电子与通信工程系列教材

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图书在版编目(C I P)数据

电磁场与电磁波习题解答/马汉炎主编.—哈尔滨：哈尔滨工业大学出版社，2002.5

电子与通信工程系列教材

ISBN 7-5603-1719-7

Ⅰ.电… Ⅱ.马… Ⅲ.①电磁场-高等学校-解题②电磁波-高等学校-解题 Ⅳ.0441.4-44

中国版本图书馆 CIP 数据核字(2002)第 028457 号

出版发行 哈尔滨工业大学出版社

社　　址 哈尔滨市南岗区教化街 21 号　邮编 150006

传　　真 0451—6414749

印　　刷 地矿部黑龙江测绘印制中心印刷厂印刷

开　　本 850×1168　1/32　印张 7.125　字数 192 千字

版　　次 2002 年 5 月第 1 版　2002 年 5 月第 1 次印刷

书　　号 ISBN 7-5603-1719-7/TM·37

印　　数 1～4 000

定　　价 13.00 元

前　言

《电磁场与电磁波》课程是电子信息工程、通信工程、电子科学与技术等专业的一门重要技术基础课。该课程概念比较抽象，问题的分析求解所用数学知识较多，学生遇到习题时常感到无处着手。然而，解题是学习电磁场与电磁波的重要环节，能加深和巩固对基本理论的理解，提高培养学生理论联系实际的能力。而编写此书的目的是期望学生通过阅读本书，在解题思路和方法上能有所启迪，从而达到举一反三的效果。

本书对哈尔滨工业大学出版社出版的《电磁场与电磁波》一书中的 187 道习题作了详细解答，包括矢量分析，宏观电磁运动的普遍规律、平面电磁波、平面波的反射与折射、导行电磁波、电磁波的辐射、静态场、稳恒场的解法共八章。附录收录了最近两年哈尔滨工业大学电磁场与电磁波考试试题及参考答案，硕士研究生入学考试电磁场试题及参考答案。习题加试题共计 240 道。

本书第一、二、六章由马汉炎教授编写，第三、四、五章由邱景辉副教授编写，第七、八章由王宏讲师编写。全书由马汉炎教授审核定稿。本书出版得到哈尔滨工业大学出版社，哈尔滨工业大学电子与通信工程系的大力支持，编者一并致以谢意。

限于编者水平，书中有不当之处，恳请批评指正。

编　者

2002 年 4 月

目　录

第一章　矢量分析

1.1　证明两个矢量 $\boldsymbol{A} = 2\boldsymbol{a}_x + 5\boldsymbol{a}_y + 3\boldsymbol{a}_z$ 和 $\boldsymbol{B} = 4\boldsymbol{a}_x + 10\boldsymbol{a}_y + 6\boldsymbol{a}_z$ 是相互平行的。

解:(1) 利用 $\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{0}$,证明 $\boldsymbol{A}//\boldsymbol{B}$

$$\boldsymbol{A} \times \boldsymbol{B} = (2\boldsymbol{a}_x + 5\boldsymbol{a}_y + 3\boldsymbol{a}_z) \times (4\boldsymbol{a}_x + 10\boldsymbol{a}_y + 6\boldsymbol{a}_z) =$$

$$20\boldsymbol{a}_z - 12\boldsymbol{a}_y - 20\boldsymbol{a}_z + 30\boldsymbol{a}_x + 12\boldsymbol{a}_y - 30\boldsymbol{a}_x = \boldsymbol{0}$$

所以 $\boldsymbol{A}//\boldsymbol{B}$

(2) 若 $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$, 则 $\boldsymbol{A}//\boldsymbol{B}$

本题　$l_1 = 2,\ m_1 = 5,\ n_1 = 3$

$l_2 = 4,\ m_2 = 10,\ n_3 = 6$

代入有　$\frac{2}{4} = \frac{5}{10} = \frac{3}{6}$

所以 $\boldsymbol{A}//\boldsymbol{B}$

1.2　证明下列三个矢量在同一平面上,$\boldsymbol{A} = \frac{11}{3}\boldsymbol{a}_x + 3\boldsymbol{a}_y + 6\boldsymbol{a}_z$, $\boldsymbol{B} = \frac{17}{3}\boldsymbol{a}_x + 3\boldsymbol{a}_y + 9\boldsymbol{a}_z$, $\boldsymbol{C} = 4\boldsymbol{a}_x - 6\boldsymbol{a}_y + 5\boldsymbol{a}_z$。

解:矢量 $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ 共面的充要条件是 $\boldsymbol{A}\cdot(\boldsymbol{B} \times \boldsymbol{C}) = 0$ 或 $B \cdot (C \times A) = C \cdot (A \times B) = 0$

$$\boldsymbol{A} \times \boldsymbol{B} = (\frac{11}{3}\boldsymbol{a}_x + 3\boldsymbol{a}_y + 6\boldsymbol{a}_z) \times (\frac{17}{3}\boldsymbol{a}_x + 3\boldsymbol{a}_y + 9\boldsymbol{a}_z) =$$

$$11\boldsymbol{a}_z - 33\boldsymbol{a}_y - 17\boldsymbol{a}_z + 17\boldsymbol{a}_x + 34\boldsymbol{a}_y - 18\boldsymbol{a}_x =$$

$$9\boldsymbol{a}_x + \boldsymbol{a}_y - 6\boldsymbol{a}_z$$

$$C \cdot (A \times B) = (4a_x - 6a_y + 5a_z) \cdot (9a_x + a_y - 6a_z) = 36 - 6 - 30 = 0$$

