Eigenvalue Problems on Manifolds

（流形上的特征值问题）

Jing Mao Feng Du Chuan-Xi Wu

（毛 井 杜 锋 吴传喜）

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In memory of Mr. Xu-Gui Mao, Prof.Jing Mao's father, who left us in October 2015 physically but would stay with us mentally forever.

Preface

The main content of this book is based on the Ph.D. thesis of Prof. Jing Mao (the first author of this book) submitted to University of Lisbon, Portugal and some of his published joint-works (on Spectral Geometry) with collaborators (for instance, the second and the third authors of this book, Prof. Isabel Salavssa, Prof. Pedro Freitas, Prof. Q.-L. Wang, and so on).

In the first four chapters, we mainly consider eigenvalue problems of some self-adjoint elliptic operators (e.g., the Laplace operator, the drifting Laplacian, the p-Laplacian, the bi-harmonic operator, the bi-drifting Laplacian, the Paneitz operator, and so on), which appeared in the study of Physics, on bounded connected domains (with or without suitable boundary constraints). As we know, in the boundedness setting, these operators have the discrete spectrum, and then one can try to estimate elements (i.e., eigenvalues) in the discrete spectrum. Based on this fact, some *new* physical isoperimetric inequalities, estimations for lower eigenvalues, and universal inequalities have been shown to readers. Meanwhile, some latest progresses and possible questions desired to study have also been mentioned. Chapters 3 and 4 were written by Prof. Feng Du. In December 2016, Prof. C.-X. Wu and Prof. Jing Mao have made careful check and necessary revisions to the first manuscript of this book.

In the last chapter, based on the first author's research experience, we issue several open problems which might be interesting to readers from three aspects (i.e., spectral problems concerning quantum strips and quantum layers, connection between curvature flows and eigenvalue problem, and eigenvalue problem in Finsler geometry). Besides, some basic knowledge about warped products, which have been used many times in this book, have been introduced in Appendix A.

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Jing Mao
Faculty of Mathematics and Statistics of Hubei University
December, 2016

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Chapter 1

Introduction

Let Ω be a bounded connected domain on an n-dimensional complete Riemannian manifold (M, g). The so-called *Dirichlet eigenvalue problem* is to find all possible real numbers λ such that there exists a nontrivial solution u to the boundary value problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta u = \operatorname{div}(\nabla u)$ is the Laplacian of u given by

$$\Delta u = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det(g_{ij})} g^{ij} \frac{\partial u}{\partial x_j} \right)$$

in a system of local coordinates $\{x_1, \ldots, x_n\}$ on M, with $(g^{ij}) = (g_{ij})^{-1}$ the inverse of the metric matrix. The desired real numbers λ are called the eigenvalues of Δ. For a given λ, the space of solutions of (1.1) is a vector space, since the first equation of (1.1) is linear in u. This vector space is called the eigenspace of λ. The non-zero elements of each eigenspace are called eigenfunctions. Denote by $L^2(\Omega)$ the space of all measurable functions f on Ω satisfying

$$\int_\Omega |f|^2 d\Omega < \infty,$$

with $d\Omega$ the volume element of Ω. We can define the usual inner product and induced norm on $L^2(\Omega)$ by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f g d\Omega, \qquad \|f\|^2 = (f, f)_{L^2(\Omega)}$$

for $f, g \in L^2(\Omega)$. Under this inner product, $L^2(\Omega)$ is a Hilbert space. By the boundary condition in (1.1) and the Green's formula, it follows that Δ is a self-adjoint operator on the Hilbert space $L^2(\Omega)$. Furthermore, by the spectral theory of a self-adjoint compact operator, we know that the Laplacian Δ in (1.1) has eigenvalues listed by

