Variational Approach and Its Applications

—Differential，Impulsive and Difference Equations

田 玉 郭玉翠 著

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内 容 简 介

微分方程边值问题具有悠久的研究历史，是微分方程理论的一个重要分支。目前，国内外主要研究各类边值问题的解的存在性和唯一性、多解存在性、正解存在性，为具体问题的求解提供理论基础，这提供了研究微分方程边值问题的必要性。研究微分方程边值问题的解的存在性理论的传统方法有：拓扑度理论、上下解方法。变分法越来越多地应用在解存在性研究中，逐渐成为研究非线性微分方程的主要工具，此方法能得到不同于其他传统方法的结果。

本书介绍变分法的主要结论和最新进展，以及如何应用变分法到微分方程、脉冲微分方程、差分方程定解问题中，得到解的存在性、多解性、变号解和正解存在性。为了进一步研究解的性态，介绍了如何将变分法与上下解方法结合得到变号解存在性。这些研究丰富了解的存在性理论，扩展了变分法的应用范围。

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Foreword

As a young scholar engaged in the research of differential equations, Professor Yu Tian has been teaching and studying at Beijing University of Posts and Telecommunications for a long period. Her research interest rests on different fields of differential equations, including ordinary differential equations, difference equations and impulsive differential equations. By the creative establishing of variational constructions , she obtained a series of valuable achievements. Her fruitful results attracted the mention of both home and abroad experts. Several years ago I attended an international conference on differential equations and dynamic systems in Atlanta, USA. At that meeting a famous America mathematician quoted five papers of Prof. Tian and gave a high comment, which impressed me very deeply.

Recently Professor Tian and her cooperator, Professor Yucui Guo, completed a monograph, titled Variaional Approach and Its Applications in Differential, Impulsive and Difference Equations, in which the authors summarized their research achievements in the past 15 years. I believe that the publication of their book is of much benefit to young researchers in those fields mentioned above.

Weigao Ge
Professor in Mathematics
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September 2018

Preface

Variational approach is a new content of nonlinear analysis which was born and has matured from abundant research developed in studying nonlinear problems. In the past forty years, variational approach has undergone rapid growth and become part of the mainstream research fields in nonlinear analysis.

Many differential equations, impulsive differential equations and difference equations are motivated by problems arising from models of chemotherapy, population dynamics, optimal control, ecology, physical phenomena. It is necessary and important to discuss the existence of solutions once the mathematical models are established. Many books on the topic have been published. This book gives a systematic study to the existence of solutions from the view point of variational approach. In light of the content and the type of equations this book is divided into five chapters.

Chapter 1 gives a survey to the background and research development of differential, impulsive differential and difference equations.

Chapter 2 summarizes the variational approaches, including the basic definitions and properties from such topics of mathematical analysis as functional spaces, Sobolev space and some critical point theories and nonsmooth critical point theories.

Chapter 3 systematically studies ordinary differential equation systems by using variational approach. Section 3.1 is devoted to the periodic solutions of Hamiltonian system with a p-Laplacian. Section 3.2 and Section 3.3 study separately the second-order differential system with resonance and non-resonance. The existence results of anti-periodic solutions are established. The study of higher order differential equations, especially $2n$th-order differential equations is in Section 3.4. The variational structure is established and the existence results are obtained.

Chapter 4 systematically studies the impulsive differential equations by using variational approach. The variational structures of impulsive differential equations with mixed boundary conditions, Sturm-Liouville boundary conditions are established. The variational structure make it possible to research solutions under the framework of variational approach. Section 4.1 is devoted to the impulsive differential equations with mixed boundary condition. In Section 4.2-4.5, the linear and nonlinear impulsive differential equations with Sturm-Liouville boundary condition are studied. The existence results of positive solutions, sign-changing solutions and multiple solutions are established by using variational approaches combined with lower and upper solution method, invariant sets of descending flow. In Section 4.6, we consider the variational structure and the existence results of one solution, three solutions and infinitely many solutions for fourth-order impulsive differential equations. In Section 4.7, second order impulsive differential inclusion with Sturm-Liouville boundary condition is investigated by using non-smooth critical point theorem.

In Chapter 5, we investigate the applications of variational approach with a view to establishing the solutions to the difference equations. The variational structures for difference equations with Sturm-Liouville boundary condition, anti-periodic boundary condition are established in Section 5.1 and Section 5.2, which also develops the existence conditions for solutions. For the generalization, two solutions, four solutions and $2n$ solutions for $2n$th-order difference equations with anti-periodic boundary conditions are established in Section 5.3.

