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A Course in Metric Geometry

度量几何学教程

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出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于1888年，是国际上极具影响力的专业学术组织，目前拥有近30000会员和580余个机构成员，出版图书3500多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统等所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

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Preface

This book is not a research monograph or a reference book (although research interests of the authors influenced it a lot)—this is a textbook. Its structure is similar to that of a graduate course. A graduate course usually begins with a course description, and so do we.

Course description. The objective of this book is twofold. First of all, we wanted to give a detailed exposition of basic notions and techniques in the theory of length spaces, a theory which experienced a very fast development in the past few decades and penetrated into many other mathematical disciplines (such as Group Theory, Dynamical Systems, and Partial Differential Equations). However, we have a wider goal of giving an elementary introduction into a broad variety of the most geometrical topics in geometry—the ones related to the notion of distance. This is the reason why we included metric introductions to Riemannian and hyperbolic geometries. This book tends to work with "easy-to-touch" mathematical objects by means of "easy-to-visualize" methods. There is a remarkable book [**Gro3**], which gives a vast panorama of "geometrical mathematics from a metric viewpoint". Unfortunately, Gromov's book seems hardly accessible to graduate students and non-experts in geometry. One of the objectives of this book is to bridge the gap between students and researchers interested in metric geometry, and modern mathematical literature.

Prerequisite. It is minimal. We set a challenging goal of making the core part of the book accessible to first-year graduate students. Our expectations of the reader's background gradually grow as we move further in the book. We tried to introduce and illustrate most of new concepts and methods by using their simplest case and avoiding technicalities that take attention away from the gist of the matter. For instance, our introduction to Riemannian geometry begins with metrics on planar regions, and we even avoid the notion of a manifold. Of course, manifolds do show up in more advanced sections. Some exercises and remarks assume more mathematical background than the rest of our exposition; they are optional, and a reader unfamiliar with some notions can just ignore them. For instance, solid background in differential geometry of curves and surfaces in $\mathbb{R}^3$ is not a mandatory prerequisite for this book. However, we would hope that the reader possesses some knowledge of differential geometry, and from time to time we draw analogies from or suggest exercises based on it. We also make a special emphasis on motivations and visualizations. A reader not interested in them will be able to skip certain sections. The first chapter is a clinic in metric topology; we recommend that the reader with a reasonable idea of metric spaces just skip it and use it for reference: it may be boring to read it. The last chapters are more advanced and dry than the first four.

Figures. There are several figures in the book, which are added just to make it look nicer. If we included all necessary figures, there would be at least five of them for each page.

• It is a must that the reader systematically studying this book makes a figure for every proposition, theorem, and construction!

Exercises. Exercises form a vital part of our exposition. This does not mean that the reader should solve all the exercises; it is very individual. The difficulty of exercises varies from trivial to rather tricky, and their importance goes all the way up from funny examples to statements that are extensively used later in the book. This is often indicated in the text. It is a very helpful strategy to perceive *every* proposition and theorem as an exercise. You should try to prove each on your own, possibly after having a brief glance at our argument to get a hint. Just reading our proof is the last resort.

Optional material. Our exposition can be conditionally subdivided into two parts: core material and optional sections. Some sections and chapters are preceded by a brief plan, which can be used as a guide through them. It is usually a good idea to begin with a first reading, skipping all optional sections (and even the less important parts of the core ones). Of course, this approach often requires going back and looking for important notions that were accidentally missed. A first reading can give a general picture of the theory, helping to separate its core and give a good idea of its logic. Then the reader goes through the book again, transforming theoretical knowledge into the genuine one by filling it with all the details, digressions, examples and experience that makes knowledge practical.

