brief description of the theories of surfaces and polyhedra each in Chapters 2 and 3 with related topics in Chapters 6, 9 and 15.

Although quotient embeddings(current graph and its dual, voltage graph) were quite active in constructing an embedding of a graph on a surface with its genus minimum in a period of decades, this book has no space for them. One reason is that some books have mentioned them such as in White, A.T.[1], Ringel, G.[3] and Liu, Y.P.[33–34], *etc.* Another reason is that only graphs with higher symmetry are suitable for quotient embeddings, or for employing the covering space method whence this book is for general graphs without such a limitation of symmetry.

In spite of refinements and simplifications for known results, this book still contains a number of new results such as in §5.2, the sufficiency in the proof of Theorem 5.2.1, §9.4, §11.3–4, §13.1–2, §13.4–5 *etc.*, only name a few. Researches were partially supported by NNSF in China under Grants No.60373030 and No.10571013.

Y. P. Liu
Beijing
December 2007

Contents

Chapter 1

Preliminaries

Throughout for the sake of brevity, the usual logical conventions are adopted: disjunction, conjunction, negation, implication, equivalence, universal quantification and existential quantification denoted, respectively, by the familiar symbols:$\vee$, $\wedge$, $\neg$, $\Rightarrow$, $\Leftrightarrow$, $\forall$ and $\exists$. And, $\S x.y$ is for the section y in Chapter x.

In the context, $(i.j.k)$ refers to item k of section j in chapter i.

A reference $[k]$ refers to item k of the corresponding author(s) in the bibliography where k is a positive integer to distinguish publications of the same author(s).

1.1 Sets and relations

A *set* is a collection of objects with some common property which might be numbers, points, symbols, letters or whatever even sets except itself to avoid paradoxes. The objects are said to be *elements* of the set. We always denote elements by italic lower letters and sets by capital ones. The statement " x is (is not) an element of M" is written as $x \in M (x \notin M)$. A set is often characterized by a property. For example

$$M = \{x \mid x \leq 4, \text{ positive integer }\} = \{1, 2, 3, 4\}.$$

The *cardinality* of a set M (or the number of elements of M if finite) is denoted by $\mid M \mid$.

Let A, B be two sets. If $(\forall a)$ $(a \in A \Rightarrow a \in B)$, then A is said to be a *subset* of B which is denoted by $A \subseteq B$. Further, we may define the three main operations: union, intersection and subtraction respectively as $A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$, $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$ and $A \setminus B = \{x \mid (x \in A) \wedge (x \notin B)\}$.

If $B \subseteq A$, then $A \setminus B = A - B$ is denoted by $\overline{B}(A)$ which is said to be the *complement* of B in A. If all the sets are considered as subsets of Ω, then the complement of A in Ω is simply denoted by $\overline{A}$. The *empty* denoted by $\emptyset$ is the set without element. For those operations on subsets of Ω mentioned above, we have the following laws.

Idempotent law $\forall A \subseteq \Omega$, $A \cap A = A \cup A = A$.

Commutative law $\forall A, B \subseteq \Omega$, $A \cup B = B \cup A; A \cap B = B \cap A$.

Associative law $\forall A, B, C \subseteq \Omega$, $A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$.

Absorption law $\forall A, B \subseteq \Omega$, $A \cap (A \cup B) = A \cup (A \cap B) = A$.

Distributive law $\forall A, B, C \subseteq \Omega$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Universal bound law $\forall A \subseteq \Omega$, $\emptyset \cap A = \emptyset$, $\emptyset \cup A = A; \Omega \cap A = A, \Omega \cup A = \Omega$.

Unary complement law $\forall A \subseteq \Omega$, $A \cap \overline{A} = \emptyset$; $A \cup \overline{A} = \Omega$.

The unary complement law is also called the *excluded middle law* in logic.

From the laws described above, we may obtain a large number of important results. Here, only a few is listed for the usage in the context.