We wish to express our thanks to Prof. Meiqiang Feng and Prof. Xuemei Zhang for their valuable remarks and suggestions on the original manuscript. The first author wishes to express her most sincere and heartfelt thanks to Prof. Weigao Ge for his never-ending support, encouragement, guidance, wisdom, experience, and insight on her research. We would like to thank graduate students Yue Yue, Min Zhang and Mengxiang You for their help in typing the book. We would like to thank Beijing University of Posts and Telecommunications Press for the help in publishing the book.

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August 2018

Contents

CHAPTER 1

Nonlinear Differential Equations and Difference Equations

1.1 Differential equations

Differential equations first appeared with the invention of calculus by Newton and Leibniz in the 17th-century. Many mathematicians have studied differential equations and contributed to the field, including Newton, Leibniz, the Bernoulli family, Riccati, Clairaut, D'Alembert, and Euler. Differential equations play a prominent role in many disciplines including physics and astronomy (celestial mechanics), chemistry (reaction rates), biology (infectious diseases, genetic variation), ecology and population modelling (population competition), economics (stock trends, interest rates and the market equilibrium price changes) [44][51][79][85].

The existence theory is very important in the qualitative theory of differential equations. It has been developed extensively in the last 40 years, please see [1] [7] [9] [21]. Many results are obtained by using the tools such as topological degree theory [1][2][16], maximal monotone mappings [4], lower and upper solutions methods [6][7][8].

In the study of

$$\begin{cases} \ddot{u}(t) + \nabla F(t, u(t)) = 0 \quad \text{a.e. on } [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1.1}$$

J. Mawhin and M. Willem imposed a quadratic growth restriction on $F(t, \cdot)$, i.e.,

$$F(t, q) \leqslant \frac{\alpha^2}{2}|q|^2 + \gamma(t), \quad \text{for a.e. } t \in [0, T], \tag{1.1.2}$$

where $\alpha \in [0, 2\pi/T]$ and $\gamma \in L^2(0, T; \mathbb{R}^+), q \in \mathbb{R}^N$, and a coercivity condition on the average Hamiltonian

$$\frac{1}{T}\int_0^T F(t, q)\mathrm{d}t \to +\infty \quad \text{as } |q| \to +\infty, \quad q \in \mathbb{R}^N. \tag{1.1.3}$$

Then they obtained an existence result (Theorem 3.5 in [12]).

Recently there are extensive literatures related to the existence of periodic solutions of second order systems by use of variational method. To identify a few, we refer the reader to [86]-[89], [101] and the references therein.

The study of anti-periodic solutions for nonlinear evolution equations was initiated by Okochi[14]. Okochi studied the nonlinear parabolic equation in a real Hilbert space H, which is of the form

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t}+\partial\varphi(u(t))\ni f(t),$$

where $f\in L^2_{1\mathrm{oc}}(R;H)$, φ is a proper l.s.c. (lower semi-continuous) convex functional on H and $\partial\varphi$ is the subdifferential of φ. By using fixed point theory, the existence of anti-periodic solutions was obtained in the case $\partial\varphi$ is odd and f is T-anti-periodic. Inspired by [14], anti-periodic problems for second-order and higher-order differential equations have been extensively studied, see [6]-[11], [15] and the references therein.

In [4], S. Aizicovici, N.H. Pavel established the existence, uniqueness and continuous dependence upon data of anti-periodic solutions to some first- and second-order evolution equations. In [30], Wang and Li studied the existence of solutions of the following antiperiodic boundary value problem for second-order conservative system:

$$\begin{cases} q''=u(t,q), & t\in[0,T],\\ q(0)=-q(T), & q'(0)=-q'(T). \end{cases}\tag{1.1.4}$$

By using fixed point theory together with the Green's function, the existence result is as follows. Assume that there exist constants $0\leqslant c<8$ and $M>0$, such that

$$|u(t,q)|\leqslant\frac{c}{T^2}|q|+M,$$

for all $t\in[0,T], q\in\mathbf{R}^1$. The problem (1.1.5) has at least one solution.

In [97], Wang and Shen studied antiperiodic boundary value problem as follows

$$\begin{cases} x''+f(t,x(t))=0, & t\in[0,T],\\ x(0)=-x(T), & x'(0)=-x'(T). \end{cases}\tag{1.1.5}$$

By using Schauder's fixed point theorem and the lower and upper solutions method, some sufficient conditions for the existence of solutions are obtained.

Assume that there exist constants $0<r<2, l>0$, and functions $p,q,h\in C[0,T]$ such that

$$uf(t,u)\leqslant p(t)u^2+q(t)|u|^r+h(t),$$

for $t\in[0,T], |u|>1$. Further suppose that $\int_0^T p^+(s)\mathrm{d}s<4$, where $p^+(t)=\max\{p(t),0\}$. Then (1.1.5) has at least one solution.