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \uparrow \infty,$$

and each associated eigenspace has finite dimension (see, e.g., [30], p.169 or equivalently, Theorem 1.6 below). λ_i ($i \geqslant 1$) is called the ith Dirichlet eigenvalue of Δ. If $\lambda = 0$ in (1.1), then together with the boundary condition $u = 0$ we know that u vanishes identically. This contradicts the fact that u is nontrivial on Ω. So, the lowest Dirichlet eigenvalue λ_1 is strictly positive. By Rayleigh's theorem and Max-min principle (see, e.g., [30], p.16-17), we know that (1.1) has positive weak solutions u in the space $W_0^{1,2}(\Omega)$, the completion of the set $C_0^\infty(\Omega)$ of smooth functions compactly supported on Ω under the Sobolev norm $\|u\|_{1,2} = \left\{\int_\Omega (|u|^2 + |\nabla u|^2) d\Omega\right\}^{\frac{1}{2}}$, and the first Dirichlet eigenvalue $\lambda_1(\Omega)$ can be characterized by

$$\lambda_1(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla u|^2 d\Omega}{\int_\Omega |u|^2 d\Omega} \middle| u \neq 0, u \in W_0^{1,2}(\Omega) \right\}. \tag{1.2}$$

Remark 1.1 (1) In fact, according to the different situations of boundary $\partial\Omega$, one can consider different eigenvalue problems of the Laplacian. If $\partial\Omega = \varnothing$ (i.e., Ω is compact without boundary), then one can consider the following *closed eigenvalue problem*

$$\Delta u + \lambda u = 0 \qquad \text{in } \Omega.$$

As in the case of the Dirichlet eigenvalue problem, it is not difficult to get that $-\Delta$ only has the discrete spectrum and all the eigenvalues can be listed increasingly also. However, in this case, the first closed eigenvalue $\lambda_1^c(\Omega)$ satisfies $\lambda_1^c(\Omega) = 0$ and the corresponding eigenfunction should be nonzero constant function. Moreover, by Rayleigh's theorem and Max-min principle, the first *nonzero* closed eigenvalue $\lambda_1^c(\Omega)$ can be characterized as follows

$$\lambda_1^c(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla u|^2 d\Omega}{\int_\Omega |u|^2 d\Omega} \middle| u \neq 0, u \in W^{1,2}(\Omega), \int_\Omega u d\Omega = 0 \right\},$$

where $W^{1,2}(\Omega)$ is the completion of the set $C_0^\infty(\Omega)$ of smooth functions under the Sobolev norm $\|u\|_{1,2}$, and the constraint $\int_\Omega u d\Omega = 0$ should be assured because of Courant's minimum principle. If $\partial\Omega \neq \varnothing$, except the Dirichlet eigenvalue problem, one can consider the *Neumann eigenvalue problem*

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial v} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\boldsymbol{v}$ is the outward unit normal vector field on $\partial\Omega$, and the *mixed eigenvalue problem*

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega - N, & \dfrac{\partial u}{\partial \boldsymbol{v}} = 0 \quad \text{on } N, \end{cases} \tag{1.4}$$

where N is an open submanifold of $\partial\Omega$. Clearly, in (1.3) and (1.4), $-\Delta$ only has the discrete spectrum and all the eigenvalues can be listed increasingly. But by the boundary conditions and the maximum principle of second-order partial differential equations (PDEs for short), we know that the first Neumann eigenvalue $\lambda_1^N(\Omega)$ satisfies $\lambda_1^N(\Omega) = 0$ with nonzero constant function as its eigenfunction, and the first mixed eigenvalue $\lambda_1^M(\Omega)$ must be positive, i.e., $\lambda_1^M(\Omega) > 0$. Moreover, the first nonzero Neumann and mixed eigenvalues are the infimums of the Rayleigh's quotient $\dfrac{\int_\Omega |\nabla u|^2 d\Omega}{\int_\Omega |u|^2 d\Omega}$ in different functional spaces.

(2) One can consider the following nonlinear Dirichlet eigenvalue problem

$$\begin{cases} \Delta_p u + \lambda |u|^{p-2} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|_g^{p-2} \nabla u)$ is the p-Laplacian with $1 < p < \infty$, and Ω is a bounded domain on an n-dimensional Riemannian manifold (M, g). In local coordinates $\{x_1, \ldots, x_n\}$ on M, we have

$$\Delta_p u = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det(g_{ij})} g^{ij} |\nabla u|^{p-2} \frac{\partial u}{\partial x_j} \right), \tag{1.6}$$