所以 $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ 共面。

1.3 在球坐标系中,试求点 $M(6,\frac{2\pi}{3},\frac{2\pi}{3})$ 与点 $N(4,\frac{\pi}{3},0)$ 之间的距离。(提示:换成直角坐标求解)

解:转换成直角坐标

$$M(6\sin\frac{2\pi}{3}\cos\frac{2\pi}{3},\ 6\sin\frac{2\pi}{3}\sin\frac{2\pi}{3},\ 6\cos\frac{2\pi}{3})$$

$$N(4\sin\frac{\pi}{3}\cos 0,\ 4\sin\frac{\pi}{3}\sin 0,\ 4\cos\frac{\pi}{3})$$

即 $M(-2.598,\ 4.5,\ -3)$

$N(3.464,\ 0,\ 2)$

距离 $$d = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{1/2} = [(-2.598 - 3.464)^2 + (4.5)^2 + (-3-2)^2]^{1/2} = 81.998^{1/2} = 9.055$$

1.4 求下列矢量场的散度和旋度:

(1) $\boldsymbol{A} = (3x^2y + z)\boldsymbol{a}_x + (y^3 - xz^2)\boldsymbol{a}_y + 2xyz\boldsymbol{a}_z$

(2) $\boldsymbol{A} = yz^2\boldsymbol{a}_x + zx^2\boldsymbol{a}_y + xy^2\boldsymbol{a}_z$

(3) $\boldsymbol{A} = P(x)\boldsymbol{a}_x + Q(y)\boldsymbol{a}_y + R(z)\boldsymbol{a}_z$

解:(1) $\mathrm{div}\boldsymbol{A} = \nabla \cdot \boldsymbol{A} =$

$$\frac{\partial}{\partial x}(3x^2y + z) + \frac{\partial}{\partial y}(y^3 - xz^2) + \frac{\partial}{\partial z}(2xyz) = 6xy + 3y^2 + 2xy = 8xy + 3y^2$$

$$\mathrm{rot}\boldsymbol{A} = \nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y + z & y^3 - xz^2 & 2xyz \end{vmatrix} =$$

$$\boldsymbol{a}_x[\frac{\partial}{\partial y}(2xyz) - \frac{\partial}{\partial z}(y^3 - xz^2)] + \boldsymbol{a}_y[\frac{\partial}{\partial z}(3x^2y + z) -$$

$$\frac{\partial}{\partial x}(2xyz)] + \boldsymbol{a}_z[\frac{\partial}{\partial x}(y^3 - xz^2) - \frac{\partial}{\partial y}(3x^2y + z)] =$$

$$\boldsymbol{a}_x[2xz + 2xz] + \boldsymbol{a}_y[1 - 2yz] + \boldsymbol{a}_z[-z^2 - 3x^2] =$$

$$4xz\boldsymbol{a}_x + (1 - 2yz)\boldsymbol{a}_y - (3x^2 + z^2)\boldsymbol{a}_z$$

(2) $\mathrm{div}\boldsymbol{A} = \nabla \cdot \boldsymbol{A} = \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(zx^2) + \frac{\partial}{\partial z}(xy^2) = 0$

$$\mathrm{rot}\boldsymbol{A} = \nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & zx^2 & xy^2 \end{vmatrix} =$$

$$\boldsymbol{a}_x[\frac{\partial}{\partial y}(xy^2) - \frac{\partial}{\partial z}(zx^2)] + \boldsymbol{a}_y[\frac{\partial}{\partial z}(yz^2) - \frac{\partial}{\partial x}(xy^2)] +$$

$$\boldsymbol{a}_z[\frac{\partial}{\partial x}(zx^2) - \frac{\partial}{\partial y}(yz^2)] =$$

$$\boldsymbol{a}_x[2xy - x^2] + \boldsymbol{a}_y[2yz - y^2] + \boldsymbol{a}_z[2xz - z^2]$$

(3) $\mathrm{div}\boldsymbol{A} = \nabla \cdot \boldsymbol{A} = \frac{\partial}{\partial x}P(x) + \frac{\partial}{\partial y}Q(y) + \frac{\partial}{\partial z}R(z) =$

$$P'(x) + Q'(y) + R'(z)$$

$$\mathrm{rot}\boldsymbol{A} = \nabla \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x) & Q(x) & R(x) \end{vmatrix} =$$

$$\boldsymbol{a}_x[\frac{\partial}{\partial y}R(z) - \frac{\partial}{\partial z}Q(y)] + \boldsymbol{a}_y[\frac{\partial}{\partial z}P(x) - \frac{\partial}{\partial x}R(z)] +$$

$$\boldsymbol{a}_z[\frac{\partial}{\partial x}Q(y) - \frac{\partial}{\partial y}P(x)] = 0$$