Theorem 1.1.1 $\forall A \subseteq \Omega$,

$$\begin{cases} (\forall X \subseteq \Omega)\big((A \cap X = A) \vee (A \cup X = X)\big) \\ \qquad \Rightarrow A = \emptyset; \\ (\forall X \subseteq \Omega)\big((A \cap X = X) \vee (A \cup X = A)\big) \\ \qquad \Rightarrow A = \Omega. \end{cases} \tag{1.1.1}$$

Theorem 1.1.2 $\forall A, B \subseteq \Omega$,

$$A \cap B = A \Leftrightarrow A \cup B = B. \tag{1.1.2}$$

Theorem 1.1.3 $\forall A, B, C \subseteq \Omega$,

$$(A \cap B = A \cap C) \wedge (A \cup B = A \cup C) \Leftrightarrow B = C. \tag{1.1.3}$$

Theorem 1.1.4 $\forall A \subseteq \Omega$,

$$\overline{\overline{A}} = A. \tag{1.1.4.}$$

Theorem 1.1.5 $\forall A, B \subseteq \Omega$,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}; \overline{A \cap B} = \overline{A} \cup \overline{B}. \tag{1.1.5}$$

From those described above, it is seen that $\overline{\emptyset} = \Omega$ and $\overline{\Omega} = \emptyset$. Further, the symmetry (or duality) that any proposition related to $\cup, \cap, \emptyset, \Omega$ can be transformed into another by interchanging $\cup$ and $\cap, \emptyset$ and Ω.

For $A, B \subseteq \Omega$, an *injection* (or 1 - to - 1 correspondence) between A and B is a mapping $\alpha: A \rightarrow B$ such that $\forall a, b \in A,\ a \neq b \Rightarrow \alpha(a) \neq \alpha(b)$. A *surjection* between A and B is a mapping $\beta: A \rightarrow B$ such that $(\forall b \in B)(\exists a \in A)(\beta(a) = b)$. If a mapping is both an injection and a surjection, then it is called a *bijection*. Two sets are said to be *isomorphic* if there is a bijection between them. Two isomorphic sets A and B, or write $A \sim B$, are always treated as the same. Of course, for finite sets, it is trivial to justify if two sets are isomorphic by the fact: $\forall A, B \subseteq \Omega,\ A \sim B \Leftrightarrow \mid A \mid = \mid B \mid$.

For a set M, let $M \times M = \{\prec x, y \succ \mid \forall x, y \in M\}$ which is said to be the *Cartesian product* of M. Here, $\prec x, y \succ \neq \prec y, x \succ$ in general.

A *binary relation* R on M is a subset of $M \times M$. The adjective "binary" of the relation will often be omitted in the context. If the relation R holds for $x, y \in M$, then we write $\prec x, y \succ \in R$, or xRy. An *order*, denoted by $\preceq$, is a relation R which satisfies the following three laws:

Reflective law $\forall x \in M, xRx$.

Antisymmetry law $\forall x, y \in M, xRy \wedge yRx \Rightarrow x = y$.

Transitive law $\forall x, y, z \in M, xRy \wedge yRz \Rightarrow xRz$.

The set M with the order $\preceq$ is said to be a *poset* (or partial order set) denoted by $(M, \preceq)$.

Theorem 1.1.6 In a poset $(M, \preceq)$, $\forall x_1, x_2, \cdots, x_n \in M$,

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_1 \Rightarrow x_1 = x_2 = \cdots = x_n. \tag{1.1.6}$$

The theorem is sometimes called the *anti-circularity law.* If a relation only satisfies Reflective law and Transitive law but not Anti-symmetry law, then it is called the *quasi-order* which is denoted by $\bullet \prec$. A set M with $\bullet \prec$ is said to be a *quoset* denoted by $(M, \bullet \prec)$.

Theorem 1.1.7 Any subset S of a quoset $(M, \bullet \prec)$, is itself a quoset with the restriction of the quasi-order to S.

If a quasi-order R on M satisfies the symmetry law described below, then it is called an *equivalent relation*, or simply an *equivalence* denoted by $\sim$.

