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Advanced Calculation Mechanics

高等计算力学

Zheming Zhu Xingyu Wang Li Ren Meng Wang
朱哲明 王兴渝 任 利 王 蒙 编著

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责任编辑:毕　潜
责任校对:蒋　玙
封面设计:墨创文化
责任印制:王　炜

图书在版编目(CIP)数据

高等计算力学：英文 / 朱哲明等编著. —成都：四川大学出版社，2019.3
ISBN 978-7-5690-2830-0

Ⅰ.①高…　Ⅱ.①朱…　Ⅲ.①计算力学-英文　Ⅳ.①O302

中国版本图书馆 CIP 数据核字（2019）第 049916 号

书名　**高等计算力学**
Advanced Calculation Mechanics

编　著	朱哲明　王兴渝　任　利　王　蒙
出　版	四川大学出版社
地　址	成都市一环路南一段 24 号 (610065)
发　行	四川大学出版社
书　号	ISBN 978-7-5690-2830-0
印　刷	郫县犀浦印刷厂
成品尺寸	185 mm×260 mm
印　张	12.75
字　数	416 千字
版　次	2019 年 3 月第 1 版
印　次	2019 年 3 月第 1 次印刷
定　价	48.00 元

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Preface

The advent of high-speed computers has given tremendous impetus to all numerical methods for solving engineering problems. Finite element method forms one of the most versatile classes of such methods, and was originally developed in the field of structural analysis. Finite element method is a computer-aided mathematical tool which can be used for obtaining approximate solutions of some parameters to those engineering problems that can be represented by physical system subjected to external influences. Such problems are in areas like solid mechanics (elasticity, plasticity, statics, dynamics, etc.), heat transfer (conduction, convection, radiation), fluid mechanics, electromagnetism and coupled interactions of the above, e. g. fluid-solid interaction. Application in solid mechanics is much more extensive, and can be classified in many different ways depending on the application types, physical system shapes or responses, loading conditions, and so on.

The purpose of this course is to introduce students (both undergraduate and graduate students) to the fundamentals of the finite element method and its applications in engineering with emphasis on solid structures or liquid. The students learn the necessary principle to help them create a basic finite element code (using FORTRAN or MATLAB or C programming language) for the stress and deformation analysis of solid structures.

This book is based on courses given by the author to both undergraduate and graduate students of engineering mechanics at Sichuan University. A prior knowledge of the FORTRAN or MATLAB or C programming language and solid mechanics is assumed. The level of continuum mechanics, numerical analysis, matrix algebra and other mathematics employed in this book is that normally taught in undergraduate engineering courses. The book is therefore suitable for engineering undergraduates and other students at an equivalent level. Postgraduates and practising engineers may also find it useful if they are comparatively new to finite element methods.

Sichuan University, Zheming Zhu

Chengdu, China

Contents

Chapter 1 Discretization and element stiffness

The limitations of the human mind are such that it cannot grasp the behaviour of its complex surroundings and creations in one operation. Therefore, the process of subdividing all systems into their individual components or 'elements', whose behavior is readily understood, and then rebuilding the original system from such components to study its behavior, is a natural way in which the engineer, the scientist, or even the economist proceeds.

In many situations an adequate model is obtained using a finite number of well-defined components, and we shall term such problems discrete. In others the subdivision is continued indefinitely and the problem can only be defined using the mathematical fiction of an infinitesimal. This leads to differential equations or equivalent statements which imply an infinite number of elements, and we shall term such systems continuous.

With the advent of digital computers, discrete problems can generally be solved readily even if the number of elements is very large. As the capacity of all computers is finite, continuous problems can only be solved exactly by mathematical manipulation. Here, the available mathematical techniques usually limit the possibilities to oversimplified situations.

To overcome the intractability of realistic types of continuum problems, various methods of discretization have been proposed from time to time both by engineers and mathematicians. All involve an approximation which, hopefully, approaches in the limit the true continuum solution as the number of discrete variables increases.

