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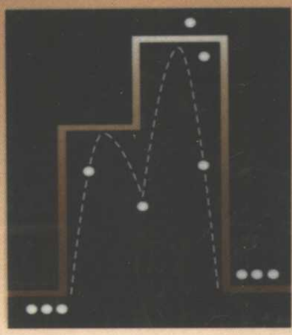

An Introduction to Stochastic Processes

Edward P.C. Kao

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This is an introductory book on stochastic processes—a subject about modeling and analysis of random phenomena occurring over time or space. Many years ago, we could not *do* stochastic processes in a serious way in the context of real-world problem solving. The rapid advancements in numerical methods and computing facilities have profoundly changed the landscape. This text responds to the challenges of incorporating computer use in the teaching and learning of stochastic process.

This book is written for *students* who are interested in learning concepts, models, and computational approaches in stochastic processes. The intended audience includes upper-level undergraduates and first-year graduate students in operations research, management science, finance, engineering, statistics, computer science, and applied mathematics. The prerequisites for the text are intermediate-level calculus, elementary linear algebra, and an introductory course in probability with an emphasis in operational skills on conditioning.

This book takes an application and computation oriented approach instead of the standard formal and mathematically rigorous approach. The emphasis is on the development of operational skills in stochastic modeling and analysis through a variety of examples drawn from diverse areas while relegating the burden of computation to its rightful master—the computer. Following our approach, we are able to present many topics of practical importance in detail at a very early stage. One such example is the study of a time-dependent service system covered in Chapter 2. There we see that once the model is constructed the time-dependent solutions of the system of differential equations with time-varying parameters can be obtained rather conveniently on a computer.

Organization and Coverage

The book covers standard topics in a first course in stochastic processes. It also includes some additional materials reflecting recent development in computational probability. The first chapter reviews some preliminary materials. They include a brief introduction, transform methods, and some basic concepts in mathematical analysis. Chapters 2 to 6 are organized in a logical sequence. We start with a Poisson process and its variants in Chapter 2, and move to a more general counting process called the renewal process in Chapter 3. To model dependency in random phenomena, we study discrete-time and continuous-time Markov chains in Chapters 4 and 5, respectively. The Markov renewal process presented in Chapter 6 can be considered as the generalization of most models studied earlier. The capstone of all these is the semi-regenerative process in Section 6.4. The last chapter is about Brownian motion, diffusion processes, and Ito's lemmas. The chapter also contains applications of diffusion process in finance.

The book provides a great deal of flexibility for instructors. For students in business and management, Chapters 1–5 should provide a good introduction to stochastic models in management science. For students majoring in finance, the first few sections of Chapters 2–5 along with the last chapter will give them the preliminary background in stochastic processes for further study in continuous-time finance. For students in computer science, electrical and computer engineering, or operations management who want to acquire some knowledge about Markovian service systems necessary for performance evaluations of communication systems, computer networks, or automated manufacturing systems they will find Chapters 2, 4, and 5 useful. In order to reach Section 5.8 on queueing networks in a one-semester course, instructors may choose to skip Sections 5.5–5.7. For students in industrial and systems engineering and operation research who eventually will study queueing theory beyond Markovian models, knowing the materials in Chapter 6 would be helpful. To cover the entire book, a two-semester sequence can be considered with Chapters 1–4 in the first semester and Chapters 5–7 in the second. More difficult examples and problems are marked with an asterisk (*).

A solution manual is available from the publisher for instructors who adopt this text for a course. Readers are welcome to use the perforated card in the back of the book to contact MathWorks, Inc. for a diskette containing the MATLAB programs listed in the appendices.

Notation

We use a capitalized italic letter to denote a matrix, and a bold-italic lower-case letter to denote a vector. The letters I and O denote the identity and zero matrices, respectively. Column and row vectors will not be distinguished but will be stated as such if their types are not clear from the context. We use $exp(\mu)$ to denote an exponential density with parameter μ, $U(0, 1)$ a uniform density over the interval $(0, 1)$, $pos(n; \lambda, t)$ a Poisson probability mass at n whose parameter is λt, $Erlang(n, \mu)$ an Erlang density with parameters n and μ, and $N(\mu, \sigma^2)$ a normal density with mean μ and variance σ^2. We follow the MATLAB notation that "j: i: k" denotes $\{j, j+i, j+2i, \ldots, k\}$. For example, $t = 1{:}1{:}10$ means $t = 1, 2, \ldots, 10$. MATLAB commands and function calls are stated in **`bold courier`** fonts.

