

高等学校信息领域全英文课程

“十三五”系列规划教材

Random Signal
Processing

随机信号分析

英文版

杨 洁 刘聪锋 编著



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北 京

内 容 简 介

本书讨论随机信号的基本概念和分析方法。其特色可归纳为：注重随机信号处理领域整体知识体系的构建、强化重点知识的分析推导、理论分析与实际应用相结合。全书共分为六章，主要内容包括概率论知识回顾、随机过程及其特征描述、随机过程的线性变换、平稳窄带随机过程、平稳随机过程的非线性变换、非平稳随机过程的特征描述等。

本书可作为高等院校工科电子信息类专业的专业基础课教材，也可作为从事信号处理相关专业的科技人员的参考书。

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总 序

自我国实施“一带一路”倡议以来，教育资源之间的国际交流更加频繁。世界各国纷纷看好了我国教育市场这个平台，很多外国年轻人选择中国作为攻读本科、硕士或者博士学位的第一选择。据教育部相关统计显示，目前我国已经成为世界第三大留学生输入国。

来华留学教育是我国高等教育的重要组成部分，留学生教育是经济全球化时代高等教育产业调整的新亮点。为贯彻落实《国家中长期教育改革和发展规划纲要（2010年—2020年）》，实施《留学中国计划》，打造我国留学生教育的国际品牌，促进来华留学教育质量的进一步提升，教育部办公厅启动了来华留学英语授课品牌课程评选工作，信息领域有多门课程入选。经认真研究，科学出版社依托西安电子科技大学等信息领域的优势高校，整理和总结各校在电子信息工程、通信工程、计算机等专业全英文授课课程的教学成果，在西安电子科技大学教育部来华留学示范基地的配套支持下，合作出版本套高质量的“高等学校信息领域全英文课程‘十三五’系列规划教材”。这套英文教材不仅可以作为来华留学生教育的教学用书，也可作为我国高校双语课程的教材或者参考书。

本套教材主要针对电子信息、通信工程、计算机等信息领域专业的学生，其组织情况和编写特色如下。

（1）强化项目顶层设计。本套教材项目得到教育部国际合作与交流司、中国高等教育学会外国留学生教育管理分会的大力支持，并入选科学出版社2016年度社级重大项目。与此同时，西安电子科技大学教务处和国际教育学院协助项目的前期调研及编委会的组建工作，同时推荐了部分优秀主编。

（2）精选优秀编写团队。本套教材的作者为具有丰富教学经验的高校教师，其中部分作者为来华留学英语授课品牌课程、国家级双语教学示范课程、国家级精品资源共享课或省级精品资源共享课的负责人。


（3）打造精品英文教材。本套教材为全英文编写，以通俗易懂的语言介绍信息领域各课程的专业知识，摒弃以往英文教材晦涩难懂的缺点，帮助读者（特别是来华留学生和国内双语课程的学生）提高学习效果。

为了更好地推广本套教材的先进教学理念和教学成果，科学出版社积极推进相关版权输出工作。目前本项目教材除了在国内出版发行外，大部分教材也由德国德古意特出版社同步出版发行，进一步提高了我国教材在国际上的影响力。国

II — 总序

内版教材的图文编排参考了国际版的版式体例。

本套教材的出版得到西安电子科技大学教育部来华留学示范基地的支持。感谢西安电子科技大学及所有参与本套教材组织与编写工作的其他各高校领导、老师的大力支持和辛勤付出，感谢科学出版社和德古意特出版社提供的优秀出版平台和优质出版服务。希望本套教材的出版能够对我国高校来华留学生教育有积极的推动作用。

A handwritten signature in black ink, reading '高新波' (Gao Xinbo) in a cursive style.

2018年11月

Preface

Random signals (stochastic signals) are also known as random processes (stochastic processes). It is a quantitative description of the dynamic relationship of a series of random events. Random research and other branches of mathematics such as potential theory, differential equations, the mechanics and theory of complex functions, and so on, are closely linked in natural science, engineering science and social science research in every field of random phenomena is an important tool. Random signal research has been widely used in areas such as weather forecasting, statistical physics, astrophysics, management decision-making, economic mathematics, safety science, population theory, reliability and many fields such as computer science often use random process theory to establish mathematical models.

In the study of random processes, people accidentally came to describe the inherent law of necessity and to describe these laws in probability form, realizing that the inevitability is the charm of this discipline.

The theoretical basis of the whole discipline of stochastic processes was laid by Kolmogorov and Dub. This discipline first originated from the study of physics, such as by Gibbs, Boltzmann, Poincare and others studying statistical mechanics, and later Einstein, Wiener, Levy and others with the pioneering work of the Brownian movement. Around 1907, Markov studied a series of random variables with specific dependencies, which were later called Markov chains. In 1923, Wiener gave the mathematical definition of Brown's movement, and this process is still an important research topic today.