The discretization of continuous problems has been approached differently by mathematicians and engineers. Mathematicians have developed general techniques applicable directly to differential equations governing the problem, such as finite difference approximations, various weighted residual procedures, or proximate techniques for determining the stationarity of properly defined 'functionals'. The engineer, on the other hand, often approaches the problem more intuitively by creating an analogy between real discrete elements and finite portions of a continuum domain.

Since the early 1960s much progress has been made, and today the purely mathematical and 'analogy' approaches are fully reconciled. It is the object of this text to present a view of the finite element method as a general discretization procedure of continuum problems posed by mathematically defined statements.

In the analysis of problems of a discrete nature, a standard methodology has been developed over the years. The civil engineer, dealing with structures, first calculates force-displacement relationships for each element of the structure and then proceeds to assemble the whole by following a well-defined procedure of establishing local equilibrium at each 'node' or connecting point of the structure. The resulting equations can be solved for the unknown displacements. Similarly, the electrical or hydraulic engineer, dealing with a network of electrical components (resistors, capacitances, etc.) or hydraulic conduits, first establishes a relationship between currents (flows) and potentials for individual elements and then proceeds to assemble the system by ensuring continuity of flows.

All such analyses follow a standard pattern which is universally adaptable to discrete systems. It is thus possible to define a standard discrete system, and this chapter will be primarily concerned with establishing the processes applicable to such systems. Much of what is presented here will be known to engineers, but some reiteration at this stage is advisable. As the treatment of elastic solid structures has been the most developed area of activity, this will be the focus of this book.

1.1 Discretization of a domain by elements

Forstructures with a complex configuration and subjected to multi-loads as shown in Fig. 1-1, it is very difficult, and sometimes it is impossible, to obtain analytical solutions of the stresses or the displacements by using the knowledge of solid mechanics. Under such scenario, a better approach is to discretize the structure or the domain into a finite number of small triangular (or quadrilateral) elements, as shown in Fig. 1-2, and then try to solve the stresses and/or strains of each element. This process is called discretization of a domain. For three dimensional problems, we use tetrahedral elements or brick elements to discretize the domain.

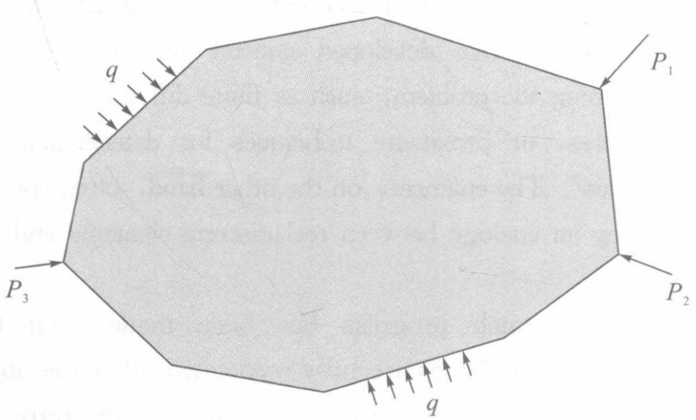
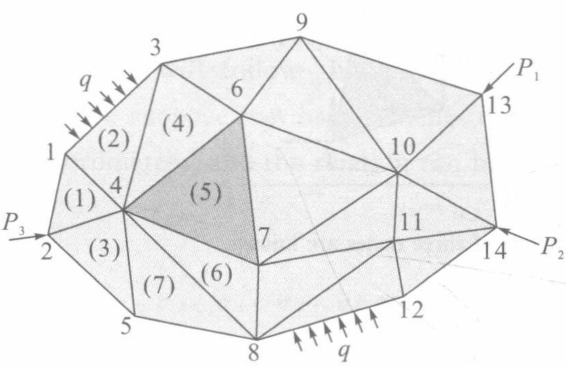


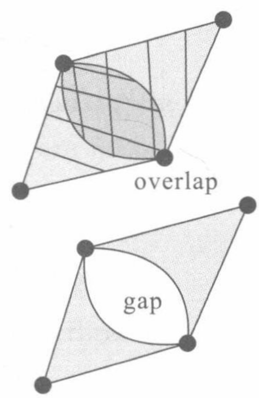
Fig. 1-1 A domain with a complex configuration and multi-loads

Assume that:

1. The continuum (or domain) is separated by imaginary lines or surfaces (in 3D cases) into a number of 'finite elements', and the nodes are numbered. The number of the nodes are arbitrary, but for saving the calculation time, the difference between a node number with its adjacent node numbers should be as small as possible.
2. The elements are interconnected at a discrete number of nodes, not the imaginary lines. If there is action between two adjacent elements, only the conjunct nodes can transfer the force or the displacement.
3. The phenomena of overlap and gap between two adjacent elements as shown in Fig. 1−2 are not allowed.



