

# Graduate Texts in Mathematics

**Peter Walters**

## **An Introduction to Ergodic Theory**

**遍历性理论引论**

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# Graduate Texts in Mathematics 79

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(continued after index)

Peter Walters

# An Introduction to Ergodic Theory

With 8 Illustrations



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## Preface

In 1970 I gave a graduate course in ergodic theory at the University of Maryland in College Park, and these lectures were the basis of the Springer Lecture Notes in Mathematics Volume 458 called "Ergodic Theory—Introductory Lectures" which was published in 1975. This volume is now out of print, so I decided to revise and add to the contents of these notes. I have updated the earlier chapters and have added some new chapters on the ergodic theory of continuous transformations of compact metric spaces. In particular, I have included some material on topological pressure and equilibrium states. In recent years there have been some fascinating interactions of ergodic theory with differentiable dynamics, differential geometry, number theory, von Neumann algebras, probability theory, statistical mechanics, and other topics. In Chapter 10 I have briefly described some of these and given references to some of the others. I hope that this book will give the reader enough foundation to tackle the research papers on ergodic theory and its applications.

I would like to dedicate this volume to the memory of Rufus Bowen who died on July 30, 1978 at the age of 31. He made outstanding contributions to ergodic theory and his friendship enhanced the lives of all who knew him.

April, 1981

PETER WALTERS

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so that we get an action of  $R$  on each energy surface. One is interested in the asymptotic properties of the action i.e. in  $T_t$  for large  $t$ . The transformations  $T_t|H^{-1}(e)$  are continuous and are smooth if  $H^{-1}(e)$  is smooth. Measure theory enters this picture via a theorem of Liouville which tells us that if the forces are of a certain type one can choose coordinates in the state space so that the usual  $6k$ -dimensional measure in these coordinates is preserved by each transformation  $T_t$ .

The word “ergodic” was introduced by Boltzmann to describe a hypothesis about the action of  $\{T_t|t \in R\}$  on an energy surface  $H^{-1}(e)$  when the Hamiltonian  $H$  is of the type that arises in statistical mechanics. Boltzmann had hoped that each orbit  $\{T_t(x)|t \in R\}$  would equal the whole energy surface  $H^{-1}(e)$  and he called this statement the ergodic hypothesis. The word “ergodic” is an amalgamation of the Greek words *ergon* (work) and *odos* (path). Boltzmann made the hypothesis in order to deduce the equality of time means and phase means which is a fundamental algorithm in statistical mechanics. The ergodic hypothesis, as stated above, is false. The property the flow needs to satisfy in order to equate time means and phase means of real-valued functions is what is now called ergodicity.

It is common to use the name ergodic theory to describe only the qualitative study of actions of groups on measure spaces. The actions on topological spaces and smooth manifolds are often called topological dynamics and differentiable dynamics. This measure theoretic study began in the early 1930's and the ergodic theorems of Birkhoff and von Neumann were proved then. The next major advance was the introduction of entropy by Kolmogorov in 1958. The proof, by Ornstein in 1969, that entropy was complete for Bernoulli shifts revitalised the work on the isomorphism problem. During recent years ergodic theory had been used to give important results in other branches of mathematics.

We shall study actions of the group  $Z$  of integers on a space  $X$  i.e. we study a transformation  $T: X \rightarrow X$  and its iterates  $T^n$ ,  $n \in Z$ . This is simpler than studying the actions of  $R$ . Of course, if  $\{T_t|t \in R\}$  is an action of  $R$  on  $X$ , then by choosing  $t_0 \neq 0$  and observing the system at the times  $\dots, -t_0, 0, t_0, 2t_0, 3t_0, \dots$ , we are considering  $(T_{t_0})^n$ ,  $n \in Z$ .

In the following sections we summarise some of the background ideas and notation we shall be using.

We shall use  $Z$  to denote the set of integers,  $Z^+$  to denote the non-negative integers,  $R$  to denote the real numbers,  $R^+$  to denote the non-negative reals, and  $C$  to denote the complex numbers. The empty set will be denoted by  $\emptyset$ .

If  $A, B$  are subsets of a set  $X$ , then  $B \setminus A$  denotes the difference set  $\{x \in X | x \in B, x \notin A\}$ , and  $A \triangle B$  denotes the symmetric difference  $(A \setminus B) \cup (B \setminus A)$ . We use  $2^X$  to denote the collection of all subsets of  $X$ .