MATLAB

The software chosen for this text is MATLAB®. MATLAB is easy to learn and numerically reliable. It is most suitable for solving problems involving matrices. In many homework problems, students are expected to experiment with their models and solution procedures with the aid of MATLAB. Of course, Mathematica® or Maple® can also be used to accomplish the same for those who are conversant with and have access to these software.

A brief tutorial on MATLAB is given at the end of the text. For more information about the MATLAB software, readers may contact: The MathWorks, Inc., 24 Prime Park Way, Natick, MA 01760–1500, E-Mail: info@mathworks.com, WWW: http://www.mathworks.com. We emphasize that the MATLAB programs shown at the end of each chapter were for illustrative purposes and no attempts were made to optimize the codes.

Acknowledgments

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While I was fortunate to receive the help from many people in writing and improving this text, I bear responsibility for any errors and would appreciate hearing about them.

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March 1996

CONTENTS

1

Introduction

Tips for Chapter 1

- The details about numerical inversion of transforms (the paragraphs immediately following Examples 1.2.8 and 1.3.4) are there for the curious. These materials are not relevant to subsequent exposition.
- Motivations of Examples 1.2.6 and 1.3.4: One of the most frequently used approaches in modeling a probabilistic system is to (i) develop a system of difference or differential equations by conditioning on all the possible outcomes of the first step (or the last step) of a sequence of experiments, (ii) obtain the transform of the aforementioned system, (iii) invert the transform algebraically or numerically to solve the problem. In these examples, we consider two somewhat elaborate cases to illustrate this problem-solving approach. A good grasp of the ideas underlying the two examples will give readers a head start for many derivations in subsequent chapters. Readers who find the exposition lengthy or the steps hard to follow may want to consider these two examples as supplementary reading materials for later chapters. For instance, Example 1.3.4 can be read after covering Section 2.1 or 5.2.
- The Riemann-Stieltjes integral and Riemann-Stieltjes transform discussed in Section 1.4 are useful concepts for handling a continuous random variable that contains discrete components. When a random variable is either continuous or discrete, these concepts *only* provide unified notations. The latter scenario applies to most subjects considered in the text. Examples 1.4.9 and 1.4.10 give two illustrations of applying Leibniz's rule in renewal theory (Section 3.3). Easier examples can be found in most calculus books. The last

section (Section 1.4) can be used as quick reference for the various mathematical concepts involved in the text.

- The moment generating function (shown at the end of Section 1.3) and Taylor-series expansion (given at the end of Section 1.4) are introduced here but will only be needed in Chapter 7.

1.0 Overview

This chapter introduces the subject of stochastic processes, reviews transform techniques to facilitate problem solving and analysis in applied probability, and presents some mathematical background needed in the sequel. In the first section, we define what is meant by a stochastic process and the ideas of stationary and independent increments. The section also gives an overview of the text. The next two sections review generating functions and Laplace transforms. They are quite useful in handling discrete and continuous random variables that we will encounter in the study of stochastic processes. In addition to inversion by algebraic means (manageable only for problems of small size and simple structure), we also present approaches for inverting probability generating functions and Laplace transforms numerically. In this age of computers, numerical inversion enlarges the domain of applicability of transform methods. Readers who have experiences in using generating functions and Laplace transforms in other contexts can go through Sections 1.2 and 1.3 rather quickly. The last section lists a minimal set of results in mathematical analysis that are needed for the text. The section is written primarily for readers who do not have training in mathematics beyond calculus. For others, the section can serve as a source for quick reference. Those who already have had a course in advanced calculus or elementary analysis (say at a level of Rudin [1976] or Bartle [1976]) can skip the last section and go directly to the next chapter.