where $|\nabla u|^2 = |\nabla u|_g^2 = \sum\limits_{i,j=1}^{n} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$, and $(g^{ij}) = (g_{ij})^{-1}$ is the inverse of the metric matrix. Clearly, the p-Laplacian is a generalization of the linear Laplacian. Although many results about the linear Laplacian ($p = 2$) have been obtained, many rather basic questions about the spectrum of the nonlinear p-Laplacian remain to be solved. A well-known result about the above nonlinear eigenvalue problem states that it has a positive weak solution, which is unique modulo the scaling, in the space $W_0^{1,p}(\Omega)$, the completion of the set $C_0^\infty(\Omega)$ under the Sobolev norm $\|u\|_{1,p} = \left\{ \int_\Omega (|u|^p + |\nabla u|^p) d\Omega \right\}^{\frac{1}{p}}$. For a bounded simply connected domain with sufficiently smooth boundary in Euclidean space, one can get a simple proof of this fact in [15]. Moreover, the first Dirichlet eigenvalue $\lambda_{1,p}(\Omega)$ of the p-Laplacian can be characterized by

$$\lambda_{1,p}(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla u|^p d\Omega}{\int_\Omega |u|^p d\Omega} \middle| u \neq 0, u \in W_0^{1,p}(\Omega) \right\}. \tag{1.7}$$

(3) In fact, except the Laplace and the p-Laplace operators, one can also consider eigenvalue problems for other elliptic operators (such as, the drifting Laplacian, the bi-harmonic operator, the bi-drifting Laplacian, the Paneitz operator, the poly-harmonic operator, and so on) on bounded connected domains, which will be investigated in the following chapters.

Optimal domains in isoperimetric inequalities relating eigenvalues to geometrical quantities such as volume and surface area quite often display some degree of symmetry. In many instances, this symmetry is actually the maximal possible, such as in the Rayleigh-Faber-Krahn and the Szegö-Weinberger inequalities, corresponding to Dirichlet and Neumann boundary conditions for Euclidean domains, respectively. It is thus quite natural that symmetrization plays a fundamental role in this aspect of spectral theory and is at the heart of many isoperimetric inequalities of this type. The Rayleigh-Faber-Krahn inequality, for instance, is a consequence of the fact that Schwarz symmetrization does not increase the Dirichlet integral while leaving the L^2 norm unchanged. Even in some cases where the minimiser is not one but two balls, this symmetrization plays a role, as happens not only in the case of the second Dirichlet eigenvalue, but also when other restrictions are enforced—see, for instance, [21] and [71].

However, Schwarz and other similar symmetrization procedures are mostly Euclidean techniques, and do not extend to manifolds in general. But, as stated above, this is not to say that symmetry does not play a similar fundamental role in isoperimetrical eigenvalue inequalities on manifolds. This can be seen, for instance, from Hersch's result for two-dimensional spheres [86], which states that among all surfaces with the same area which are homeomorphic to $\mathbb{S}^2$, the round sphere (canonical metric) maximises the first nontrivial eigenvalue.

The purpose of Chapter 2 is to develop the usage of symmetrization techniques in the case of manifolds, allowing us to derive comparison isoperimetric inequalities there. To this end, we shall consider a symmerization procedure based on curvature. More precisely, given a complete n-manifold M and a point p in M such that we have lower and upper bounds for the radial Ricci and sectional curvatures within a geodesic disk of radius r_0, which depend only on the distance t to the point p, we build two spherically symmetric manifolds centered at a point p^* and whose curvatures are determined by the respective bounds. In this way, we are then able to obtain that the first eigenvalue with Dirichlet boundary conditions is bounded from above and below by the first Dirichlet eigenvalue on geodesic disks centered at p^* on these two manifolds—see Theorems 2.8 and 2.14 for the precise statements of these results.