The general theory of stochastic processes is generally considered to have begun in the 1930s. In 1931, Kolmogorov published the *Analytical Methods in Probability Theory*. In 1934, Khintchine published *The theory of smooth process*, which laid the theoretical basis of the Markov process and the stationary process. In 1953, Dub published the famous "random process theory," systematically and strictly describing the basic theory of random processes. At this point, the stochastic process developed into a systematic scientific theory.

In our daily lives, because of the presence of noise and interference, the signal we receive is no longer a clear signal, but a random process; usually we call this a random signal. A random signal is a kind of signal that is prevalent in the objective world. It is very important for college students in the information technology field to have a deep understanding of the statistical characteristics and to master the corresponding processing and analysis methods. Therefore, random signal analysis is an important basic course in the field of electronic information. Through the study of the course, students are taught to understand the basic concepts of random signals, to master the basic theory of random signals, statistical characteristics and analytical methods, to learn "statistical signal processing" or "signal detection and valuation," with other follow-up courses which lay a solid foundation for future developments.

The book was written on the basis of the textbook *Random Signal Analysis* compiled by Professor Zhang Qianwu from Xidian University, which absorbed the experience of similar teaching materials in brother colleges and universities, and which was finalized after a number of discussions in the project group. The textbook characteristics can be summarized as follows.

(1) Focus on the construction of the whole knowledge system in the field of signal processing.

From the point of view of the knowledge system, the mathematical basis of random signal analysis is “higher mathematics,” “probability theory,” and “linear algebra,” and a professional background from “signals and systems,” and the following courses are “statistical signal processing,” or “signal detection and estimation.” Therefore, it continues to strengthen students’ foundation of mathematical analysis and the known basic concept of signal analysis, basic principles and basic analysis and processing methods, and at the same time helps students to understand the application of random signal analysis methods to signal detection and parameter estimation with noise in the background. The textbook emphasizes the knowledge system and the structure of signal processing in its entirety, so as to avoid students learning and understanding random signal processing in isolation.

(2) Continuous random signal and discrete random sequence analysis.

Traditional random signal analysis materials mostly focus on the description, characterization and analysis of continuous stochastic process, often ignoring the introduction of discrete random sequences, so that the students taking this course can only carry out theoretical analysis and derivation and cannot use computers for simulation and emulation. However, making full use of computers to process and analyze random signals, on the one hand, is beneficial for students to obtain an intuitive understanding, and, on the other hand, is helpful for students to apply their knowledge, truly combining theoretical research and practical applications. Therefore, in the course of compiling the textbook, the analysis of the discrete random sequence was also taken into consideration in detail while the continuous random process is analyzed.

(3) The combination of theoretical analysis and experimental practice.

Random signal analysis is a practical course, and most current textbooks only focus on theoretical teaching instead of experimental practice. This textbook will design the corresponding experimental content for each chapter, so that students can understand and grasp basic concepts, basic principles and basic methods through computer simulation experiments.

(4) Introduction of the latest research results.

Random signal analysis of existing teaching material is mainly limited to the characterization and analysis of stationary random processes lacks a description of nonsta-

tionary random processes and related analysis of random processes after passing nonlinear systems. With the advancement of modern signal processing, nonstationary, aperiodic, non-Gaussian and nonlinear stochastic signal processing problems have led to a lot of research results; these results should be the basis of a preliminary understanding of today's undergraduates. Therefore, this textbook will devote a chapter to the introduction of time-frequency analysis and basic knowledge of wavelet analysis.

The book is divided into six chapters: Chapter 0 is an introduction, which reviews and summarizes the basic knowledge of probability theory and introduces random variables and their related digital features and characteristic functions, as well as other knowledge points. Chapter 1 introduces the basic concept of random signals. It discusses their basic characteristics and methods to describe them, complex stochastic process. The discrete-time stochastic process is also detailed, and the normal stochastic process and its spectral analysis and white noise process are introduced as well. Chapter 2 introduces the linear transformation of the stationary stochastic process, and reviews the linear transformation and linear system. Moreover, the process of differential and integral of random process is also introduced therein. The stochastic process is analyzed by continuous and discrete-time systems. White noise is analyzed by a linear system and the method of solving the probability density after the linear transformation of the random process. In Chapter 3, we discuss the stationary narrowband stochastic process and first introduce the analytical process and Hilbert transform, narrowband stochastic representation, and the analytic complex stochastic process. We then discuss the probability density of the envelope and phase of the narrowband normal process and the probability density of the square of its envelope. Chapter 4 discusses the nonlinear transformation method of stationary random process, including the direct method, transformation and the analysis of the stochastic process through limiters and the method of calculating the signal-to-noise ratio at the output of the nonlinear system are also detailed. The characteristic description and analysis method of the nonstationary stochastic process is given in Chapter 5. First, the definition and description of the nonstationary stochastic process are introduced, and then the correlation function and power spectral density are discussed. Finally, the analysis method of the nonstationary stochastic process in modern signal processing, such as Wigner–Ville distribution and wavelet analysis are introduced. The book incorporates a large number of examples and illustrations, and at the end of each chapter there are enough exercises for practice.