About metric geometry. Whereas the borderlines between mathematical disciplines are very conditional, geometry historically began from very "down-to-earth" notions (even literally). However, for most of the last century it was a common belief that "geometry of manifolds" basically boiled down to "analysis on manifolds". Geometric methods heavily relied on differential machinery, as it can be guessed even from the name "Differential geometry". It is now understood that a tremendous part of geometry essentially belongs to metric geometry, and the differential apparatus can be used just to define some class of objects and extract the starting data to feed into the synthetic methods. This certainly cannot be applied to all geometric notions. Even the curvature tensor remains an obscure monster, and the geometric meaning of only some of its simplest appearances (such as the sectional curvature) are more or less understood. Many modern results involving more advanced structures still sound quite analytical. On the other hand, expelling analytical machinery from a certain sphere of definitions and arguments brought several major benefits. First of all, it enhanced mathematical understanding of classical objects (such as smooth Riemannian manifolds) both ideologically, and by concrete results. From a methodological viewpoint, it is important to understand what assumptions a particular result relies on; for instance, in this respect it is more satisfying to know that geometrical properties of positively curved manifolds are based on a certain inequality on distances between quadruples of points rather than on some properties of the curvature tensor. This is very similar to two ways of thinking about convex functions. One can say that a function is convex if its second derivative is nonnegative (notice that the definition already assumes that the function is smooth, leaving out such functions as $f(x) = |x|$). An alternative definition says that a function is convex if its epigraph (the set $\{(x, y) : y \geq f(x)\}$) is; the latter definition is equivalent to Jensen's inequality $f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$ for all nonnegative α, β with $\alpha + \beta = 1$, and it is robust and does not rely on the notion of a limit. From this viewpoint, the condition $f'' \geq 0$ can be regarded as a convenient criterion for a smooth function to be convex.

As a more specific illustration of an advantage of this way of thinking, imagine that one wants to estimate a certain quantity over all metrics on a sphere. It is so tempting to study a metric for which the quantity attains its maximum, but alas this metric may fail exist within smooth metrics, or even metrics that induce the same topology. It turns out that it still may exist if we widen our search to a class of more general length spaces. Furthermore, mathematical topics whose study used to lie outside the range of noticeable applications of geometrical technique now turned out to be traditional objects of methods originally rooted in differential geometry. Combinatorial group theory can serve as a model example of this situation. By now the scope of the theory of length spaces has grown quite far from its cradle (which was a theory of convex surfaces), including most of classical Riemannian geometry and many areas beyond it. At the same time, geometry of length spaces perhaps remains one of the most "hands-on" mathematical techniques. This combination of reasons urged us to write this "beginners' course in geometry from a length structure viewpoint".

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Contents

Chapter 1

Metric Spaces

The purpose of the major part of the chapter is to set up notation and to refresh the reader's knowledge of metric spaces and related topics in point-set topology. Section 1.7 contains minimal information about Hausdorff measure and dimension.

It may be a good idea to skip this chapter and use it only for reference, or to look through it briefly to make sure that all examples are clear and exercises are obvious.

1.1. Definitions

Definition 1.1.1. Let X be an arbitrary set. A function $d : X \times X \to \mathbb{R} \cup \{\infty\}$ is a *metric* on X if the following conditions are satisfied for all $x, y, z \in X$.

(1) Positiveness: $d(x, y) > 0$ if $x \neq y$, and $d(x, x) = 0$.

(2) Symmetry: $d(x, y) = d(y, x)$.

(3) Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.

A *metric space* is a set with a metric on it. In a formal language, a metric space is a pair (X, d) where d is a metric on X. Elements of X are called *points* of the metric space; $d(x, y)$ is referred to as the *distance* between points x and y.

When the metric in question is clear from the context, we also denote the distance between x and y by $|xy|$.

Unless different metrics on the same set X are considered, we will omit an explicit reference to the metric and write "a metric space X" instead of "a metric space (X, d)."

In most textbooks, the notion of a metric space is slightly narrower than our definition: traditionally one consider metrics with finite distance between points. If it is important for a particular consideration that d takes only finite values, this will be specified by saying that d is a *finite metric*. There is a very simple relation between finite and infinite metrics, namely a metric space with possibly infinite distances splits canonically into subspaces that carry finite metrics and are separated from one another by infinite distances:

Exercise 1.1.2. Show that the relation $d(x, y) \neq \infty$ is an equivalence relation. Each of its equivalence classes together with the restriction of d is a metric space with a finite metric.

Definition 1.1.3. Let X and Y be two metric spaces. A map $f : X \to Y$ is called *distance-preserving* if $|f(x)f(y)| = |xy|$ for any two points $x, y \in X$. A bijective distance-preserving map is called an *isometry*. Two spaces are *isometric* if there exists an isometry from one to the other.