Symmetry law $\forall x, y \in M,\ xRy \Rightarrow yRx.$

For the equivalence $\sim$ on M, we are allowed to define the set $x(M) = \{y \mid \forall y \in M, y \sim x\}$ which is said to be the *equivalent class* for $x \in M$. The set which consists of all the equivalent classes is called the *quotient set* of $(M, \sim)$ denoted by $M/\sim$. In a quoset $(M, \bullet \prec)$, let $\sim_{\bullet\prec}$ be defined by

$$\forall x, y \in M, x \sim_{\bullet\prec} y \Leftrightarrow (x \bullet \prec y) \wedge (y \bullet \prec x). \qquad (1.1.7)$$

Then, it is easily seen that $\sim_{\bullet\prec}$ is an equivalence on M and that $(M/\sim_{\bullet\prec}, \bullet \prec)$ is also a quoset.

Theorem 1.1.8 A quoset $(M, \bullet \prec)$ is a poset if, and only if, $M/\sim_{\bullet\prec} = M$, or say, it satisfies the anti- circularity law.

In a poset $(M, \preceq)$, we define the *strict inclusion*, denoted by $\prec$, of the order by the *anti-reflective law*: $\neg x \in M, x \prec x$ and the *transitive law*: $(x \prec y) \wedge (y \prec z) \Rightarrow x \prec z$ while noticing that $x \preceq y \Leftrightarrow (x \prec y) \vee (x = y)$. If an order $\preceq$ on M satisfies the alternative law described below, then it is called a *total order*, or a *linear order*.

Alternative law $\forall x, y \in M,\ x \npreceq y \Rightarrow y \preceq x.$

A set with a total order is said to be a *chain*. The *length* of a chain with n elements is defined to be $n-1$. From Theorem 1.1.7 and the definitions, we may have

Theorem 1.1.9 Any subset of a poset is a poset and any subset of a chain is a chain.

The *converse* of a relation R on M is, by definition, the relation $R^* : \forall x, y \in M, xR^*y \Leftrightarrow yRx$. It is obvious from inspection of the three laws for order to have

Theorem 1.1.10 (Duality principle) The converse of any order is itself an order.

In a poset $(M, \preceq)$, there may have an element $a : \forall x \in M, a \preceq x$. Because of Antisymmetry law, such an element, if it exists, is a unique one which is called the *least element* denoted by O. In dual case, the *greatest element*, if it exists, denoted by I. The elements O and I, when they exist, are called *universal bound* of the poset.

Theorem 1.1.11 A chain has the universal bounds if it is finite.

In a poset $(M, \preceq)$, an element $a \in M : \forall x \in M, x \preceq a \Rightarrow x = a$ is called a *minimal element*. Dually, a *maximal element* is defined as $a \in M : \forall x \in M, a \preceq x \Rightarrow a = x$.

Theorem 1.1.12 Any finite nonempty poset $(M, \preceq)$ has minimal and maximal elements.

A mapping $\tau : M \to N$ from a poset $(M, \preceq)$ to a poset $(N, \preceq)$ is called *order- preserving*, or *isotone* if it satisfies

$$\forall x, y \in M, \ x \preceq y \Leftrightarrow \tau(x) \preceq \tau(y). \tag{1.1.8}$$

Further, if an isotone $\tau : M \to N$ satisfies

$$\forall x, y \in M, \tau(x) \preceq \tau(y) \Rightarrow x \preceq y, \tag{1.1.9}$$

then it is called an *isomorphism*. Two posets $(M, \preceq)$ and $(N, \preceq)$ are said to be *isomorphic*, that is $(M, \preceq) \cong (N, \preceq)$, if there is an isomorphism between them. All isomorphic posets are treated as the same. However, it is not trivial as for sets to justify if two posets are isomorphic in general.

An *upper bound* of a subset X of a poset $(M, \preceq)$ is an element $a : \forall x \in X, x \preceq a$. The *least upper bound* (or l.u.b.) is an upper bound b : $a \preceq b \Rightarrow a = b$, where a is another upper bound of X. Dually, a *lower bound* and the *greatest lower bound*(g.l.b.). The *length* of a poset is the l.u.b. of the lengths of chains in the poset. A *lattice* is a poset any two x and y of whose elements has a g.l.b. or *meet* denoted by $x \wedge y$ and an l.u.b. or *join* denoted by $x \vee y$. A lattice $L = (M, \preceq; \vee, \wedge)$ is *complete* if each of its subset X has an l.u.b. and a g.l.b.. Moreover, we have known that all finite length lattices are complete.