1.5 在圆柱坐标中,一点的位置由$(4,2\pi/3,3)$定出,求该点在(1) 直角坐标中;(2) 球坐标中的坐标。

解:圆柱坐标某点$(\rho,\varphi,z) = (4,2\pi/3,3)$

(1) 直角坐标

$$\begin{cases} x = \rho\cos\varphi = 4\cos(2\pi/3) = -2 \\ y = \rho\sin\varphi = 4\sin(2\pi/3) = 2\sqrt{3} \\ z = 3 \end{cases}$$

(2) 球坐标

$$\begin{cases} r = \sqrt{\rho^2 + z^2} = \sqrt{4^2 + 3^2} = 5 \\ \theta = \arcsin\dfrac{\rho}{\sqrt{\rho^2 + z^2}} = \arcsin\dfrac{4}{5} = 53.1° \\ \varphi = \dfrac{2\pi}{3} = 120° \end{cases}$$

1.6 求 div$\boldsymbol{A}$ 在给定点处的值。

(1) $\boldsymbol{A} = x^3\boldsymbol{a}_x + y^3\boldsymbol{a}_y + z^3\boldsymbol{a}_z$，在点 $M(1,0,-1)$ 处。

(2) $\boldsymbol{A} = 4x\boldsymbol{a}_x + 2xy\boldsymbol{a}_y + z^2\boldsymbol{a}_z$，在点 $M(1,1,3)$ 处。

解：(1)
$$\operatorname{div}\boldsymbol{A}\Big|_M = \nabla\cdot\boldsymbol{A}\Big|_M = [\frac{\partial}{\partial x}x^3 + \frac{\partial}{\partial y}y^3 + \frac{\partial}{\partial z}z^3]\Big|_M =$$
$$(3x^2 + 3y^2 + 3z^2)\Big|_{M(1,0,-1)} =$$
$$3 + 0 + 3 = 6$$

(2)
$$\operatorname{div}\boldsymbol{A}\Big|_M = \nabla\cdot\boldsymbol{A}\Big|_M = [\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(z^2)]\Big|_M =$$
$$(4 + 2x + 2z)\Big|_{M(1,1,3)} = 4 + 2 + 6 = 12$$

1.7 一球面 S，半径为 5，球心在原点上，计算 $\oint_s(\boldsymbol{a}_r 3\sin\theta)\cdot \mathrm{d}\boldsymbol{s}$ 的值。

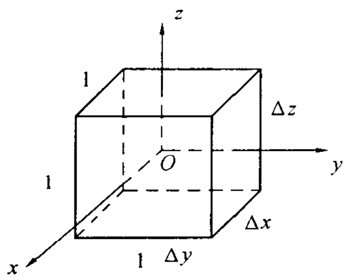
解：用球坐标

$$\oint_s(\boldsymbol{a}_r 3\sin\theta)\cdot\mathrm{d}\boldsymbol{s} = \int_0^{2\pi}\int_0^{\pi}(\boldsymbol{a}_r 3\sin\theta)\cdot(\boldsymbol{a}_r r^2\sin\theta\mathrm{d}\theta\mathrm{d}\varphi) =$$
$$6\pi r^2\int_0^{\pi}\sin^2\theta\mathrm{d}\theta = 3\pi^2 r^2$$

代入 $r = 5$，

$$\oint_s (\boldsymbol{a}_r 3\sin\theta)\cdot \mathrm{d}\boldsymbol{s} = 75\pi^2$$

1.8 求(1) 矢量 $\boldsymbol{A} = \boldsymbol{a}_x x^2 + \boldsymbol{a}_y (xy)^2 + \boldsymbol{a}_z 24x^2y^2z^3$ 的散度;(2) 求$\nabla\cdot\boldsymbol{A}$对中心在原点的一个单位立方体的积分;(3) 求 $\boldsymbol{A}$ 对此立体表面的积分,验证散度定理。



题 1.8

解:(1) $\mathrm{div}\boldsymbol{A} = \nabla\cdot\boldsymbol{A} =$

$$\frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}(xy)^2 + \frac{\partial}{\partial z}(24x^2y^2z^3) =$$

$$2x + 2x^2y + 72x^3y^2z^2$$

(2) $$\int_v \nabla\cdot\boldsymbol{A}\,\mathrm{d}v = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}(2x + 2x^2y + 72x^2y^2z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z =$$

$$x^2\Big|_{-\frac{1}{2}}^{\frac{1}{2}} y\Big|_{-\frac{1}{2}}^{\frac{1}{2}} z\Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \frac{1}{3}x^3\Big|_{-\frac{1}{2}}^{\frac{1}{2}} y^2\Big|_{-\frac{1}{2}}^{\frac{1}{2}} z\Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \frac{8}{3}x^3\Big|_{-\frac{1}{2}}^{\frac{1}{2}} y^3\Big|_{-\frac{1}{2}}^{\frac{1}{2}} z^3\Big|_{-\frac{1}{2}}^{\frac{1}{2}} =$$

$$0 + 0 + \frac{8}{3}\times\frac{1}{4}\times\frac{1}{4}\times\frac{1}{4} = \frac{1}{24}$$