(a) Discretization of a domain



(b) Two cases of dislocation

Fig. 1−2 Discretization of a domain with triangular elements and numbered nodes

After the domain is discretized by a finite number of triangle elements, as shown in Fig. 1−3, we will try to solve the stresses and/or the strains of each element. First, we arbitrarily take one triangular element from the domain divided by the triangular elements, for example, the element number (5), and for generality, the node numbers are replaced by $i\,(x_i,\ y_i)$, $j\,(x_j,\ y_j)$ and $k\,(x_k,\ y_k)$ in an anticlockwise sequence, which is a strictly rule we must obey in the following study. Each node has two coordinates $(x,\ y)$ which are known because they were fixed during meshing.

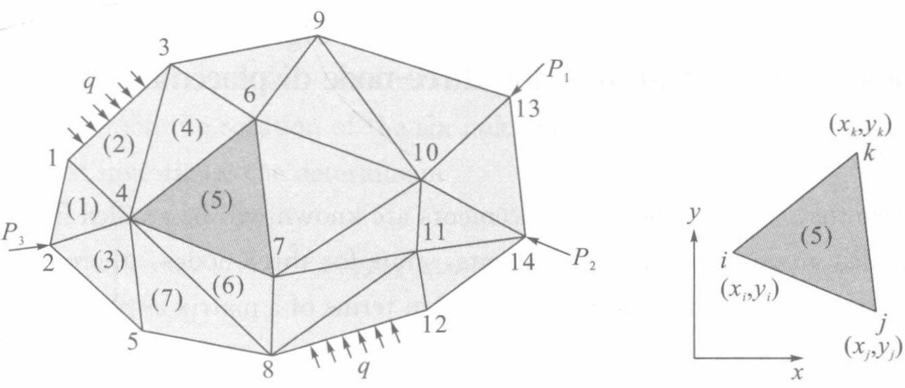


Fig. 1−3 One element is selected arbitrarily and its nodes are numbered by *i*, *j* and *k*

Let's think about: if one can find the solution of the stresses of the arbitrarily selected element, then one may take the same procedure to get the solutions of all the elements, which means you can obtain all the stresses and the strains of the whole domain or structure. Therefore, in the following study, we will first consider the simplest case only one element, which is also rational because one could use only one element to divide a domain in some simple situations. Then we will consider multi-elements, and through the relation between one element and multi-elements, one can easily get the solutions of stresses and strains of the structures.

Based on one element, two cases will be considered in this chapter: (1) the displacements of the three nodes are known, as shown in Fig. 1−4; (2) the loads acting on the three nodes are known, as shown in Fig. 1−5.

$$\{\delta\}^e = \begin{bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{bmatrix}$$

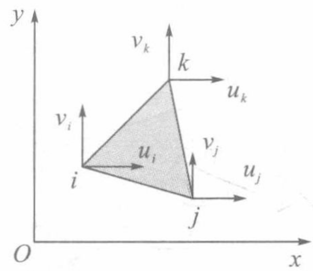


Fig. 1−4 The displacements of three nodes are known

$$\{F\}^e = \begin{bmatrix} F_i^x \\ F_i^y \\ F_j^x \\ F_j^y \\ F_k^x \\ F_k^y \end{bmatrix}$$

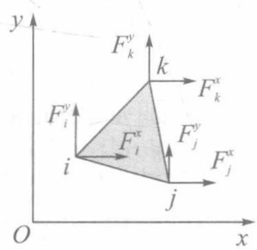


Fig. 1−5 The loads of three nodes are known

In the following, we will present detailed procedures of solving the stresses and strains for the two cases shown in Figs. 1−4 and 1−5.