1.1 Introduction

Let $X(t)$ denote the state of a system at time t. For example, the state $X(t)$ can be the closing price of an IBM stock on day t. The collection of the random variables $\boldsymbol{X} = \{X(t), t \in \boldsymbol{T}\}$ is called a *stochastic process*, in which the set $\boldsymbol{T}$ is called the *index set*. When the index set is countable, $\boldsymbol{X}$ is called a discrete-time process. Thus the daily closing prices of an IBM stock form a discrete-time stochastic process, in which $\boldsymbol{T} = \{0, 1, \dots\}$. When the index set is an interval of the real line, the stochastic process is called a continuous-time process. If $X(t)$ denotes the price of an IBM stock at time t on a given day, then the process $\boldsymbol{X} = \{X(t), t \in \boldsymbol{T}\}$ is a continuous-time process, in which $\boldsymbol{T}$ is the interval covering a trading day.

If we assume that $X(t)$ takes values in a set $\boldsymbol{S}$ for every $t \in \boldsymbol{T}$, then $\boldsymbol{S}$ is called the state space of the process $\boldsymbol{X}$. When $\boldsymbol{S}$ is countable, we say that the process has a discrete state space. The two stochastic processes involving the price of an IBM stock both have discrete state spaces whose elements are dollars in increment of

1/8. When S is an interval of a real line, the process has a continuous state space. As an example, if $X(t)$ denotes the temperature at Houston Intercontinental Airport at time t, then, in principle, $X(t)$ can assume any value in an interval S.

A realization of a stochastic process X is called a *sample path* of the process. In Figure 1.1, we depict a sample path associated with a discrete-time process with a discrete state space—namely, the daily closing prices of an IBM stock. In Figure 1.2, we do the same for a continuous-time process with a discrete state space—namely, the price at any time t on a given day. Similarly, in Figure 1.3 we plot a sample path for a continuous-time process with a continuous state space representing the uninterrupted temperature readings at Houston Intercontinental Airport over a given period. If these temperature readings are taken at a set of

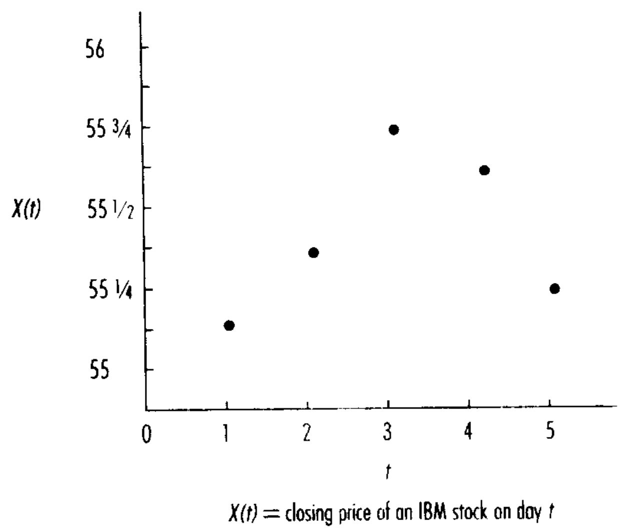

FIGURE
1.1 A sample path of a discrete-time process with a discrete state space.

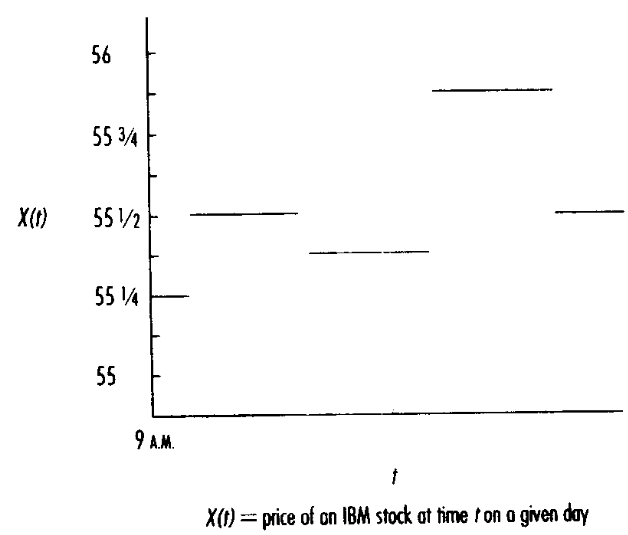

FIGURE
1.2 A sample path of a continuous-time process with a discrete state space.