We use “iff” to denote “if and only if.” We number lemmas and theorems in a single sequence (Theorem 5.6 is the sixth theorem in Chapter 5) but give a corollary the same number as the corresponding theorem (Corollary 5.6.2 is the second corollary of Theorem 5.6).

## §0.2 Measure Spaces

We shall generally refer to Kingman and Taylor [1] and Parthasarathy [2]. Let  $X$  be a set. A  $\sigma$ -algebra of subsets of  $X$  is a collection  $\mathcal{B}$  of subsets of  $X$  satisfying the following three conditions: (i)  $X \in \mathcal{B}$ ; (ii) if  $B \in \mathcal{B}$  then  $X \setminus B \in \mathcal{B}$ ; (iii) if  $B_n \in \mathcal{B}$  for  $n \geq 1$  then  $\bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$ .

We then call the pair  $(X, \mathcal{B})$  a *measurable space*. A *finite measure* on  $(X, \mathcal{B})$  is a function  $m: \mathcal{B} \rightarrow \mathbb{R}^+$  satisfying  $m(\emptyset) = 0$  and  $m(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} m(B_n)$  whenever  $\{B_n\}_1^{\infty}$  is a sequence of members of  $\mathcal{B}$  which are pairwise disjoint subsets of  $X$ . (Actually the latter condition implies  $m(\emptyset) = 0$  since  $m$  is finite-valued.) A *finite measure space* is a triple  $(X, \mathcal{B}, m)$  where  $(X, \mathcal{B})$  is a measurable space and  $m$  is a finite measure on  $(X, \mathcal{B})$ . We say  $(X, \mathcal{B}, m)$  is a *probability space*, or a *normalised measure space*, if  $m(X) = 1$ . We then say  $m$  is a *probability measure* on  $(X, \mathcal{B})$ . We shall usually consider only probability spaces.

A *finite signed measure* on a measurable space  $(X, \mathcal{B})$  is a function  $m: \mathcal{B} \rightarrow \mathbb{R}$  satisfying  $m(\emptyset) = 0$  and  $m(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} m(B_n)$  whenever  $\{B_n\}_1^{\infty}$  is a sequence of members of  $\mathcal{B}$  which are pairwise disjoint subsets of  $X$ . The *Jordan decomposition* says that a finite signed measure  $m$  on  $(X, \mathcal{B})$  can be written as the difference  $m = m_1 - m_2$  of two finite measures on  $(X, \mathcal{B})$  which are uniquely determined by  $m$  (see Kingman and Taylor [1], pp. 62 and 64).

Measurable spaces are usually constructed by having a collection  $\mathcal{S}$  of interesting subsets of a set  $X$  (such as the collection of all subintervals of  $[0, 1]$ ) and then considering the smallest  $\sigma$ -algebra  $\mathcal{B}$  containing all these subsets. This makes sense because  $2^X$  is a  $\sigma$ -algebra and any intersection of  $\sigma$ -algebras of subsets of  $X$  is also a  $\sigma$ -algebra of subsets of  $X$ . It is then usually difficult to decide which subsets of  $X$  are in  $\mathcal{B}$ . When constructing a measure on a measurable space  $(X, \mathcal{B})$  obtained in this way, one usually knows what values the measure should take on members of  $\mathcal{S}$  and then one needs to extend it to be defined on  $\mathcal{B}$ . We now describe the basic extension theorem of this type. This involves discussion of the properties the collection  $\mathcal{S}$  should have.

A collection  $\mathcal{S}$  of subsets of  $X$  is called a *semi-algebra* if the following three conditions hold: (i)  $\emptyset \in \mathcal{S}$ ; (ii) if  $A, B \in \mathcal{S}$ , then  $A \cap B \in \mathcal{S}$ ; (iii) if  $A \in \mathcal{S}$ , then  $X \setminus A = \bigcup_{i=1}^n E_i$  where each  $E_i \in \mathcal{S}$  and  $E_1, \dots, E_n$  are pairwise disjoint subsets of  $X$ . For example, the collection of all subintervals of  $[0, 1]$  is a semi-algebra. Also, the collection of all subintervals of  $[0, 1]$  of the forms  $[0, b]$  and  $(a, b]$ , with  $0 \leq a < b \leq 1$ , forms a semi-algebra.