1.2 Solution to the case that the three-node displacements are known

The case that the three-node displacements are known will be studied in this chapter. For each node, there are two displacements, thus for three nodes, there are totally six displacements $(\delta)^e$, and they can be expressed in terms of a matrix as shown in Fig. 1−6.

$$\{\delta\}^e = \begin{bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{bmatrix}$$

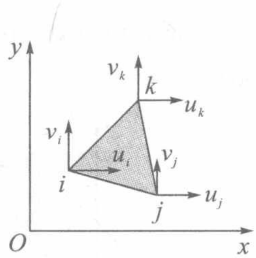


Fig. 1—6 A triangle element and its six known displacement components

For the case that the displacements of the three nodes are known, we first try to obtain the displacements (u and v) of any point inside the triangular element based on the three node displacements. We use the strain-displacement relationship (i. e. $\varepsilon = \frac{\partial u}{\partial x}$) to obtain the strains of this element, and finally we use the stress-strain relationship (i. e. $\sigma_x = \frac{E}{1-\nu}(\varepsilon_x + \nu\varepsilon_y)$) to obtain the stresses of this element.

Next, we will follow this procedure to get the solution to the stresses of this element. We suppose the displacements insider this element are linearly related to the x and y coordinates, and the relation can be written as

$$\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ v &= \alpha_4 + \alpha_5 x + \alpha_6 y \end{aligned} \tag{1—1}$$

where α_1, α_2, α_3, α_4, α_5, α_6 are unknown coefficients. In a three-node element, there are totally six displacement components which are known and can be employed to determine the six unknown coefficients. Substituting the three node coordinates x and y into Eq. (1—1), one can get six linear simultaneous equations as

$$\begin{cases} u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i \\ u_j = \alpha_1 + \alpha_2 x_j + \alpha_3 y_j \\ u_k = \alpha_1 + \alpha_2 x_k + \alpha_3 y_k \end{cases} \Rightarrow \begin{Bmatrix} u_i \\ u_j \\ u_k \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \tag{1—2}$$

$$\begin{cases} v_i = \alpha_4 + \alpha_5 x_i + \alpha_6 y_i \\ v_j = \alpha_4 + \alpha_5 x_j + \alpha_6 y_j \\ v_k = \alpha_4 + \alpha_5 x_k + \alpha_6 y_k \end{cases} \Rightarrow \begin{Bmatrix} v_i \\ v_j \\ v_k \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{Bmatrix} \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} \tag{1—3}$$

1.2.1 The area of a triangle element

In order to obtain the solution of the six unknown coefficients α_1, α_2, α_3, α_4, α_5, α_6, we will first investigate the determinant

$$D = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = x_j y_k + x_k y_i + x_i y_j - x_j y_i - x_i y_k - x_k y_j \tag{1—4}$$

Combined with Fig. 1—7, Eq. (1—4) can be written in another form as

$$D = y_k(x_j - x_i) + y_j(x_i - x_k) + y_i(x_k - x_j) = 2A \tag{1—5}$$

where A is the triangle area in Fig. 1−7. This indicates that the determinant D is just the double triangle area.

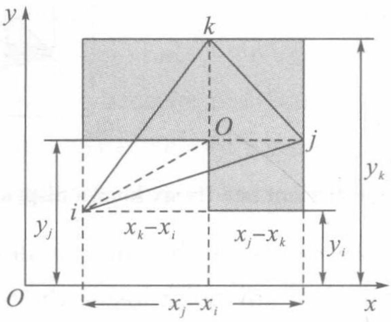


Fig. 1−7 Sketch of a triangle

1.2.2 The solution of the six unknown coefficients

According to the knowledge of Linear Algebra, from Eq. (1−2), the coefficients α_1, α_2 and α_3 can be expressed as

$$\alpha_1 = \frac{1}{D}\begin{vmatrix} u_i & x_i & y_i \\ u_j & x_j & y_j \\ u_k & x_k & y_k \end{vmatrix},\ \alpha_2 = \frac{1}{D}\begin{vmatrix} 1 & u_i & y_i \\ 1 & u_j & y_j \\ 1 & u_k & y_k \end{vmatrix},\ \alpha_3 = \frac{1}{D}\begin{vmatrix} 1 & x_i & u_i \\ 1 & x_j & u_j \\ 1 & x_k & u_k \end{vmatrix} \tag{1−6}$$