Now, we would like to recall the history of radial curvature briefly and also mention some comparison theorems for radial curvature partially. It was the first time that Klingenberg introduced the notion of radial curvature in [101] to study compact Riemannian manifolds with radial curvatures pinched between 1/4 and 1. After that, mathematicians have been paying attention to the radial curvatures. In general, the reference manifolds for comparison theorems are space forms. However, Elerath [69] employed a Von Mangoldt surface of revolution (i.e., a complete surface of revolution homeomorphic to Euclidean plane whose Gaussian curvature is non-increasing along each meridian) $\widetilde{Z} \subset \mathbb{R}^3$ with nonnegative Gaussian curvature as the reference surface to prove the generalized Toponogov comparison theorem (we write GTCT for short) successfully for complete open Riemannian manifolds with radial curvatures bounded from below by that of $\widetilde{Z}$. For complete open Riemannian manifolds whose radial Ricci curvatures are bounded from below by a nonnegative smooth function $\zeta(t)$ of the distance parameter w.r.t. some point (as described in Definition 2.2), together with other constraints for $\zeta(t)$, Abresch proved the GTCT in [1] (these special manifolds were called "*asymptotically non-negatively curved*" manifolds therein). Of course, there are other types of GTCT, which we do not need to also mention here. From these facts, we know that mathematicians have investigated manifolds with radial curvatures bounded by some continuous function of the distance parameter (of the original manifolds), and generalized some classical comparison theorems.

Theorems 2.8 and 2.14 may be seen as extensions of Cheng's bounds for the first eigenvalue, where the comparison is made between a geodesic disk on M and those on spaces of constant curvature which are obtained by taking lower and upper bounds of the curvature [50, 51]. More precisely, Cheng has proved the following conclusions.

Theorem 1.2 ([50]) *Suppose M is a complete Riemannian manifold and Ricci curvature of $M \geqslant (n-1)k$, with* $\dim M = n$. *Then, for $x_0 \in M$ we have*

$$\lambda_1\left(B(x_0, r_0)\right) \leqslant \lambda_1\left(V_n(k, r_0)\right),$$

and equality holds if and only if $B(x_0, r_0)$ is isometric to $V_n(k, r_0)$, where $B(x_0, r_0)$ denotes the open geodesic ball with center x_0 and radius r_0 on M, and $V_n(k, r_0)$ is a geodesic ball with radius r_0 in the n-dimensional simply connected space form with constant curvature k; moreover, $\lambda_1(\cdot)$ denotes the first Dirichlet eigenvalue of the Laplacian on the corresponding geodesic ball.

Theorem 1.3 ([51]) *Let M be a complete Riemannian manifold all of whose sectional curvatures are less than or equal to a given constant k, and* $\dim M = n$. *Then, for $p \in M$ and $\delta > 0$ for which $B(p, \delta)$ is within the cut-locus of p, we have*

$$\lambda_1 (B(p, \delta)) \geqslant \lambda_1 (V_n(k, \delta)),$$

where symbols $B(\cdot, \cdot)$, $V_n(\cdot, \cdot)$, and $\lambda_1(\cdot)$ have the same meanings as those in Theorem 1.2.

The starting point behind Theorems 2.8 and 2.14 is twofold. On the one hand, it should be possible to replace the constant curvature spaces in Cheng's results by spherically symmetric spaces, in such a way that these still yield curvature bounds which imply the desired eigenvalue bounds. On the other hand, spherically symmetric manifolds posses a relatively simple characterization and the first Dirichlet eigenvalue on a geodesic disk is given by the zero of a solution to a second order ordinary differential equation (see (2.16)). Thus, there are many bounds for these eigenvalues, some of which providing quite accurate bounds—see [11], [12], [20], [77], for instance.

The heat equation, which can be used to describe the conduction of heat through a given medium, and related deformations of the heat equation, like the diffusion equation, the Fokker-Planck equation, and so on, are of basic importance in variable scientific fields. Given an n-dimensional Riemannian manifold M with the Laplace-Beltrami operator Δ. Then we are able to define a differential operator L, which is known as the heat operator, by

$$L = \Delta - \frac{\partial}{\partial t}$$

acting on functions in $C^0 (M \times (0, \infty))$, which are C^2 w.r.t. the variable x, varying on M, and C^1 w.r.t. the variable t, varying on $(0, \infty)$. Correspondingly, the heat equation is given by

$$Lu = 0 \quad \left(\text{equivalently, } \Delta u - \frac{\partial u}{\partial t} = 0\right),$$

with $u \in C^0 (M \times (0, \infty))$. If we want to get the existence, or even give an explicit expression, of the solution for this heat equation with a prescribed initial condition or (Dirichlet or Neumann) boundary condition, we need to use a tool named *heat kernel*.