(3) $$\oint_s \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s} = \int_{s前}\boldsymbol{A}\Big|_{x=\frac{1}{2}}\cdot\boldsymbol{a}_x\mathrm{d}s + \int_{s左}\boldsymbol{A}\Big|_{x=\frac{1}{2}}\cdot(-\boldsymbol{a}_x\mathrm{d}s) + \int_{s后}\boldsymbol{A}\Big|_{y=-\frac{1}{2}}\cdot(-\boldsymbol{a}_y\mathrm{d}s) +$$

$$\int_{s右}\boldsymbol{A}\Big|_{y=\frac{1}{2}}\cdot\boldsymbol{a}_y\mathrm{d}s + \int_{s上}\boldsymbol{A}\Big|_{z=\frac{1}{2}}\cdot\boldsymbol{a}_z\mathrm{d}s + \int_{s下}\boldsymbol{A}\Big|_{z=\frac{1}{2}}\cdot(-\boldsymbol{a}_z\mathrm{d}s) =$$

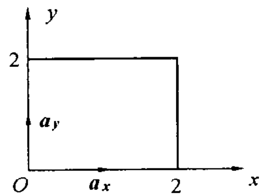
$$(\frac{1}{2})^2\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\mathrm{d}y\mathrm{d}z - (-\frac{1}{2})^2\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\mathrm{d}y\mathrm{d}z -$$

$$(-\frac{1}{2})^2\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}x^2\mathrm{d}x\mathrm{d}z + (\frac{1}{2})^2\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}x^2\mathrm{d}x\mathrm{d}z + 24\times$$

$$(\frac{1}{2})^3\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}x^2y^2\mathrm{d}x\mathrm{d}y - 24\times(-\frac{1}{2})^3\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}x^2y^2\mathrm{d}x\mathrm{d}y =$$

$$\frac{1}{3}x^3\Big|_{-\frac{1}{2}}^{\frac{1}{2}}y^3\Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \frac{1}{3}x^3\Big|_{-\frac{1}{2}}^{\frac{1}{2}}y^3\Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{24}$$

由上述结果$\int_v \nabla\cdot \boldsymbol{A}\mathrm{d}v = \oint_s \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s} = \frac{1}{24}$,故验证了散度定量。



题 1.9

1.9 求矢量 $\boldsymbol{A} = \boldsymbol{a}_x x + \boldsymbol{a}_y x^2 + \boldsymbol{a}_z y^2 z$ 沿 xy 平面上的一个边长 2 的正方形回路的线积分,此正方形的两个边分别与 x 轴、与 y 轴相重合。再求$\nabla\times\boldsymbol{A}$ 对此回路所包围的表面积分,验证斯托克斯定理。

解:
$$\oint_c \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l} = \int_0^2 \boldsymbol{A}\cdot\boldsymbol{a}_x\Big|_{y=0}\mathrm{d}x + \int_0^2 \boldsymbol{A}\cdot\boldsymbol{a}_y\Big|_{x=2}\mathrm{d}y + \int_0^2 \boldsymbol{A}\cdot(-\boldsymbol{a}_x)\Big|_{y=2}\mathrm{d}x + \int_0^2 \boldsymbol{A}\cdot(-\boldsymbol{a}_y)\Big|_{x=0}\mathrm{d}y =$$

$$\int_0^2 x\mathrm{d}x + \int_0^2 4\mathrm{d}y - \int_0^2 x\mathrm{d}x - \int_0^2 0\mathrm{d}y = 8$$

$$\nabla\times\boldsymbol{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & x^2 & y^2 z \end{vmatrix} = \boldsymbol{a}_x[\frac{\partial}{\partial y}(y^2 z) - \frac{\partial}{\partial z}(x^2)] +$$

$$\boldsymbol{a}_y[\frac{\partial}{\partial z}x - \frac{\partial}{\partial x}(y^2 z)] + \boldsymbol{a}_z[\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}x] =$$

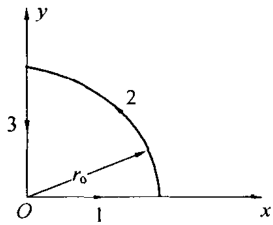
$$\boldsymbol{a}_x 2yz + \boldsymbol{a}_z 2x$$

$$\int_s \nabla\times\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s} = \int_0^2\int_0^2(\boldsymbol{a}_x 2yz + \boldsymbol{a}_z 2x)\cdot\boldsymbol{a}_z\mathrm{d}x\mathrm{d}y =$$

$$\int_0^2\int_0^2 2x\mathrm{d}x\mathrm{d}y = 2\int_0^2\mathrm{d}x^2 = 8$$

由此可见$\oint_c \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \int_s \nabla \times \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{s} = 8$,故验证了斯托克斯公式。

1.10 给定矢量$\boldsymbol{A} = 4\boldsymbol{a}_r + 3\boldsymbol{a}_\theta - 2\boldsymbol{a}_\varphi$,试求矢量$\boldsymbol{A}$沿着图示的闭合路径的线积分。路径的曲线部分是以原点为圆心,以r_0为半径的圆弧;求$\nabla \times \boldsymbol{A}$在它所封闭的面上的面积分。并将这两个积分结果相比较。