It is clear that being isometric is an equivalence relation. Isometric spaces share all properties that can be expressed completely in terms of distances.

Semi-metrics.

Definition 1.1.4. A function $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is called a *semi-metric* if it satisfies all properties from Definition 1.1.1 of a metric except the requirement that $d(x, y) = 0$ implies $x = y$. This means that we allow zero distance between different points.

There is an obvious relation between semi-metrics and metrics, namely identifying points with zero distance in a semi-metric leads to a usual metric:

Proposition 1.1.5. *Let d be a semi-metric on X. Introduce an equivalence relation R_d on X: set xR_dy iff $d(x, y) = 0$. Since $d(x, y) = d(x_1, y_1)$ whenever xR_dx_1 and yR_dy_1, the projection $\hat{d}$ of d onto the quotient space $\hat{X} = X/R_d$ is well-defined. Then $(\hat{X}, \hat{d})$ is a metric space.*

Proof. Trivial (exercise). □

We will often abuse notation, writing $(X/d, d)$ rather than $(X/R_d, \hat{d})$, with X/d instead of X/R_d and using the same letter d for its projection $\hat{d}$.

Example 1.1.6. Let the distance between two points (x, y), (x', y') in $\mathbb{R}^2$ be defined by $d((x, y), (x', y')) = |(x - x') + (y - y')|$. Check that it is a semi-metric. Prove that the quotient space $(\mathbb{R}^2/d, d)$ is isometric to the real line.

1.2. Examples

Various examples of metric spaces will appear everywhere in the course. In this section we only describe several important ones to begin with. For many of them, verification of the properties from Definition 1.1.1 is trivial and is left for the reader.

Example 1.2.1. One can define a metric on an arbitrary set X by

$$|xy| = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

This example is not particularly interesting but it can serve as the initial point for many constructions.

Example 1.2.2. The real line, $\mathbb{R}$, is canonically equipped with the distance $|xy| = |x - y|$, and thus can be considered as a metric space. There is an immense variety of other metrics on $\mathbb{R}$; for instance, consider $d_{\log}(x, y) = \log(|x - y| + 1)$.

Example 1.2.3. The Euclidean plane, $\mathbb{R}^2$, with its standard distance, is another familiar metric space. The distance can be expressed by the Pythagorean formula,

$$|xy| = |x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

where (x_1, x_2) and (y_1, y_2) are coordinates of points x and y. The triangle inequality for this metric is known from elementary Euclidean geometry. Alternatively, it can be derived from the Cauchy inequality.

Example 1.2.4 (direct products)**.** Let X and Y be two metric spaces. We define a metric on their direct product $X \times Y$ by the formula

$$|(x_1, y_1)(x_2, y_2)| = \sqrt{|x_1x_2|^2 + |y_1y_2|^2}.$$

In particular, $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

Exercise 1.2.5. Derive the triangle inequality for direct products from the triangle inequality on the Euclidean plane.

Example 1.2.6. Recall that the coordinate n-space $\mathbb{R}^n$ is the vector space of all n-tuples $(x_1, \dots, x_n)$ of real numbers, with component-wise addition and multiplication by scalars. It is naturally identified with the multiple direct product $\mathbb{R} \times \dots \times \mathbb{R}$ (n times). This defines the standard Euclidean distance,

$$|xy| = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$.

Example 1.2.7 (dilated spaces). This simple construction is similar to obtaining one set from another by means of a homothety map. Let X be a metric space and $\lambda > 0$. The metric space λX is the same set X equipped with another distance function $d_{\lambda X}$ which is defined by $d_{\lambda X}(x, y) = d_X(x, y)$ for all $x, y \in X$, where d_X is the distance in X. The space λX is referred to as X *dilated* (or *rescaled*) by λ.

Example 1.2.8 (subspaces). If X is a metric space and Y is a subset of X, then a metric on Y can be obtained by simply restricting the metric from X. In other words, the distance between points of Y is equal to the distance between the same points in X.