A collection  $\mathcal{A}$  of subsets of  $X$  is called an *algebra* if the following three conditions hold: (i)  $\emptyset \in \mathcal{A}$ ; (ii) if  $A, B \in \mathcal{A}$ , then  $A \cap B \in \mathcal{A}$ ; (iii) if  $A \in \mathcal{A}$ , then  $X \setminus A \in \mathcal{A}$ . Clearly every algebra is a semi-algebra and every  $\sigma$ -algebra is an algebra. In the definition of an algebra we can replace (ii) by the condition that whenever  $A_1, \dots, A_n \in \mathcal{A}$  then  $\bigcup_{i=1}^n A_i \in \mathcal{A}$ .

Since the intersection of any family of algebras of a set  $X$  is again an algebra of subsets of  $X$  it makes sense to speak of the algebra generated by

any given collection of subsets of  $X$ . There is the following simple theorem (Parthasarathy [2], p. 19)

**Theorem 0.1.** *Let  $\mathcal{S}$  be a semi-algebra of subsets of  $X$ . The algebra,  $\mathcal{A}(\mathcal{S})$ , generated by  $\mathcal{S}$  consists precisely of those subsets of  $X$  that can be written in the form  $E = \bigcup_{i=1}^n A_i$  where each  $A_i \in \mathcal{S}$  and  $A_1, \dots, A_n$  are disjoint subsets of  $X$ .*

Suppose  $\mathcal{S}$  is a semi-algebra of subsets of  $X$ . A function  $\tau: \mathcal{S} \rightarrow R^+$  is called *finitely additive* if  $\tau(\emptyset) = 0$  and  $\tau(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n \tau(E_i)$  whenever  $E_1, \dots, E_n$  are members of  $\mathcal{S}$  which are pairwise disjoint subsets of  $X$  and  $\bigcup_{i=1}^n E_i \in \mathcal{S}$ . Such a map  $\tau$  is called *countably additive* if the second condition is replaced by the requirement that  $\tau(\bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty \tau(E_i)$  whenever  $\{E_i\}_1^\infty$  are members of  $\mathcal{S}$  which are pairwise disjoint subsets of  $X$  and  $\bigcup_{i=1}^\infty E_i \in \mathcal{S}$ . If  $\mathcal{S}$  is the semi-algebra of all subintervals of  $[0, 1]$  of the form  $[0, b]$  and  $(a, b]$  then the length function is countably additive. A simple application of Theorem 0.1 gives the following (Parthasarathy [2], pp. 20 and 59).

**Theorem 0.2.** *If  $\mathcal{S}$  is a semi-algebra of subsets of  $X$  and  $\tau: \mathcal{S} \rightarrow R^+$  is finitely additive then there is a unique finitely additive function  $\tau_1: \mathcal{A}(\mathcal{S}) \rightarrow R^+$  which is an extension of  $\tau$  (i.e.  $\tau_1 = \tau$  on  $\mathcal{S}$ ). If  $\tau$  is countably additive then so is  $\tau_1$ .*

It could be that  $X \notin \mathcal{S}$  but we always have that  $X$  is a disjoint union of a finite number of members  $E_1, \dots, E_n$  of  $\mathcal{S}$  so  $\tau_1(X) = 1$  if  $\sum_{i=1}^n \tau(E_i) = 1$ .

There is the following theorem on extension from an algebra  $\mathcal{A}$  to the  $\sigma$ -algebra  $\mathcal{B}(\mathcal{A})$  generated by  $\mathcal{A}$ . ( $\mathcal{B}(\mathcal{A})$  is the intersection of all  $\sigma$ -algebras that contain  $\mathcal{A}$ .) (See Parthasarathy [2], pp. 70 and 71).

**Theorem 0.3.** *Let  $\mathcal{A}$  be an algebra of subsets of  $X$  and let  $\tau_1: \mathcal{A} \rightarrow R^+$  be countably additive and  $\tau_1(X) = 1$ . Then there is a unique probability measure  $\tau_2$  on  $(X, \mathcal{B}(\mathcal{A}))$  which extends  $\tau_1$ .*

By combining Theorems 0.2 and 0.3 we see that a countably additive function  $\tau$  on a semi-algebra  $\mathcal{S}$  can be uniquely extended to a probability measure on  $(X, \mathcal{B}(\mathcal{S}))$  if  $\sum_{i=1}^n \tau(E_i) = 1$  when  $X = \bigcup_{i=1}^n E_i$  is a disjoint union of members of  $\mathcal{S}$ . As an example the length function defined on the semi-algebra of all subintervals of  $[0, 1]$  of the form  $[0, b]$  and  $(a, b]$  can be uniquely extended to a probability measure, called the Lebesgue measure, defined on the Borel subsets of  $[0, 1]$ .