Now we present questions.

Q1. How to establish the variational structure for periodic Hamiltonian system with a p-Laplacian and establish the existence results?

Q2. Is it OK to apply the dual least action principle into resonance anti-periodic boundary value problem since it has been successfully applied to periodic Hamiltonian system?

Q3. How to establish the variational structure for anti-periodic boundary value problem?

1.2 Impulsive differential equations

During many evolution processes, evolution is subjected to a rapid change, that is, a jump in their states. This phenomenon is described by impulsive differential equations in mathematics. For the background, theory and applications of impulsive differential equations, we refer the readers to the monographs and some recent contributions as [15] [25] [48] [55] [57] [63] [66] [67] [71] [73] [82] [119] [120] [122] [124].

Motivated by the wide applications in evolution process, impulsive differential equations are studied extensively, we refer the readers to the monographs and some recent contributions as [48] [55] [66] [67] [71] [73] [82]. Main results are obtained by using the tools such as fixed point theorems in cones [5][37][54], the method of lower and upper solutions [33]. On the other hand, also critical point theory is a powerful tool to study differential and difference equations (see, for instance, [8] [59] [72] [74] [77] [93] [96]). However, besides [95] , there are only a few papers where impulsive differential equations are studied by means of critical point theory.

In [95], Tian and Ge studied the following equations with impulsive effects

$$\begin{cases} -(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) = f(t, x(t)), \quad t \neq t_i, \quad \text{a.e.} \quad t \in [a, b], \\ -\Delta(\rho(t_i)\Phi_p(x'(t_i))) = I_i(x(t_i)), \quad i = 1, 2, \cdots, l, \\ \alpha x'(a) - \beta x(a) = A, \quad \gamma x'(b) + \sigma x(b) = B. \end{cases} \tag{1.2.1}$$

They essentially proved that when f and I_i satisfy some conditions, (1.2.1) has at least two positive solutions via variational approach.

In [68], Nieto studied linear Dirichlet impulsive problem

$$\begin{cases} -u''(t) + \lambda u(t) = \sigma(t), \quad t \neq t_j, \quad t \in [0, T], \\ \Delta u'(t_j) = d_j, \quad i = 1, 2, \cdots, p, \\ u(0) = u(T) = 0 \end{cases}$$

and nonlinear Dirichlet impulsive problem

$$\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)), \quad t \neq t_j, t \in [0, T], \\ \Delta u'(t_j) = I_j(u(t_j^-)), \quad i = 1, 2, \cdots, p, \\ u(0) = u(T) = 0. \end{cases}$$

By using variational approach, the existence of solutions was obtained.

Invariant sets of descending flow defined by a pseudogradient vector field of a functional in a Banach space plays an important role in the existence of sign-changing solutions. The method was proposed in [80] and properties of invariant sets of descending flow and applications can be found in [39] [49] [51] [81] [85] [86] [105]. Mao and Zhang [62] studied the existence of solutions for a Kirchhoff type problem using minimax methods and invariant sets of descending flow. Zhang and Perera [118] obtained sign changing solutions for a class of nonlocal quasilinear elliptic boundary value problems using variational approach and invariant sets of descending flow.

Some questions are

Q1. How to establish the sign-changing solutions for impulsive differential equations?

Q2. Can the existence results for impulsive differential equations be established by using the combination of variational approach and lower and upper solutions?

In [104], the authors studied the existence of solutions of the following equations

$$\begin{cases} u^{(iv)}(t) + Au''(t) + Bu(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ -\Delta u''(t_j) = I_{1j}(u'(t_j)), \quad j = 1, 2, \cdots, l, \\ -\Delta u'''(t_j) = I_{2j}(u(t_j)), \quad j = 1, 2, \cdots, l, \\ u(0) = u(T) = u''(0^+) = u''(T^-) = 0. \end{cases} \tag{1.2.2}$$

They essentially proved that when f, I_{1j} and I_{2j} satisfy some conditions, (1.2.2) has at least one solution or infinitely many classical solutions via variational methods.

In [42], L. Kong investigated eigenvalues for a fourth-order elliptic problem by using Ekeland's variational principle and some recent theory on the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$.