Let 2^Ω be the set which consists of all subsets of Ω. From §1.1, we may see that $(2^\Omega, \subseteq; \cup, \cap)$ is a lattice. In fact, we have

Theorem 1.1.13 A poset is a lattice if, and only if, it satisfies the idempotent, commutative, associative and absorption laws.

Two lattices $(M, \preceq; \vee, \wedge)$ and $(N, \preceq; \vee, \wedge)$ are *isomorphic* if there is an isomorphism τ between $(M, \preceq)$ and $(M, \preceq)$ such that, $\forall x, y \in M$,

$$(\tau(x \vee y) = \tau(x) \vee \tau(y)) \wedge (\tau(x \wedge y) = \tau(x) \wedge \tau(y)). \qquad (1.1.10)$$

Of course, it is nontrivial as well to justify if two lattices are isomorphic in general.

1.2 Partitions and permutations

A *partition* of a set X is such a set of subsets of X that any two subsets are without common element and the union of all the subsets is X.

Theorem 1.2.1 A partition $P(X)$ of a set X determines an equivalence on X such that the subsets in $P(X)$ are the equivalent classes. □

Let $P(X) = \{p_1, p_2, \cdots, p_{k_1}\}$ and $Q(X) = \{q_1, q_2, \cdots, q_{k_2}\}$ be two partitions of X. If for any q_j, $1 \leq j \leq k_1$, there exists a p_i, $1 \leq i \leq k_2$ such that $q_j \subset p_i$, then $Q(X)$ is called a *refinement* of $P(X)$ and $P(X)$, an *enlargement* of $Q(X)$ except only for $P(X) = Q(X)$. The partition of X with each subset of a single element, or only one subset which is

X in its own right is, respectively, called the 0-*partition*, or 1-*partition* and denoted by $0(X)$, or $1(X)$.

Theorem 1.2.2 For a set X and its partition $P(X)$, the 0-partition $1(X)$ (or 1-partition $1(X)$) can be obtained by refinements (or enlargements) for at most $O(\log|X|)$ times in the worst case.

Proof In the worst case, it suffices to consider $P(X) = 1(X)$(or $0(X)$) and only one more subset produced in a refinement. Because of

$$1 + 2 + 2^2 + \cdots + 2^{\log|X|} = \frac{2^{1+\log|X|} - 1}{2 - 1} = O(|X|), \qquad (1.2.1)$$

the times of refinements(or enlargements) needed for getting $0(X)$ (or $1(X)$) is $O(\log|X|)$. The theorem is obtained. □

For two partitions $P = \{p_1, p_2, \cdots, p_s\}$ and $Q = \{q_1, q_2, \cdots, q_t\}$ of a set X, the *family intersection* of P and Q is defined to be

$$P \cap Q = \bigcup_{i=1}^{s} \{p_i \cap q_1, p_i \cap q_2, \cdots, p_i \cap q_t\}. \qquad (1.2.2)$$

Actually, $\{p_i \cap q_1, p_i \cap q_2, \cdots, p_i \cap q_t\}$ for $i = 1, 2, \cdots, l$ are partitions of p_i.

Theorem 1.2.3 The family intersection satisfies the commutative and associate laws. And further, $P \cap Q$ is a refinement of both P and Q. □

A *permutation* of a set X is a bijection of X to itself. Because elements in a set are no distinction, they are allowed to be distinguished by natural numbers as $X = \{x_1, x_2, \cdots\}$, or simply $X = \{1, 2, \cdots\}$. So, a permutation of set $L = \{1, 2, \cdots, l\}$ can be expressed as

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & l \\ i_1 & i_2 & i_3 & \cdots & i_l \end{pmatrix}. \qquad (1.2.3)$$

If $i_j = j$ for all $1 \leq j \leq l$, the the permutation is call the *identity*. From Theorem 1.1.4, the identity is unique.

Theorem 1.2.4 Let π be a permutation of set $L = \{1, 2, \cdots, l\}$, then for any $i \in L$ there is an integer $n \geq 0$ such that $p^n i = i$.