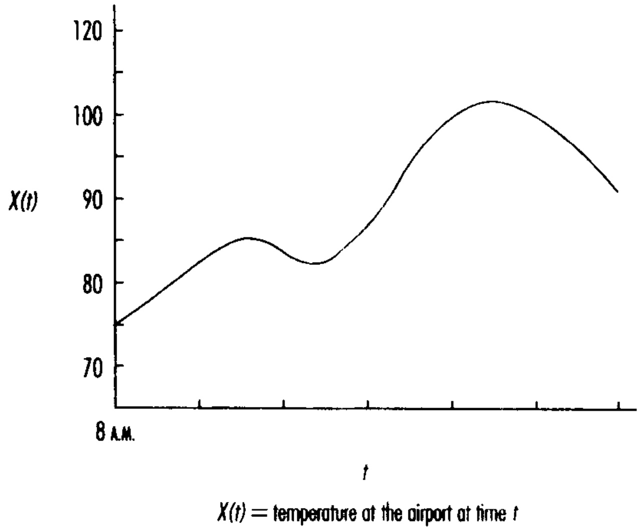

FIGURE 1.3 A sample path of a continuous-time process with a continuous state space.

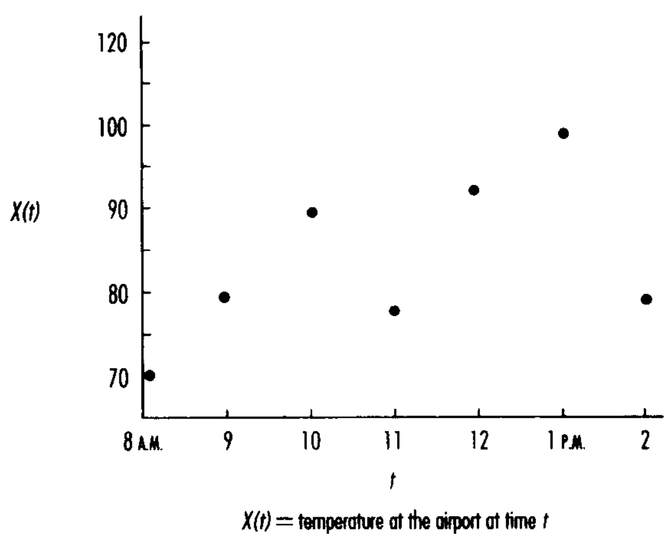

FIGURE 1.4 A sample path of a discrete-time process with a continuous state space.

distinct epochs, say every hour on the hour, then the process becomes a discrete-time process with a continuous state space. The latter is depicted in Figure 1.4.

Without structural properties, little can be said or done about a stochastic process. Two important properties are the independent-increment and stationary-increment properties of a stochastic process. A process $X = \{X(t), t \geq 0\}$ possesses the *independent-increment* property if for all $t_0 < t_1 < \cdots < t_n$, random variables $X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are independent (the time indices will either be discrete or continuous depending on the context). Hence in a process with independent increments, the magnitudes of state change over nonoverlapping intervals are mutually independent. A process possesses the *stationary-increment* property if the random variable $X(t + s) - X(t)$ possesses the same probability distribution for all t and any $s > 0$. In other words, the probability distribution governing the magnitude of state change depends only on the difference in the lengths of the time indices and is independent of the time origin used for the indexing variable.

Let $N(t)$ denote the number of arrivals of a given event by time t (e.g., car arrivals to a toll booth). The stochastic process $N = \{N(t), t \geq 0\}$ is called a *counting* process. The Poisson process studied in Chapter 2 is a counting process in which interarrival times of successive events are independently and identically distributed (i.i.d.) exponential random variables. The process possesses both the independent-increment and stationary-increment properties. Poisson processes are used extensively in modeling arrival processes to service systems and demand processes in inventory systems. There are many useful variants of Poisson processes. An important extension is the nonhomogeneous Poisson process in which we assume that the arrival rate is time dependent. This extension makes Poisson a versatile process for real-world applications.