In checking that the extension works for particular examples the most difficult part is usually showing countable additivity. This can sometimes be done for  $\tau$  on the semi-algebra  $\mathcal{S}$  (as for Lebesgue measure) but it is sometimes more convenient to prove that  $\tau$  is finitely additive and that the finitely additive extension  $\tau_1: \mathcal{A}(\mathcal{S}) \rightarrow R^+$  is countably additive. The main tool for this is the following theorem (Kingman and Taylor [1], p. 56).

**Theorem 0.4.** Let  $\mathcal{A}$  be an algebra of subsets of  $X$  and let  $\tau_1: \mathcal{A} \rightarrow R^+$  be finitely additive and let  $\tau_1(X) = 1$ . Then  $\tau_1$  will be countably additive if for every decreasing sequence  $E_1 \supset E_2 \supset E_3 \supset \cdots$  of members of  $\mathcal{A}$  with  $\bigcap_{n=1}^{\infty} E_n = \emptyset$  we have  $\tau_1(E_n) \rightarrow 0$ .

One situation where this theorem is used is in defining product measures on a countable product of probability spaces. For  $i \in Z$  let  $(X_i, \mathcal{B}_i, m_i)$  be a probability space. Let  $X = \prod_{i=-\infty}^{\infty} X_i$ . So a point of  $X$  is a bisequence  $\{x_i\}_{i=-\infty}^{\infty}$  with  $x_i \in X_i$  for each  $i$ . We now define a  $\sigma$ -algebra  $\mathcal{B}$  of subsets of  $X$  called the product of the  $\sigma$ -algebras  $\mathcal{B}_i$ . Let  $n \geq 0$ , let  $A_j \in \mathcal{B}_j$  for  $|j| \leq n$ , and consider the set

$$\prod_{i=-\infty}^{-(n+1)} X_i \times \prod_{j=-n}^n A_j \times \prod_{i=n+1}^{\infty} X_i = \{(x_i)_{i=-\infty}^{\infty} \in X \mid x_j \in A_j \text{ for } |j| \leq n\}.$$

Such a set is called a *measurable rectangle* and the collection of all such subsets of  $X$  forms a semi-algebra  $\mathcal{S}$ . The  $\sigma$ -algebra  $\mathcal{B}$  is the  $\sigma$ -algebra generated by  $\mathcal{S}$ . We write  $(X, \mathcal{B}) = \prod_{i=-\infty}^{\infty} (X_i, \mathcal{B}_i)$ . If we define  $\tau: \mathcal{S} \rightarrow R^+$  by giving the above rectangle the value  $\prod_{j=-n}^n m_j(A_j)$ , then one can use Theorems 0.2 and 0.4 (see Kingman and Taylor [1], p. 140) to extend  $\tau$  to a probability measure  $m$  on  $(X, \mathcal{B})$ . The probability space  $(X, \mathcal{B}, m)$  is called the *direct product* of the spaces  $(X_i, \mathcal{B}_i, m_i)$  and is sometimes denoted  $\prod_{i=-\infty}^{\infty} (X_i, \mathcal{B}_i, m_i)$ . The corresponding construction holds for a product  $\prod_{i=0}^{\infty} (X_i, \mathcal{B}_i, m_i)$ .