where $D = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix} = 2A$, and more details of the coefficients α_1, α_2 and α_3 can be written as

$$\alpha_1 = \frac{1}{D}\begin{vmatrix} u_i & x_i & y_i \\ u_j & x_j & y_j \\ u_k & x_k & y_k \end{vmatrix} = \frac{1}{2A}\left(u_i\begin{vmatrix} x_j & y_j \\ x_k & y_k \end{vmatrix} - u_j\begin{vmatrix} x_i & y_i \\ x_k & y_k \end{vmatrix} + u_k\begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix}\right)$$

$$= \frac{1}{2A}(a_i u_i + a_j u_j + a_k u_k)$$

$$\alpha_2 = \frac{1}{D}\begin{vmatrix} 1 & u_i & y_i \\ 1 & u_j & y_j \\ 1 & u_k & y_k \end{vmatrix} = \frac{1}{2A}\left(-u_i\begin{vmatrix} 1 & y_j \\ 1 & y_k \end{vmatrix} + u_j\begin{vmatrix} 1 & y_i \\ 1 & y_k \end{vmatrix} - u_k\begin{vmatrix} 1 & y_i \\ 1 & y_j \end{vmatrix}\right)$$

$$= \frac{1}{2A}(b_i u_i + b_j u_j + b_k u_k)$$

$$\alpha_3 = \frac{1}{D}\begin{vmatrix} 1 & x_i & u_i \\ 1 & x_j & u_j \\ 1 & x_k & u_k \end{vmatrix} = \frac{1}{2A}\left(u_i\begin{vmatrix} 1 & x_j \\ 1 & x_k \end{vmatrix} - u_j\begin{vmatrix} 1 & x_i \\ 1 & x_k \end{vmatrix} + u_k\begin{vmatrix} 1 & x_i \\ 1 & x_j \end{vmatrix}\right)$$

$$= \frac{1}{2A}(c_i u_i + c_j u_j + c_k u_k)$$

where $a_i = \begin{vmatrix} x_j & y_j \\ x_k & y_k \end{vmatrix} = x_j y_k - x_k y_j$; $b_i = -\begin{vmatrix} 1 & y_j \\ 1 & y_k \end{vmatrix} = y_j - y_k$; $c_i = \begin{vmatrix} 1 & x_j \\ 1 & x_k \end{vmatrix} = -x_j + x_k$;

(i, j, k), and here the subscripts i, j and k circulate in the same manner. For example, according to the i, j and k sequence, as shown in Fig. 1−8, a_j can be written as $a_j = x_k y_i - x_i y_k$. Similarly, all the other coefficients a_k, b_j, b_k, c_j and c_k can be obtained, and the students should write out them as an assignment.

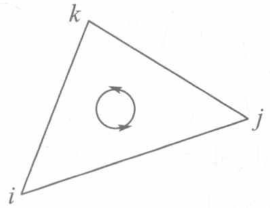


Fig. 1−8 Sketch of a triangle

It should be noted that all the a, b and c coefficients are constants because the nodal coordinates (x, y) are known and have a fixed value for a specific triangle element.

Similarly, from Eq. (1−3), one can gain the solution of the coefficients α_4, α_5 and α_6, and they can be expressed as

$$\alpha_4 = \frac{1}{2A}(a_i v_i + a_j v_j + a_k v_k)$$

$$\alpha_5 = \frac{1}{2A}(b_i v_i + b_j v_j + b_k v_k)$$

$$\alpha_6 = \frac{1}{2A}(c_i v_i + c_j v_j + c_k v_k)$$

It can be seen that the six coefficients α_1, α_2, α_3, α_4, α_5, α_6 are related to the node displacements (u_i, v_i) (i, j, k). The six coefficients can be written in another form as

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_k \end{Bmatrix}, \quad \begin{Bmatrix} \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix}\begin{Bmatrix} v_i \\ v_j \\ v_k \end{Bmatrix} \tag{1−7}$$