Restricting the distance is the simplest but not the only way to define a metric on a subset. In many cases it is more natural to consider an *intrinsic metric*, which is generally not equal to the one restricted from the ambient space. The notion of intrinsic metric will be explained further in the course, but its intuitive meaning can be illustrated by the following example of the intrinsic metric on a circle.

Example 1.2.9. The unit circle, S^1, is the set of points in the plane lying at distance 1 from the origin. Being a subset of the plane, the circle carries the restricted Euclidean metric on it. We define an alternative metric by setting the distance between two points as the length of the shorter arc between them. For example, the arc-length distance between two opposite points of the circle is equal to π. The distance between adjacent vertices of a regular n-gon (inscribed into the circle) is equal to $2\pi/n$.

Exercise 1.2.10. (a) Prove that any circle arc of length less or equal to π, equipped with the above metric, is isometric to a straight line segment.

(b) Prove that the entire circle with this metric is not isometric to any subset of the plane (regarded with the restriction of Euclidean distance onto this subset).

1.2.1. Normed vector spaces.

Definition 1.2.11. Let V be a vector space. A function $|\cdot| : V \to \mathbb{R}$ is a *norm* on V if the following conditions are satisfied for all $v, w \in V$ and $k \in \mathbb{R}$.

(1) Positiveness: $|v| > 0$ if $v \neq 0$, and $|0| = 0$.

(2) Positive homogeneity: $|kv| = |k||v|$.

(3) Subadditivity (triangle inequality): $|v + w| \leq |v| + |w|$.

A *normed space* is a vector space with a norm on it. Finite-dimensional normed spaces are also called *Minkowski spaces*. The distance in a normed space $(V, |\cdot|)$ is defined by the formula

$$d(v,w) = |v - w|.$$

It is easy to see that a normed space with the above distance is a metric space. The norm is recovered from the metric as the distance from the origin.

The Euclidean space $\mathbb{R}^n$ described in Example 1.2.6 is a normed space whose norm is expressed by

$$|(x_1, \ldots, x_n)| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

There are other natural norms in $\mathbb{R}^n$.

Example 1.2.12. The space $\mathbb{R}_1^n$ is the coordinate space $\mathbb{R}^n$ with a norm $\|\cdot\|_1$ defined by

$$\|(x_1, \ldots, x_n)\|_1 = |x_1| + \cdots + |x_n|$$

(where $|\cdot|$ is just the absolute value of real numbers).

Example 1.2.13. Similarly, the space $\mathbb{R}_\infty^n$ is $\mathbb{R}^n$ with a norm $\|\cdot\|_\infty$ where

$$\|(x_1, \ldots, x_n)\|_\infty = \max\{|x_1|, \ldots, |x_n|\}.$$

Exercise 1.2.14. Prove that

(a) $\mathbb{R}_1^2$ and $\mathbb{R}_\infty^2$ are isometric;

(b) $\mathbb{R}_1^n$ and $\mathbb{R}_\infty^n$ are not isometric for any $n > 2$.

Example 1.2.15. Let X be an arbitrary set. The space $\ell_\infty(X)$ is the set of all bounded functions $f : X \to \mathbb{R}$. This is naturally a vector space with respect to pointwise addition and multiplication by scalars. The standard norm $\|\cdot\|_\infty$ on $\ell_\infty(X)$ is defined by

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

Exercise 1.2.16. Show that $\mathbb{R}_\infty^n = \ell_\infty(X)$ for a suitable set X. *Hint*: an n-tuple $(x_1, \ldots, x_n)$ is formally a map, isn't it?

1.2.2. Euclidean spaces. Let X be a vector space. Recall that a *bilinear form* on X is a map $F : X \times X \to \mathbb{R}$ which is linear in both arguments. A bilinear form F is *symmetric* if $F(x,y) = F(y,x)$ for all $x, y \in X$. A symmetric bilinear form F can be recovered from its associated *quadratic form* $Q(x) = Q_F(x) = F(x,x)$, e.g., by means of the formula $4F(x,y) = Q(x+y) - Q(x-y)$.

Definition 1.2.17. A *scalar product* is a symmetric bilinear form F whose associated quadratic form is positive definite, i.e., $F(x,x) > 0$ for all $x \neq 0$. A *Euclidean space* is a vector space with a scalar product on it.