A special type of product space will be important for us. Here each space  $(X_i, \mathcal{B}_i, m_i)$  is the same space  $(Y, \mathcal{C}, \mu)$  and  $Y$  is the finite set  $\{0, 1, \dots, k-1\}$ ,  $\mathcal{C} = 2^Y$ , and  $\mu$  is given by a probability vector  $(p_0, p_1, \dots, p_{k-1})$  where  $p_i = \mu(\{i\})$ . We can take elementary rectangles where each  $A_j$  (in the description above) is taken to be one point of  $Y$ . So if  $n \geq 0$  and  $a_j \in Y$ ,  $|j| \leq n$ , such an elementary rectangle has the form  $\{(x_i)_{i=-\infty}^{\infty} \mid x_j = a_j \text{ for } |j| \leq n\}$ . We shall denote this set by  ${}_{-n}[a_{-n}, a_{-(n+1)}, \dots, a_{n-1}, a_n]_n$  and call it a *block* with end points  $-n$  and  $n$ . The collection of all these sets form a semi-algebra which generates the product  $\sigma$ -algebra  $\mathcal{B}$ . We have  $m({}_{-n}[a_n, \dots, a_n]_n) = \prod_{j=-n}^n p_{a_j}$ . The measure  $m$  is called the  $(p_0, \dots, p_{k-1})$ -product measure. Sometimes we consider *blocks* with end points  $h$  and  $l$  where  $h \leq l$ . Such a set is one of the form  ${}_h[a_h, \dots, a_l]_l = \{(x_i)_{i=-\infty}^{\infty} \mid x_i = a_i \text{ for } h \leq i \leq l\}$ . It has measure  $\prod_{i=h}^l p_{a_i}$ .

Theorem 0.4 can also be used to obtain further measures on the space  $(X, \mathcal{B})$  where  $X = \prod_{i=-\infty}^{\infty} Y$ ,  $Y = \{0, 1, \dots, k-1\}$ , and  $\mathcal{B}$  is the product  $\sigma$ -algebra described above. The following is a special case of the Daniell-Kolmogorov consistency theorem (Parthasarathy [2], p. 119).

**Theorem 0.5.** Fix  $k \geq 1$  and let  $Y = \{0, 1, \dots, k-1\}$  and  $(X, \mathcal{B}) = \prod_{i=-\infty}^{\infty} (Y, 2^Y)$ . For each natural number  $n$  and  $a_0, \dots, a_n \in Y$  suppose a non-negative real number  $p_n(a_0, \dots, a_n)$  is given so that

$$(a) \quad \sum_{a_0 \in Y} p_0(a_0) = 1$$

and

$$(b) \quad p_n(a_0, \dots, a_n) = \sum_{a_{n+1} \in Y} p_{n+1}(a_0, \dots, a_n, a_{n+1}).$$

Then there is a unique probability measure  $m$  on  $(X, \mathcal{B})$  with  $m([a_h, \dots, a_l]_l) = p_{l-h}(a_h, \dots, a_l)$  for all  $h \leq l$  and all  $a_i \in Y$ ,  $h \leq i \leq l$ .

The proof boils down to showing that the function naturally defined on the algebra of all finite unions of elementary rectangles is countable additive by using Theorem 0.4.

There is another way, which is useful in some proofs, of describing the  $\sigma$ -algebra  $\mathcal{B}(\mathcal{A})$  generated by an algebra  $\mathcal{A}$ . A collection  $M$  of subsets of  $X$  is called a *monotone class* if whenever  $E_1 \subset E_2 \subset E_3 \subset \dots$  all belong to  $M$  then so does  $\bigcup_{n=1}^{\infty} E_n$  and whenever  $F_1 \supset F_2 \supset F_3 \supset \dots$  all belong to  $M$  then so does  $\bigcap_{n=1}^{\infty} F_n$ . Since the intersection of any family of monotone classes is a monotone class, we can speak of the monotone class generated by any given collection of subsets of  $X$ .

**Theorem 0.6.** *Let  $\mathcal{A}$  be an algebra of subsets of  $X$ . Then  $\mathcal{B}(\mathcal{A})$  equals the monotone class generated by  $\mathcal{A}$ .*

As we have seen we usually know the elements of an algebra  $\mathcal{A}$  but we do not know which subsets of  $X$  belong to  $\mathcal{B}(\mathcal{A})$ . This problem can sometimes be overcome by using the following approximation theorem (Kingman and Taylor [1], p. 84).

**Theorem 0.7.** *Let  $(X, \mathcal{B}, m)$  be a probability space and let  $\mathcal{A}$  be an algebra of subsets of  $X$  with  $\mathcal{B}(\mathcal{A}) = \mathcal{B}$ . Then for each  $\varepsilon > 0$  and each  $B \in \mathcal{B}$  there is some  $A \in \mathcal{A}$  with  $m(A \triangle B) < \varepsilon$ .*

Note that when  $m(A \triangle B) < \varepsilon$  then  $|m(A) - m(B)| < \varepsilon$  because  $m(A) = m(A \setminus B) + m(A \cap B)$  and  $m(B) = m(B \setminus A) + m(A \cap B)$ , so that  $|m(A) - m(B)| \leq m(A \triangle B)$ .