1.2.3 The displacement expression and shape function

Substituting the six coefficients in Eq. (1−7) into the displacement expression in Eq. (1−1), one can have

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y = [1 \quad x \quad y]\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \frac{1}{2A}[1 \quad x \quad y]\begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_k \end{Bmatrix}$$

$$= \begin{bmatrix} N_i \\ N_j \\ N_k \end{bmatrix}^{\mathrm{T}}\begin{Bmatrix} u_i \\ u_j \\ u_k \end{Bmatrix} = N_i u_i + N_j u_j + N_k u_k$$

$$v = \alpha_4 + \alpha_5 x + \alpha_6 y = [1 \quad x \quad y]\begin{Bmatrix} \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} = \frac{1}{2A}[1 \quad x \quad y]\begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix}\begin{Bmatrix} v_i \\ v_j \\ v_k \end{Bmatrix}$$

$$= \begin{bmatrix} N_i \\ N_j \\ N_k \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} v_i \\ v_j \\ v_k \end{Bmatrix} = N_i v_i + N_j v_j + N_k v_k \qquad (1-8)$$

where $\begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} = \frac{1}{2A}[1\ x\ y] \begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} = \frac{1}{2A} \begin{Bmatrix} a_i + b_i x + c_i y \\ a_j + b_j x + c_j y \\ a_k + b_k x + c_k y \end{Bmatrix}$; and where $\begin{cases} a_i = x_j y_k - x_k y_j \\ b_i = y_j - y_k \\ c_i = -x_j + x_k \end{cases}$

(i, j, k)

where N_i, N_j and N_k are called Shape Function of this triangle element, and for simplification, they can be written as

$$N_i = \frac{1}{2A}(a_i + b_i x + c_i y) \qquad (i, j, k) \qquad (1-9)$$

For a triangular element, the area A and the coefficients a_i, b_i and c_i are constants, and therefore, the shape function is only the function of the coordinates x and y.

The displacements in Eq. (1−8) can be simply expressed as

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_k u_k \\ N_i v_i + N_j v_j + N_k v_k \end{Bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_k & 0 \\ 0 & N_i & 0 & N_j & 0 & N_k \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{Bmatrix} = [N]\{\delta\}^e \qquad (1-10)$$

where $[N]$ is the matrix of shape function. It can be seen that the displacements within each element and on its boundaries can be expressed in terms of the shape function and its nodal displacements.

1.2.4 Strain and strain matrix

After the displacement functions of the element have been determined, we can use the general relation of strain-displacement to get the strains inside this element. The relation of strain-displacement for a plane problem can be expressed as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad (1-11)$$

Substituting Eq. (1−10) into Eq. (1−11), we have

$$\{\varepsilon\}=\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}N_i & 0 & N_j & 0 & N_k & 0\\ 0 & N_i & 0 & N_j & 0 & N_k\end{bmatrix}\begin{Bmatrix}u_i\\ v_i\\ u_j\\ v_j\\ u_k\\ v_k\end{Bmatrix}=\begin{bmatrix}\frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_j}{\partial x} & 0 & \frac{\partial N_k}{\partial x} & 0\\ 0 & \frac{\partial N_i}{\partial y} & 0 & \frac{\partial N_j}{\partial y} & 0 & \frac{\partial N_k}{\partial y}\\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial y} & \frac{\partial N_k}{\partial x}\end{bmatrix}\begin{Bmatrix}u_i\\ v_i\\ u_j\\ v_j\\ u_k\\ v_k\end{Bmatrix}$$

Substituting the shape function $N_i=\frac{1}{2A}(a_i+b_ix+c_iy)(i,j,k)$ into the above equation, we have

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\frac{1}{2A}\begin{bmatrix}b_i & 0 & b_j & 0 & b_k & 0\\ 0 & c_i & 0 & c_j & 0 & c_k\\ c_i & b_i & c_j & b_j & c_k & b_k\end{bmatrix}\begin{Bmatrix}u_i\\ v_i\\ u_j\\ v_j\\ u_k\\ v_k\end{Bmatrix}=[B]\{\delta\}^e \qquad (1-12)$$

where $b_i=y_j-y_k, c_i=-x_j+x_k(i,j,k)$, $[B]$ is strain matrix, and $(\delta)^e$ is element displacements. It can be seen that the matrix $[B]$ is a constant matrix, which means that the strains inside the whole element are the same because no x and y are related in the $[B]$ matrix.