题 1.10

解:采用球坐标,设 ds 为l围线所围面积元,微分线元

$$\mathrm{d}\boldsymbol{l} = \mathrm{d}r\boldsymbol{a}_r + r\mathrm{d}\theta\boldsymbol{a}_\theta + r\sin\theta\mathrm{d}\varphi\boldsymbol{a}_\varphi$$

本题$\theta = \frac{\pi}{2}$, $\sin\theta = 1$, $\mathrm{d}\theta = 0$, $\mathrm{d}\boldsymbol{l} = \mathrm{d}r\boldsymbol{a}_r + r\mathrm{d}\varphi\boldsymbol{a}_\varphi$

将围线l分成1、2、3三段,则

$$\oint_c \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \oint_c (4\boldsymbol{a}_r + 3\boldsymbol{a}_\theta - 2\boldsymbol{a}_\varphi) \cdot (\mathrm{d}r\boldsymbol{a}_r + r_0\mathrm{d}\varphi\boldsymbol{a}_\varphi) =$$

$$\int_1 4\mathrm{d}r - \int_2 2r_0\mathrm{d}\varphi + \int_3 4\mathrm{d}r =$$

$$\int_0^{r_0} 4\mathrm{d}r - \int_2^{\frac{\pi}{2}} r_0\mathrm{d}r + \int_{r_0}^{0} 4\mathrm{d}r =$$

$$4r_0 - \pi r_0 - 4r_0 = -\pi r_0$$

$$\nabla \times \boldsymbol{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \boldsymbol{a}_r & r\boldsymbol{a}_\theta & r\sin\theta\boldsymbol{a}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix} =$$

$$\frac{1}{r^2\sin\theta}\begin{vmatrix} \boldsymbol{a}_r & r\boldsymbol{a}_\theta & r\sin\theta\boldsymbol{a}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ 4 & 3r & -2r\sin\theta \end{vmatrix} =$$

$$\frac{1}{r^2\sin\theta}\Big\{\boldsymbol{a}_r[\frac{\partial}{\partial\theta}(-2r\sin\theta)-\frac{\partial}{\partial\varphi}(3r)]+r\boldsymbol{a}_\theta[\frac{\partial}{\partial\varphi}4-\frac{\partial}{\partial r}(-2r\sin\theta)]+r\sin\theta\boldsymbol{a}_\varphi[\frac{\partial}{\partial r}(3r)-\frac{\partial}{\partial\theta}4]\Big\}=$$

$$\frac{1}{r^2\sin\theta}\Big\{\boldsymbol{a}_r(-2r\cos\theta)+\boldsymbol{a}_\theta(2r\sin\theta)+\boldsymbol{a}_\varphi(3r\sin\theta)\Big\}=$$

$$-\frac{2}{r}\frac{\cos\theta}{\sin\theta}\boldsymbol{a}_r+\frac{2}{r}\boldsymbol{a}_\theta+\frac{3}{r}\boldsymbol{a}_\varphi$$

$$\mathrm{d}\boldsymbol{s}=-\boldsymbol{a}_\theta\mathrm{d}s_\theta=-\boldsymbol{a}_\theta r\sin\theta\mathrm{d}r\mathrm{d}\varphi\Big|_{\theta=\frac{\pi}{2}}=-\boldsymbol{a}_\theta r\mathrm{d}r\mathrm{d}\varphi$$

$$\int_s\nabla\times\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s}=\int_s[-\frac{2}{r}\frac{\cos\theta}{\sin\theta}\boldsymbol{a}_r+\frac{2}{r}\boldsymbol{a}_\theta+\frac{3}{r}\boldsymbol{a}_\varphi]\cdot[-\boldsymbol{a}_\theta r\mathrm{d}r\mathrm{d}\varphi]=-2\int_0^{r_0}\int_0^{\pi/2}\mathrm{d}r\mathrm{d}\varphi=-2r_0\frac{\pi}{2}=-\pi r_0$$

由此可见，$\oint_c\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}=\int_s\nabla\times\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{s}=-\pi r_0$

1.11 设 $\boldsymbol{F}=-\boldsymbol{a}_x a\sin\theta+\boldsymbol{a}_y b\cos\theta+\boldsymbol{a}_z c$

式中 a,b,c 为常数，求积分 $\boldsymbol{S}=\frac{1}{2}\int_0^{2\pi}(\boldsymbol{F}\times\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}\theta})\mathrm{d}\theta$。

解：$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}\theta}=-\boldsymbol{a}_x a\cos\theta-\boldsymbol{a}_y b\sin\theta$，代入积分 $\boldsymbol{s}$ 中

$$\boldsymbol{S}=\frac{1}{2}\int_0^{2\pi}(\boldsymbol{F}\times\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}\theta})\mathrm{d}\theta=$$

$$\frac{1}{2}\int_0^{2\pi}[-\boldsymbol{a}_x a\sin\theta+\boldsymbol{a}_y b\cos\theta+\boldsymbol{a}_z c]\times(-\boldsymbol{a}_x a\cos\theta-\boldsymbol{a}_y b\sin\theta)\mathrm{d}\theta=$$

$$\frac{1}{2}\int_0^{2\pi}[\boldsymbol{a}_z(ab\sin^2\theta+ab\cos^2\theta)-\boldsymbol{a}_y ac\cos\theta+\boldsymbol{a}_x bc\sin\theta]\mathrm{d}\theta=$$

$$\frac{1}{2}\int_0^{2\pi}[\boldsymbol{a}_z ab-\boldsymbol{a}_y ac\cos\theta+\boldsymbol{a}_x bc\sin\theta]\mathrm{d}\theta=\boldsymbol{a}_z ab\pi+\boldsymbol{a}_x bc$$