Definition 1.4 *A fundamental solution, which is called the heat kernel, of the heat equation on a prescribed Riemannian manifold M is a continuous function $H(x, y, t)$, defined on $M \times M \times (0, \infty)$, which is C^2 with respect to x, C^1 with respect to t, and* *which satisfies*

$$L_x H = 0, \qquad \lim_{t\to 0} H(x,y,t) = \delta_y(x),$$

where $\delta_y(x)$ *is the Dirac delta function, that is, for all bounded continuous function* f *on* M, *we have, for every* $y \in M$,

$$\lim_{t\to 0} \int_M H(x,y,t) f(x) dV(x) = f(y).$$

By constructing a parametrix, the existence of the heat kernel on compact or complete Riemannian manifolds, or even manifolds with boundary subject to either Dirichlet or Neumann boundary conditions, can be obtained (see, for instance, [30]). In fact, for a complete Riemannian manifold, one can have the following.

Theorem 1.5 ([141]) *Let* M *be a complete Riemannian manifold. Then there exists a heat kernel* $H(x,y,t) \in C^\infty(M \times M \times \mathbb{R}^+)$ *such that*

(I) $H(x,y,t) = H(y,x,t)$,

(II) $\lim_{t\to 0} H(x,y,t) = \delta_x(y)$,

(III) $\left(\Delta - \frac{d}{dt}\right) H = 0$,

(IV) $H(x,y,t) = \int_M H(x,z,t-s) H(z,y,s) dV(z)$, *with* $0 < s < t$.

In Section 2.7 of Chapter 2, we would like to focus on the heat kernels of geodesic balls on complete manifolds, and successfully obtain a comparison result, which can be seen as an extension of Debiard-Gaveau-Mazet's comparison result in [55] and Cheeger-Yau's comparison result in [32], for the heat kernel with a Dirichlet or Neumann boundary condition—see Theorem 2.31 for the precise statement. There is a connection between the heat kernel and the eigenvalues of the Laplace operator. One can get a glance about this relation from the following conclusion (cf. [30], p.169).

Theorem 1.6 (The Sturm-Liouville decomposition for the Dirichlet eigenvalue problem) *Given a normal domain* Ω *in a Riemannian manifold* M, *there exists a complete orthonormal basis* $\{\phi_1, \phi_2, \phi_3, \cdots\}$ *of* $L^2(\Omega)$ *consisting of Dirichlet eigenfunctions of the Laplacian* Δ, *with* ϕ_j *having eigenvalue* λ_j *satisfying*

$$0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \uparrow \infty.$$

In particular, each eigenvalue has finite multiplicity, and

$$\phi_j \in C^\infty(\Omega) \cap \bar{C}^1(\Omega),$$

where $\bar{C}^1(\Omega)$ *is the set of functions* v *satisfying that* v *is* C^1 *on* Ω, *and can be extended to a continuous function on* $\overline{\Omega}$, *and moreover, the gradient* grad$v$ *can be extended to a continuous vector field on* $\overline{\Omega}$.

Finally, the heat kernel $H(x,y,t)$ *on* Ω *satisfies*

$$H(x,y,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t}\phi_j(x)\phi_j(y),$$

with convergence absolute, and uniform, for each $t > 0$. *In particular,*

$$\int_\Omega H(x,x,t)dV(x) = \sum_{j=1}^{\infty} e^{-\lambda_j t}.$$

By using Theorem 1.6 and the comparison result for the heat kernel, Theorem 2.31, we can supply another ways to prove the most part of Theorems 2.8 and 2.14.

As we have mentioned in Preface, we study eigenvalue problems of some elliptic operators on bounded connected domains in the first four chapters, since in this case they have the discrete spectrum. But what about the case of unbounded domains? Do they also have the discrete spectrum? The situation of unbounded domains is much complicated. For instance, the Laplace operator only has the essential spectrum in the Euclidean 3-space $\mathbb{R}^3$ or the hyperbolic 3-space $\mathbb{H}^3$, but it has the discrete spectrum on the quantum layers (in $\mathbb{R}^3$) satisfying constrains mentioned in Theorem 1 of Chapter 5.