L. Saavedra and S. Tersian [52] went a step further to study the existence and multiplicity of weak and classical solutions for higher order problem

$$\begin{cases} (\varphi_p(u^{(n)}))^{(n)} + \sum\limits_{i=1}^{n-1}(-1)^i(\varphi_p(u^{(n-i)}))^{(n-i)} + (-1)^n(f(t, u(t)) - h(t, u(t))) = 0, \\ u(T) = u(0) = \cdots = u^{(2n-1)}(T) = u^{(2n-1)}(0) = 0, \end{cases}$$

where $\varphi_p(t) = t|t|^{p-2}$, $t \in [0, T]$, $T \geqslant 0$ and $\alpha_i \geqslant 0$ for $i = 1, \cdots, n-1$, by using the minimization argument and an extended Clark's theorem.

The question is as follows.

Is it OK that the existence results for infinitely many solutions for higher-order impulsive differential equations are established when nonlinearity is oscillatory?

The study of impulsive differential equations and inclusions is linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are considered to take place in the form of impulses since the perturbations are performed discretely and their durations are negligible in comparison with the total duration of the processes and phenomena, see [30] [56]. In recent years, there has been an increasing interest in the study of differential inclusions and impulsive differential inclusions due to the fact that they often arise in models for control systems, mechanical systems, economical systems, game theory, and biological systems to name a few (see [2] [9] [12] [13] [14] [24] [44] [45] [47] [60] [84]). The theory of impulsive differential equations has been given extensive attention. We refer to the monographs of Bainov and Simeonov [10], Benchohra et al. [14], Lakshmikantham et al. [48], and Samoilenko and Perestyuk [83]. The first work dealing with partial differential inclusions with a general set-valued right-hand side via variational approach was, to the best of our knowledge, that of Frigon [34]. Ribarska et al. [76] defined a single-valued energy functional that was locally Lipschitz and proved that its critical points were just the solutions of the original problem. With this, nonsmooth variational approach can be applied to differential inclusions.

The study of anti-periodic solutions for nonlinear evolution equations was initiated by Okochi [49]. Okochi studied the nonlinear parabolic equation in a real Hilbert space. By using fixed point theory, the existence of anti-periodic solutions was obtained. Inspired by [49], anti-periodic problems for second-order and higher-order differential equations have been extensively studied, see [1] [2] [3] [21] [27] [28] [29] [37] and the references therein. Many results in the literature were obtained by using the tools such as topological degree theory, lower and upper solutions methods [1][2][3][11][13][21][27][28][29] and the maximal monotone or m-accretive operator [1][2][3].

Some questions are

Q1. How to establish the existence results for impulsive differential inclusions?

Q2. What is the variational structure for impulsive differential inclusions?

1.3 Difference equations

In recent years, a great deal of work has been done in the study of the existence of solutions for discrete boundary value problems, by which a number of physical, biological phenomena are described. For the background and results, we refer the reader to the monograph by Agarwal et al. and some recent contributions as [4] [9] [13] [14] [19] [38] [53] [54] [66].

Among them, Krasnosel'skii fixed point theorem, Leggett-Williams fixed point theorem, fixed point theorem in cones are very frequently used.

There is a trendency to study difference equations by using variational approach. Many interesting results are obtained, see for example, [7] [44] [55] [63] [64]. Agarwal et al.[12] established the existence of multiple positive solutions for the following discrete boundary value problem

$$\begin{cases} \Delta^2 y(k-1) + f(k, y(k)) = 0, & k \in [1, T], \\ y(0) = 0 = y(T+1), \end{cases} \tag{1.3.1}$$

where $[1, T]$ is the discrete interval $\{1, 2, \cdots, T\}$, $\Delta y(k) = y(k+1) - y(k)$, $f \in C([1, T] \times [0, \infty), \mathbb{R})$ satisfies $f(k, 0) \geqslant 0, \forall\ k \in [1, T]$. Guo and Yu [31] studied second-order difference equations

$$\Delta^2 x_{n-1} + f(n, x_n) = 0. \tag{1.3.2}$$

When $f(t,z)$ has superlinear growth at zero and infinity in z, (1.3.2) has multiple periodic and subharmonic solutions. In [44], Li studied the existence of solutions for the problem

$$\begin{cases} \Delta(p(k)\Delta x(k-1)) + f(k, x(k)) = g(k), \\ x(0) = x(T+1) = 0, \end{cases} \tag{1.3.3}$$

where $f \in C([0,T] \times \mathbb{R}, \mathbb{R}), p, g \in C(\mathbb{R}, \mathbb{R})$. By using variational methods, the existence of at least one non-trivial solution was obtained. For the study of discrete problem with a p-Laplacian with critical point theory, please refer to [12] [31] [64].

Now the questions are

Q1. What is variational structure of discrete Sturm-Liouville problem?

Q2. What is variational structure of discrete Neumann boundary value problem?

Q3. If the order of difference equations are 2n-order, can we investigate the existence of solutions, anti-periodic solutions, multiple solutions?

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