1.12 设 $\boldsymbol{r}=\boldsymbol{a}_x x+\boldsymbol{a}_y y+\boldsymbol{a}_z z$，$r=|\boldsymbol{r}|$，$n$ 为正整数。试求 $\nabla r,\nabla r^n,\nabla f(r)$。

解：$\nabla r = (\boldsymbol{a}_x \frac{\partial}{\partial x} + \boldsymbol{a}_y \frac{\partial}{\partial y} + \boldsymbol{a}_z \frac{\partial}{\partial z}) r =$

$$\boldsymbol{a}_x \frac{\partial r}{\partial x} + \boldsymbol{a}_y \frac{\partial r}{\partial y} + \boldsymbol{a}_z \frac{\partial r}{\partial z}$$

而 $\frac{\partial r}{\partial x} = \frac{\partial (x^2 + y^2 + z^2)^{1/2}}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} \times 2x = \frac{x}{r}$

同理可得$\frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$，代入

$$\nabla r = \frac{x}{r}\boldsymbol{a}_x + \frac{y}{r}\boldsymbol{a}_y + \frac{z}{r}\boldsymbol{a}_z = \frac{\boldsymbol{r}}{r}$$

$$\nabla r^n = (\boldsymbol{a}_x \frac{\partial}{\partial x} + \boldsymbol{a}_y \frac{\partial}{\partial y} + \boldsymbol{a}_z \frac{\partial}{\partial z}) r^n =$$

$$\boldsymbol{a}_x \frac{\partial r^n}{\partial x} + \boldsymbol{a}_y \frac{\partial r^n}{\partial y} + \boldsymbol{a}_z \frac{\partial r^n}{\partial z}$$

而 $\frac{\partial r^n}{\partial x} = \frac{\partial (x^2 + y^2 + z^2)^{n/2}}{\partial x} = \frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1} 2x =$

$$n(x^2 + y^2 + z^2)^{\frac{n-1}{2}}(x^2 + y^2 + z^2)^{-\frac{1}{2}} x =$$

$$nr^{n-1}\frac{x}{r} = nr^{n-2}x$$

同理 $\frac{\partial r^n}{\partial y} = nr^{n-2}y, \frac{\partial r^n}{\partial z} = nr^{n-2}z$，代入

$$\nabla r^n = \boldsymbol{a}_x nr^{n-2}x + \boldsymbol{a}_y nr^{n-2}y + \boldsymbol{a}_z nr^{n-2}z =$$

$$nr^{n-2}(\boldsymbol{a}_x x + \boldsymbol{a}_y y + \boldsymbol{a}_z z) = nr^{n-2}\boldsymbol{r}$$

$$\nabla f(r) = (\boldsymbol{a}_x \frac{\partial}{\partial x} + \boldsymbol{a}_y \frac{\partial}{\partial y} + \boldsymbol{a}_z \frac{\partial}{\partial z}) f(r) =$$

$$\boldsymbol{a}_x \frac{\partial}{\partial x} f(r) + \boldsymbol{a}_y \frac{\partial}{\partial y} f(r) + \boldsymbol{a}_z \frac{\partial}{\partial z} f(r)$$

而 $\frac{\partial}{\partial x} f(r) = f_x'(r) \frac{\partial r}{\partial x} = f_x'(r) \frac{x}{r}$

同理 $\frac{\partial}{\partial y} f(r) = f_y'(r) \frac{y}{r}$

$$\frac{\partial}{\partial z} f(r) = f_z'(r) \frac{z}{r}$$

代入 $\nabla f(r)=\frac{1}{r}[a_x f_x'(r)x+a_y f_y'(r)y+a_z f_z'(r)z]=$

$$\frac{1}{r}f'(r)\boldsymbol{r}=f'(r)\frac{\boldsymbol{r}}{r}$$

1.13 矢量 $\boldsymbol{A}$ 的分量是 $A_x=y\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial y}$，$A_y=z\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial z}$，$A_z=x\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial x}$，其中 f 是 x,y,z 的函数。

还有 $\boldsymbol{r}=\boldsymbol{a}_x x+\boldsymbol{a}_y y+\boldsymbol{a}_z z$。

证明：$\boldsymbol{A}=\boldsymbol{r}\times\nabla f$，$\boldsymbol{A}\cdot\boldsymbol{r}=0$，$\boldsymbol{A}\cdot\nabla f=0$。