By applying the theory of self-adjoint operators, the spectral properties of the linear Laplacian on a domain in a Euclidean space or a manifold have been studied extensively. Mathematicians are generally interested in the spectrum of the Laplacian on compact manifolds (with or without boundary) or noncompact complete manifolds, since in these two cases the linear Laplace operators can be uniquely extended to self-adjoint operators [74, 75]. However, the study of the spectrum of the Laplacian on noncompact noncomplete manifolds also attracts attention of mathematicians and physicists in the past three decades, since the study of the spectral properties of the Dirichlet Laplacian in infinitely stretched regions has applications in elasticity, acoustics, electromagnetism, quantum physics, etc. In fact, the Laplacians on some noncompact noncomplete manifolds, like quantum layers, can also be extended to self adjoint operators under suitable constraints (see, e.g., [111], [121]), and then the existing methods for the previous cases might be used. A quantum layer is actually formed by thickening an immersed oriented hypersurface of some given ambient manifold along its normal vectors to a fixed width. For instance, let Σ be an immersed oriented hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ ($n > 1$), and let $\boldsymbol{N}$ be the vector field of Σ; then the quantum layer Ω built over Σ with half width a can be seen as a differentiable manifold $\Sigma \times (-a, a)$, and it can also be given by a map

$$\Phi : \Sigma \times (-a,a) \to \mathbb{R}^{n+1}, \qquad (x,u) \to x + u\boldsymbol{N}. \tag{1.8}$$

So, from this example, we know that quantum layers are noncompact noncomplete manifolds. They maybe have the discrete spectrum or maybe not. In mathematical physics, points in the discrete spectrum are called *bound states*, which can be used to describe some physical quantities. Moreover, the lowest bound state is called the *ground state*. The integrability of the Gaussian curvature is necessary for deriving the existence of the ground state of the quantum layers built over surfaces ruled outside a compact set [112]. Lin and Lu have shown that the finite topological type makes an important role in the proof of the fact that a surface ruled outside a compact set has integrable Gaussian curvature (see Corollary 2 in [112]). From this example, we know that topological structures maybe have influence on the spectral properties of the Laplacian (see also Example 2.16 in Chapter 2).

In Chapter 5, we first recall some existing conclusions about the existence of ground state of *prescribed* quantum strips and quantum layers, which are noncompact manifolds, and also issue several open problems. This can be seen as our attempt of trying to know more information about the spectral structure of some elliptic operators on *unbounded* domains. Besides, some other open problems concerning eigenvalue problems in curvature flows and Finsler geometry will also be issued.

Chapter 2

Eigenvalue comparison theorems

The structure of the chapter is as follows. In the next section, we lay out the background to the problem and the necessary basic definitions, including the characterizations of the relevant quantities in the case of spherically symmetric manifolds. The bound for eigenvalues in the case where the *radial Ricci curvature is bounded from below w.r.t. some point* is derived in Section 2.2, together with some consequences. The case where the *radial sectional curvature is bounded from above w.r.t. some point* is dealt with in Section 2.3. This requires the extension of other comparison results to the spherically symmetric setting (as opposed to the constant curvature setting), which we believe to be interesting in their own right, such as Rauch's and Bishop's comparison theorems—see Theorem 2.11 and Theorems 2.5 and 2.12, respectively. In Section 2.4, several interesting properties of the model spaces will be discussed. Some interesting examples will be shown in Section 2.5 to explain our results clearly and intuitively. In Section 2.6, we present a criterion to judge the existence of the desired model space, a spherically symmetric manifold with a pole, for a prescribed unbounded manifold. In section 2.7, we will give a comparison result for the heat kernel, which supplies us another ways to prove the generalized eigenvalue comparison results in the previous sections. The precise statement of our main result is given in Section 2.8—see Theorem 2.34 for the detail. Besides, an interesting spectral result will also be given in this section. In Section 2.9, we will give the proof of Theorem 2.34 by using a method provided by Freitas, Mao and Salavessa in [72] (equivalently, the method shown in the proof of Theorem 2.8 in Section 2.2 of Chapter 2). By using this theorem, some estimates for the first eigenvalue of the Dirichlet p-Laplacian of a geodesic ball on a complete Riemannian manifold with *a radial Ricci curvature lower bound w.r.t. some point* will be given in the last section. The main content of this chapter is based on an independent work [122], the Ph.D. thesis [123] of the first author here (i.e., Prof. Jing Mao), and a joint-work [72] of Freitas, Mao and Salavessa.