One can find that if the displacements of the three nodes are known, i. e. $(\delta)^e$ is known, the three strain components can be easily obtained by using Eq. (1−12). The $[B]$ matrix can be written in another form as

$$[B]=[B_i \quad B_j \quad B_k]=\frac{1}{2A}\begin{bmatrix}b_i & 0 & b_j & 0 & b_k & 0\\ 0 & c_i & 0 & c_j & 0 & c_k\\ c_i & b_i & c_j & b_j & c_k & b_k\end{bmatrix} \qquad (1-13)$$

where $[B_i]=\frac{1}{2A}\begin{bmatrix}b_i & 0\\ 0 & c_i\\ c_i & b_i\end{bmatrix}(i,j,k)$.

1.2.5 Stresses and stress matrix

The relation between stresses and strains is the basic knowledge, and the students should have grasped this relationship before. It can be written as

$$\begin{cases}\sigma_x=\frac{E}{1-\nu^2}(\varepsilon_x+\nu\varepsilon_y)\\ \sigma_y=\frac{E}{1-\nu^2}(\nu\varepsilon_x+\varepsilon_y)\\ \tau_{xy}=\frac{E}{2(1+\nu)}\gamma_{xy}=\frac{E}{1-\nu^2}\cdot\frac{1-\nu}{2}\gamma_{xy}\end{cases}$$

The stress-strain relation can be rewritten in another form as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{1-14}$$

Eq. (1−14) can be simplified as

$$\{\sigma\} = [D]\{\varepsilon\} \tag{1-15}$$

where $[D]$ is the plane stress elasticity matrix and

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{1-16}$$

Substituting the strains in Eq. (1−12) into Eq. (1−15), we have

$$\{\sigma\} = [D]\{\varepsilon\} = [D][B]\{\delta\}^e = [S]\{\delta\}^e \tag{1-17}$$

where $[S]$ is stress matrix and it can be written as

$$[S] = [D][B] = \frac{E}{2A(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ 0 & c_i & 0 & c_j & 0 & c_k \\ c_i & b_i & c_j & b_j & c_k & b_k \end{bmatrix}$$

$$= \frac{E}{2A(1-\nu^2)}\begin{bmatrix} b_i & \nu c_i & b_j & \nu c_j & b_k & \nu c_k \\ \nu b_i & c_i & \nu b_j & c_j & \nu b_k & c_k \\ \frac{1-\nu}{2}c_i & \frac{1-\nu}{2}b_i & \frac{1-\nu}{2}c_j & \frac{1-\nu}{2}b_j & \frac{1-\nu}{2}c_k & \frac{1-\nu}{2}b_k \end{bmatrix} \tag{1-18}$$

where $b_i = y_j - y_k, c_i = -x_j + x_k (i, j, k)$.

Substituting Eq. (1−18) into Eq. (1−17), one can have

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{2A(1-\nu^2)}\begin{bmatrix} b_i & \nu c_i & b_j & \nu c_j & b_k & \nu c_k \\ \nu b_i & c_i & \nu b_j & c_j & \nu b_k & c_k \\ \frac{1-\nu}{2}c_i & \frac{1-\nu}{2}b_i & \frac{1-\nu}{2}c_j & \frac{1-\nu}{2}b_j & \frac{1-\nu}{2}c_k & \frac{1-\nu}{2}b_k \end{bmatrix}\begin{bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{bmatrix} \tag{1-19}$$

It can be seen that the stress matrix is also a constant matrix, which means that the stresses in this whole element are the same. Therefore, the triangle element is also called constant stress elements or constant strain elements. This is because the displacements selected in Eq. (1−1) are linearly related to x and y coordinates, and the derivatives with respect to x and y are independent of x and y. Therefore, both the $[B]$ matrix and the $[S]$ matrix are constants.