## §0.3 Integration

Let  $\mathcal{B}(R)$  denote the  $\sigma$ -algebra of Borel subsets of  $R$ . This is the  $\sigma$ -algebra generated by all open subsets of  $R$  and is also generated by the collection of all intervals, or by the collection of all intervals of the form  $(c, \infty)$ .

Let  $(X, \mathcal{B}, m)$  be a measure space. A function  $f: X \rightarrow R$  is *measurable* if  $f^{-1}(D) \in \mathcal{B}$  whenever  $D \in \mathcal{B}(R)$  or equivalently if  $f^{-1}(c, \infty) \in \mathcal{B}$  for all  $c \in R$ . A function  $f: X \rightarrow \mathbb{C}$  is measurable if both its real and imaginary parts are



measurable. If  $X$  is a topological space and  $\mathcal{B}$  the  $\sigma$ -algebra generated by the open subsets of  $X$ , then any continuous function  $f: X \rightarrow \mathbb{C}$  is measurable. We say  $f = g$  a.e. if  $m(\{x: f(x) \neq g(x)\}) = 0$ . Suppose  $X$  is a topological space,  $\mathcal{B}(X)$  its  $\sigma$ -algebra of Borel sets and  $m$  a measure on  $(X, \mathcal{B}(X))$  with the property that each non-empty open set has non-zero measure. Then for two continuous functions  $f, g: X \rightarrow \mathbb{R}$ ,  $f = g$  a.e. implies  $f = g$  because  $\{x: f(x) - g(x) \neq 0\}$  is an open set of zero measure.

Let  $(X, \mathcal{B}, m)$  be a probability space. A function  $f: X \rightarrow \mathbb{R}$  is a *simple function* if it can be written in the form  $\sum_{i=1}^n a_i \chi_{A_i}$ , where  $a_i \in \mathbb{R}$ ,  $A_i \in \mathcal{B}$ , the sets  $A_i$  are disjoint subsets of  $X$ , and  $\chi_{A_i}$  denotes the characteristic function of  $A_i$ . Simple functions are measurable. We define the integral for simple functions by:

$$\int f \, dm = \sum_{i=1}^n a_i m(A_i).$$

This value is independent of the representation  $\sum_i a_i \chi_{A_i}$ .

Suppose  $f: X \rightarrow \mathbb{R}$  is measurable and  $f \geq 0$ . Then there exists an increasing sequence of simple functions  $f_n \nearrow f$ . For example, we could take

$$f_n(x) = \begin{cases} \frac{i-1}{2^n}, & \text{if } \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \quad i = 1, \dots, n2^n \\ n, & \text{if } f(x) \geq n. \end{cases}$$

We define  $\int f \, dm = \lim_{n \rightarrow \infty} \int f_n \, dm$  and note that this definition is independent of the chosen sequence  $\{f_n\}$ . We say  $f$  is *integrable* if  $\int f \, dm < \infty$ .

Suppose  $f: X \rightarrow \mathbb{R}$  is measurable. Then  $f = f^+ - f^-$  where  $f^+(x) = \max\{f(x), 0\} \geq 0$  and  $f^-(x) = \max\{-f(x), 0\} \geq 0$ . We say that  $f$  is *integrable* if  $\int f^+ \, dm, \int f^- \, dm < \infty$  and we then define

$$\int f \, dm = \int f^+ \, dm - \int f^- \, dm.$$

We say  $f: X \rightarrow \mathbb{C}$  is *integrable* ( $f = f_1 + if_2$ ) if  $f_1$  and  $f_2$  are integrable and we define

$$\int f \, dm = \int f_1 \, dm + i \int f_2 \, dm.$$

Observe that  $f$  is integrable if and only if  $|f|$  is integrable. If  $f = g$  a.e. then one is integrable if the other is and  $\int f \, dm = \int g \, dm$ .

The two basic theorems on integrating sequences as functions are the following.

**Theorem 0.8** (Monotone Convergence Theorem). Suppose  $f_1 \leq f_2 \leq f_3 \leq \dots$  is an increasing sequence of integrable real-valued functions on  $(X, \mathcal{B}, m)$ . If  $\{\int f_n \, dm\}$  is a bounded sequence of real numbers then  $\lim_{n \rightarrow \infty} f_n$  exists a.e. and is integrable and  $\int (\lim f_n) \, dm = \lim \int f_n \, dm$ . If  $\{\int f_n \, dm\}$  is an unbounded sequence then either  $\lim_{n \rightarrow \infty} f_n$  is infinite on a set of positive measure or  $\lim_{n \rightarrow \infty} f_n$  is not integrable.