In a counting process, when interarrival times of successive events follow a probability distribution other than the exponential and yet these times are mutually independent, the resulting process is called a *renewal process*. Chapter 3 is devoted to the study of renewal and related processes. The theory of a renewal process forms a cornerstone for the development of other more complicated stochastic processes, which is accomplished by use of its extension known as the *regenerative process*. At the arrival epoch of a renewal event, the future of the process becomes independent of the past. Therefore the interval between two successive renewals forms a regeneration cycle. The regeneration cycles are probabilistic replica to one another. When we are interested in a long-run property of a stochastic process, studying it over one regeneration cycle will enable us to ascertain its asymptotic value.

In Chapter 4, we introduce Markov chains. In a Markov chain, both the state space and index set are discrete. A change of state depends probabilistically only on the current state of the system and is independent of the past given that the present state is known. A process possessing this property is known to have the *Markovian* property. The successes one can have in employing Markov chains for modeling in applications depend on proper state definitions at selected epochs to maintain the Markovian property at these epochs. When there are rewards associated with state occupancy, the resulting process is called a Markov reward process. Markov chains and Markov reward processes have been used extensively in modeling and analyses of many systems in production, inventory, computers, and communication.

In a Markov chain, we are interested in the state changes over the state space and unconcerned about the sojourn times in each state before a state change takes place. For such a chain, when sojourn times in each state follow exponential distributions with state-dependent parameters, the resulting stochastic process is called a continuous-time Markov process with a discrete state space. For the special case when transitions from a given state will only be made to states other than itself, the resulting process is called the *continuous-time Markov chain*. Various subjects relating to continuous-time Markov chains will be examined in detail in Chapter 5.

A generalization of a Markov chain allows sojourn times in each state to follow probability distributions that depend on the starting and ending states associated with each transition. Stochastic processes resulting from such a generalization are called *Markov renewal processes.* This generalization makes renewal processes and discrete-time and continuous-time Markov chains all special cases of Markov renewal processes. Subjects relating to Markov renewal processes are covered in Chapter 6.

Stochastic processes presented in Chapters 2–6 all have discrete state space. In the last chapter, we will study processes with continuous state space—particularly the Brownian motion process. The mathematics needed to handle Brownian motion and related processes is more demanding. Our coverage of the subjects involved will be relatively limited.

1.2 Discrete Random Variables and Generating Functions

Let $\{a_n\}$ denote a sequence of numbers. We define the *generating function* for the sequence $\{a_n\}$ as

$$a^g(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{1.2.1}$$

where the power series $a^g(z)$ converges in some interval $|z| < R$. $a^g(z)$ is also called the Z-transform or geometric transform for the sequence $\{a_n\}$. To illustrate, consider the case in which $a_n = \alpha^n$, $n = 0, 1, \ldots$. Then we see that $a^g(z) = 1/(1 - \alpha z)$ when $|\alpha z| < 1$. In Table 1.1, we present an abbreviated listing relating some sequences $\{a_n\}$ and their respective generating functions. For the ith pair shown in the table, we use the notation Z-i. The pairs Z-1 and Z-2 imply that the generating function is a linear operator in the sense that if a sequence is a linear combination of two sequences, the linear relation is preserved under the transform by using the generating function. The pair Z-3 implies that the convolution operation of two sequences becomes a multiplication operation if we work with the respective generating functions instead. The sequence $\{b_n\}$ in Z-6 is the sequence $\{a_n\}$ "delayed" by k units, whereas the sequence $\{b_n\}$ in Z-7 is the sequence $\{a_n\}$ "advanced" by k units. The sequences in Z-8 and Z-9 perform respectively the "summing" and "differencing" operations. They are the discrete analogs of integration and differentiation. The two pairs enable us to do these operations when the functions have first been transformed. If A_n is a square matrix with elements $\{a_{ij}(n)\}$, then the (i, j)th element of the matrix generating function $A^g(z)$ is defined as $\sum_{n=0}^{\infty} z^n a_{ij}(n)$. When the elements of matrix A are $\{a_{ij}\}$, Z-10 gives the corresponding matrix generating function.