解：先求 $\nabla f=(\boldsymbol{a}_x\frac{\partial}{\partial x}+\boldsymbol{a}_y\frac{\partial}{\partial y}+\boldsymbol{a}_z\frac{\partial}{\partial z})f(x,y,z)=$

$$\boldsymbol{a}_x\frac{\partial f}{\partial x}+\boldsymbol{a}_y\frac{\partial f}{\partial y}+\boldsymbol{a}_z\frac{\partial f}{\partial z}$$

计算 $\boldsymbol{r}\times\nabla f=(\boldsymbol{a}_x x+\boldsymbol{a}_y y+\boldsymbol{a}_z z)\times(\boldsymbol{a}_x\frac{\partial f}{\partial x}+\boldsymbol{a}_y\frac{\partial f}{\partial y}+\boldsymbol{a}_z\frac{\partial f}{\partial z})=$

$$\boldsymbol{a}_z x\frac{\partial f}{\partial y}-\boldsymbol{a}_y x\frac{\partial f}{\partial z}-\boldsymbol{a}_z y\frac{\partial f}{\partial x}+\boldsymbol{a}_x y\frac{\partial f}{\partial z}+\boldsymbol{a}_y z\frac{\partial f}{\partial x}-\boldsymbol{a}_x z\frac{\partial f}{\partial y}=$$

$$\boldsymbol{a}_x[y\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial y}]+\boldsymbol{a}_y[z\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial z}]+\boldsymbol{a}_z[x\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial x}]$$

而 $\boldsymbol{A}=\boldsymbol{a}_x[y\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial y}]+\boldsymbol{a}_y[z\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial z}]+$

$$\boldsymbol{a}_z[x\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial x}]$$

对比知 $\boldsymbol{A}=\boldsymbol{r}\times\nabla f$

计算 $\boldsymbol{A}\cdot\boldsymbol{r}=[\boldsymbol{a}_x(y\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial y})+\boldsymbol{a}_y(z\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial z})+\boldsymbol{a}_z(x\frac{\partial f}{\partial y}-$

$$y\frac{\partial f}{\partial x})]\cdot(\boldsymbol{a}_x x+\boldsymbol{a}_y y+\boldsymbol{a}_z z)=$$

$$xy\frac{\partial f}{\partial z}-xz\frac{\partial f}{\partial y}+yz\frac{\partial f}{\partial x}-xy\frac{\partial f}{\partial z}+xz\frac{\partial f}{\partial y}-$$

$$yz\frac{\partial f}{\partial x}=0$$

$$\boldsymbol{A}\cdot\nabla f = \left[\boldsymbol{a}_x\left(y\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial y}\right)+\boldsymbol{a}_y\left(z\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial z}\right)+\boldsymbol{a}_z\left(x\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial x}\right)\right]$$

$$\cdot\left(\boldsymbol{a}_x\frac{\partial f}{\partial x}+\boldsymbol{a}_y\frac{\partial f}{\partial y}+\boldsymbol{a}_z\frac{\partial f}{\partial z}\right)=$$

$$y\frac{\partial f}{\partial x}\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}-x\frac{\partial f}{\partial y}\frac{\partial f}{\partial z}+x\frac{\partial f}{\partial y}\frac{\partial f}{\partial z}-$$

$$y\frac{\partial f}{\partial x}\frac{\partial f}{\partial z}=0$$

1.14 已知 $u = 3x^2z - y^2z^3 + 4x^3y + 2x - 3y - 5$

求 $\Delta u[\Delta u = \mathrm{div}(\mathrm{grad}u)]$

解:$\Delta u = \nabla\cdot\nabla u = \nabla^2 u =$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 6z + 24xy - 2z^3 - 6y^2z$$

1.15 求矢量场 $\boldsymbol{A} = xyz(\boldsymbol{a}_x + \boldsymbol{a}_y + \boldsymbol{a}_z)$ 在点 $M(1,3,2)$ 的旋度以及在这点沿方向 $\boldsymbol{n} = \boldsymbol{a}_x + 2\boldsymbol{a}_y + 2\boldsymbol{a}_z$ 的环量面密度。

解:$\mathrm{rot}\boldsymbol{A}\Big|_M = \nabla\times\boldsymbol{A}\Big|_M = \left.\begin{vmatrix}\boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xyz & xyz & xyz\end{vmatrix}\right|_M =$

$$\left[\boldsymbol{a}_x(xz-xy)+\boldsymbol{a}_y(xy-yz)+\boldsymbol{a}_z(yz-xz)\right]\Big|_{\substack{x=1\\y=3\\z=2}} =$$

$$\boldsymbol{a}_x(2-3)+\boldsymbol{a}_y(3-6)+\boldsymbol{a}_z(6-2)=$$

$$-\boldsymbol{a}_x-3\boldsymbol{a}_y+4\boldsymbol{a}_z$$

在点 M 环量面密度 $\lim\limits_{\Delta s\to 0}\dfrac{\oint_c \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}}{\Delta S} = (\mathrm{rot}\boldsymbol{A})\Big|_M\cdot\boldsymbol{a}_n =$

$$[-\boldsymbol{a}_x-3\boldsymbol{a}_y+4\boldsymbol{a}_z]\cdot\frac{(\boldsymbol{a}_x+2\boldsymbol{a}_y+2\boldsymbol{a}_z)}{(1+2^2+2^2)^{1/2}}=$$

$$\frac{-1-6+8}{3}=\frac{1}{3}$$

1.16 设 $\boldsymbol{r} = \boldsymbol{a}_x x + \boldsymbol{a}_y y + \boldsymbol{a}_z z$,$r = |\boldsymbol{r}|$,$\boldsymbol{C}$ 为常矢