The book was completed by associate Professor Yang Jie and Liu Congfeng. Ongbwa Ollomo Armelonas, an international student of Xidian University, has made great efforts in the translation process of this book. The authors express their appreciation to Fu Panlong, Yin Chenyang, Yun Jinwei, Liu Chenchong, Sha Zhaoqun, Su Juan, and Hou Junrong for translating and correcting Chapters 0, 1, 2, 3, 4, 5 and the Appendix, respectively. The preparation process of this book was encouraged, helped and supported by the Xi'an University of Posts & Telecommunications and

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Because of the limited knowledge of editors, the book's errors and omissions are inevitable. Readers are encouraged to offer criticism and suggest corrections.

The Editors
2017.08

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0 Introduction

0.1 Probability space

In the probabilistic part of engineering mathematics, probabilities have been defined for both classical and geometric profiles. In the classical model, the possible results of the test are limited and have the same probability; for the geometric generalization, although the possible outcome of the experiment is infinite, it is still required to have some of the same probability. However, a large number of random test results in practical problems do not belong to these two types, so it is necessary to give a definite probability definition to the general stochastic phenomenon. In 1933, the Russian mathematician Kolmogorov combined his predecessors' research results, gave the axiomatic system of probability theory, and defined the basic concepts of events and probabilities, and probability theory became a rigorous branch of mathematics.

0.1.1 Randomized trials

In the probability theory, random test is a test with randomness under given conditions; E is generally used to represent randomized trials. Several random trials are presented below.

E_1 : toss a coin, observe the positive H or the negative T that appears.

E_2 : throw a die, observe the points that appear.

E_3 : a point is arbitrarily thrown on the $(0, 1)$ interval of the real axis.

E_4 : pick one out of a batch of bulbs and test its lifespan.

The above examples of several experiments have common characteristics. For example, the test E_1 has two possible results, H or T , but we do not know whether it is H or T before throwing. This test can be repeated under the same conditions. There are six possible outcomes for the test E_2 , but it is not possible to determine which outcome will occur before throwing the die, and this test can be repeated under the same conditions. Another example is the test E_4 ; we know the lamp life (in hours) $t \geq 0$ but cannot determine how long its life is before the test. This test can also be repeated under the same conditions. To sum up, these tests have the following characteristics:

- (1) They can be repeated under the same conditions.
- (2) There is more than one possible outcome of each trial, and all possible results of the test can be identified in advance.
- (3) No outcome can be determined before each trial.

In the probability theory, the experiment with these three characteristics is called a random experiment.

0.1.2 Sample space

For randomized trials, although the results of the tests cannot be predicted before each test, the set of all possible outcomes of the test is known. We refer to the set of all possible outcomes of stochastic test E as the sample space of random test E , and each possible test result is called a basic event, showing that the sample space consists of all the basic events of stochastic test E .

For example, in the random test E_1 , “positive H ” and “negative T ” are the basic events. These two basic events constitute a sample space.

In the random experiment E_2 , the respective points “1”, “2”, “3”, “4”, “5” and “6” are the basic events. These six basic events form a sample space.

In the random experiment E_3 , each point in the $(0, 1)$ interval is a basic event, and the set of all points (i.e., the $(0, 1)$ interval) constitutes a sample space.

Abstractly, the sample space is a collection of points, each of which is called a sample point. The sample space is denoted by $\Omega = \{\omega\}$, where ω represents the sample point, as defined below.

Definition 0.1.1. Set the sample space $\Omega = \{\omega\}$, a set of some subsets specified \mathcal{F} , if \mathcal{F} satisfies the following properties:

- (1) $\Omega \in \mathcal{F}$.
- (2) if $A \in \mathcal{F}$, then $\bar{A} = \Omega - A \in \mathcal{F}$.
- (3) if $A_k \in \mathcal{F}$, $k = 1, 2, \dots$, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$.

That said, \mathcal{F} is a Boral event domain or a σ event domain. A subset of each sample space Ω in the Boral event domain is called an event.

In particular, the sample Ω is called a certain event, and empty \emptyset is called an impossible event.

In the example of the three sample spaces above, each sample point is a basic event. However, generally it is not required that sample points be basic events.

0.1.3 Probability space

The statistical definition of probability and the classical probability definition have been mentioned in probability theory. The following describes the axiomatic definition of probability. This definition is abstracted from the specific probability definition described above, while retaining some of the characteristics of the specific probability definition. The probability of an event is a number corresponding to a subset of Ω in Borel field \mathcal{F} , which can be considered as the aggregation function.

Definition 0.1.2 (the axiomatic definition of probability). Set a set function in the Borel field \mathcal{F} of the sample space Ω . If $P(A)$ satisfies the following conditions:

- (1) non negativity: $\forall A \in \mathcal{F}$, we have $P(A) \geq 0$;
- (2) polarity: $P(\Omega) = 1$;