**Theorem 0.9** (Fatou's Lemma). Let  $\{f_n\}$  be a sequence of measurable real-valued functions on  $(X, \mathcal{B}, m)$  which is bounded below by an integrable function. If  $\liminf_{n \rightarrow \infty} \int f_n dm < \infty$  then  $\liminf_{n \rightarrow \infty} f_n$  is integrable and  $\int \liminf_{n \rightarrow \infty} f_n dm \leq \liminf_{n \rightarrow \infty} \int f_n dm$ .

**Corollary 0.9.1** (Dominated Convergence Theorem). If  $g: X \rightarrow \mathbb{R}$  is integrable and  $\{f_n\}$  is a sequence of measurable real-valued functions with  $|f_n| \leq g$  a.e. ( $n \geq 1$ ) and  $\lim_{n \rightarrow \infty} f_n = f$  a.e. then  $f$  is integrable and  $\lim \int f_n dm = \int f dm$ .

We denote by  $L^1(X, \mathcal{B}, m)$  (or  $L^1(m)$ ) the space of all integrable functions  $f: X \rightarrow \mathbb{C}$  where two such functions are identified if they are equal a.e. However we write  $f \in L^1(X, \mathcal{B}, m)$  to denote that  $f: X \rightarrow \mathbb{C}$  is integrable. The space  $L^1(X, \mathcal{B}, m)$  is a Banach space with norm  $\|f\|_1 = \int |f| dm$ .

If  $f \in L^1(X, \mathcal{B}, m)$ , then  $\int_A f dm$  denotes  $\int f \cdot \chi_A dm$ .

If  $m$  is a finite signed measure on  $(X, \mathcal{B})$  and  $m = m_1 - m_2$  is its unique Jordan decomposition into the difference of two finite measures, then we can define  $\int f dm = \int f dm_1 - \int f dm_2$  for  $f \in L^1(m_1) \cap L^1(m_2)$ .

## §0.4 Absolutely Continuous Measures and Conditional Expectations

Let  $(X, \mathcal{B})$  be a measurable space and suppose  $\mu, m$  are two probability measures on  $(X, \mathcal{B})$ . We say  $\mu$  is *absolutely continuous* with respect to  $m$  ( $\mu \ll m$ ) if  $\mu(B) = 0$  whenever  $m(B) = 0$ . The measures are *equivalent* if  $\mu \ll m$  and  $m \ll \mu$ . The following theorem characterises absolute continuity.

**Theorem 0.10** (Radon–Nikodym Theorem). Let  $\mu, m$  be two probability measures on the measurable space  $(X, \mathcal{B})$ . Then  $\mu \ll m$  iff there exists  $f \in L^1(m)$ , with  $f \geq 0$  and  $\int f dm = 1$ , such that  $\mu(B) = \int_B f dm \forall B \in \mathcal{B}$ . The function  $f$  is unique a.e. (in the sense that any other function with these properties is equal to  $f$  a.e.).

The function  $f$  is called the Radon–Nikodym derivative of  $\mu$  with respect to  $m$  and denoted by  $d\mu/dm$ .

The “opposite” notion to absolute continuity is as follows. Two probability measures  $\mu, m$  on  $(X, \mathcal{B})$  are said to be *mutually singular* ( $\mu \perp m$ ) if there is some  $B \in \mathcal{B}$  with  $\mu(B) = 0$  and  $m(X \setminus B) = 0$ . There is the following decomposition theorem.

**Theorem 0.11** (Lebesgue Decomposition Theorem). Let  $\mu, m$  be two probability measures on  $(X, \mathcal{B})$ . There exists  $p \in [0, 1]$  and probability measures  $\mu_1, \mu_2$  on  $(X, \mathcal{B})$  such that  $\mu = p\mu_1 + (1-p)\mu_2$  and  $\mu_1 \ll m, \mu_2 \perp m$ . ( $\mu = p\mu_1 + (1-p)\mu_2$  means  $\mu(B) = p\mu_1(B) + (1-p)\mu_2(B) \forall B \in \mathcal{B}$ ). The number  $p$  and probabilities  $\mu_1, \mu_2$  are uniquely determined.