求：(1) $\nabla \times \boldsymbol{r}$，(2) $\nabla \times [f(r)\boldsymbol{r}]$，(3) $\nabla \times [f(r)\boldsymbol{C}]$，(4) $\nabla \cdot [\boldsymbol{r} \times f(r)\boldsymbol{C}]$

解：(1) $\nabla \times \boldsymbol{r} = (\boldsymbol{a}_x \frac{\partial}{\partial x} + \boldsymbol{a}_y \frac{\partial}{\partial y} + \boldsymbol{a}_z \frac{\partial}{\partial z}) \times (\boldsymbol{a}_x x + \boldsymbol{a}_y y + \boldsymbol{a}_z z) =$

$$\boldsymbol{a}_z \frac{\partial y}{\partial x} - \boldsymbol{a}_y \frac{\partial z}{\partial x} - \boldsymbol{a}_z \frac{\partial x}{\partial y} + \boldsymbol{a}_x \frac{\partial z}{\partial y} + \boldsymbol{a}_y \frac{\partial x}{\partial z} - \boldsymbol{a}_x \frac{\partial y}{\partial z})$$

$$= \boldsymbol{0}$$

(2) $\nabla \times [f(r)\boldsymbol{r}] = \nabla f(r) \times \boldsymbol{r} + f(r) \nabla \times \boldsymbol{r} =$

$$f'(r) \frac{\boldsymbol{r}}{r} \times \boldsymbol{r} + \boldsymbol{0} = \boldsymbol{0}$$

(3) $\nabla \times [f(r)\boldsymbol{C}] = \nabla f(r) \times \boldsymbol{C} + f(r) \nabla \times \boldsymbol{C} =$

$$f'(r) \frac{\boldsymbol{r}}{r} \times \boldsymbol{C} + \boldsymbol{0} = f'(r) \frac{\boldsymbol{r}}{r} \times \boldsymbol{C}$$

(4) $\nabla \cdot [\boldsymbol{r} \times f(r)\boldsymbol{C}] = f(r)\boldsymbol{C} \cdot \nabla \times \boldsymbol{r} - \boldsymbol{r} \cdot \nabla \times [f(r)\boldsymbol{C}] =$

$$\boldsymbol{0} + \boldsymbol{r} \cdot [f'(r) \frac{\boldsymbol{r}}{r} \times \boldsymbol{C}] =$$

$$\boldsymbol{C} \cdot [\boldsymbol{r} \times f'(r) \frac{\boldsymbol{r}}{r}] = \boldsymbol{0}$$

1.17 已知 $\boldsymbol{A}(r,\theta,\varphi) = r^2 \sin\varphi \boldsymbol{a}_r + 2r\cos\theta \boldsymbol{a}_\theta + \sin\theta \boldsymbol{a}_\varphi$，求$\frac{\partial \boldsymbol{A}}{\partial \varphi}$。

解：在球坐标中，各坐标单位矢量对空间坐标变量的偏导数为

$$\frac{\partial \boldsymbol{a}_r}{\partial r} = \boldsymbol{0},\ \frac{\partial \boldsymbol{a}_r}{\partial \theta} = \boldsymbol{a}_\theta,\ \frac{\partial \boldsymbol{a}_r}{\partial \varphi} = \boldsymbol{a}_\varphi \sin\theta$$

$$\frac{\partial \boldsymbol{a}_\theta}{\partial r} = \boldsymbol{0},\ \frac{\partial \boldsymbol{a}_\theta}{\partial \theta} = -\boldsymbol{a}_r,\ \frac{\partial \boldsymbol{a}_\theta}{\partial \varphi} = \boldsymbol{a}_\varphi \cos\theta$$

$$\frac{\partial \boldsymbol{a}_\varphi}{\partial r} = \boldsymbol{0},\ \frac{\partial \boldsymbol{a}_\varphi}{\partial \theta} = \boldsymbol{0},\ \frac{\partial \boldsymbol{a}_\varphi}{\partial \varphi} = -\boldsymbol{a}_\theta \cos\theta - \boldsymbol{a}_r \sin\theta$$

则 $\frac{\partial \boldsymbol{A}}{\partial \varphi} = \frac{\partial}{\partial \varphi}[r^2 \sin\varphi \boldsymbol{a}_r + 2r\cos\theta\, \boldsymbol{a}_\theta + \sin\theta \boldsymbol{a}_\varphi] =$

$$r^2 \sin\varphi \frac{\partial}{\partial \varphi}\boldsymbol{a}_r + \boldsymbol{a}_r \frac{\partial}{\partial \varphi}(r^2 \sin\varphi) + 2r\cos\theta \frac{\partial}{\partial \varphi}\boldsymbol{a}_\theta +$$