When $\lim_{n\to\infty} a_n$ exists, we can evaluate this limit by working with the generating function using the *final value property:*

$$\lim_{n\to\infty} a_n = \lim_{z\to 1}(1-z)a^g(z).$$

TABLE 1.1 A Table of Generating Functions

The Sequence $\{a_n\}$	Generating Function $a^g(z) = \sum_{n=0}^{\infty} a_n z^n$
1. $\{\alpha a_n\}$	$\alpha a^g(z)$
2. $\{\alpha a_n + \beta b_n\}$	$\alpha a^g(z) + \beta b^g(z)$, where $b^g(z) = \sum_{n=0}^{\infty} b_n z^n$
3. $\left\{\sum_{m=0}^{n} a_m b_{n-m}\right\}$ Convolution	$a^g(z) b^g(z)$
4. $\{a^n\}$	$\dfrac{1}{1-az}$
5. $\left\{\dfrac{1}{k!}(n+1)(n+2)\cdots(n+k)a^n\right\}$	$\dfrac{1}{(1-az)^{k+1}}$
6. $\{b_n\}$, where $b_n = 0$ if $n < k$; $= a_{n-k}$ if $n \geq k$; and k is a positive integer	$z^k a^g(z)$
7. $\{b_n\}$, where $b_n = 0$ if $n < 0$; $= a_{n+k}$ if $n \geq 0$; and k is a positive integer	$\dfrac{1}{z^k}\left[a^g(z) - a_0 - a_1 z - \cdots - a_{k-1} z^{k-1}\right]$
8. $\left\{\sum_{m=0}^{n} a_m\right\}$	$\dfrac{1}{1-z} a^g(z)$
9. $\{b_n\}$, where $b_n = a_0$ if $n = 0$; $= a_n - a_{n-1}$ if $n \geq 1$	$(1-z) a^g(z)$
10. $\{A^n\}$, where A is a square matrix	$\sum_{n=0}^{\infty} (zA)^n = [I - Az]^{-1}$, where I is an identity matrix

A formal proof of the property can be found in a reference cited in the Bibliographic Notes. We leave an alternate proof based on Z-6 as an exercise.

Problem manipulations involving transforms are sometimes referred to as operations in the transform domain. When we invert a transform to its corresponding sequence $\{a_n\}$, we call the procedure an inversion of the transform to the time domain. Generating functions are quite useful in solving systems of difference equations; however, we shall focus our attention on their applications in stochastic modeling.

Let X denote a discrete random variable and $a_n = \text{Prob}\{X = n\}$. Then $P_X(z) = a^g(z) = E[z^X]$ is called the probability generating function for the random variable X. Here we impose the condition $|z| \leq 1$ so as to ensure the uniform convergence of the power series $a^g(z)$. If we know the probability generating function of X, the coefficients of the power series expansion of $a^g(z)$ give the probabilities that X assumes various values. Many times problem solving is somewhat messy in the time domain. We do our manipulations in the transform domain and then make an inversion to obtain the desired result.

We can obtain moments of a random variable X from its probability generating function $P_X(z)$. Define the kth derivative of $P_X(z)$ by

$$P_X^{(k)}(z) = \frac{d^k}{dz^k} P_X(z).$$

Then we see that

$$P_X^{(1)}(z) = \frac{d}{dz}\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \frac{d}{dz} a_n z^n = \sum_{n=0}^{\infty} n a_n z^{n-1} \qquad \text{and} \qquad E[X] = P_X^{(1)}(1).$$

Similarly, we have

$$P_X^{(2)}(z) = \frac{d}{dz} P_X^{(1)}(z) = \frac{d}{dz}\sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=1}^{\infty} n(n-1) a_n z^{n-2},$$

and $P_X^{(2)}(1) = E[X(X-1)] = E[X^2] - E[X]$. So the second derivative of $P_X(z)$ with respect to z evaluated at 1 gives the *second factorial moment* of X. The second moment of X is given by

$$E[X^2] = P_X^{(2)}(1) + P_X^{(1)}(1). \tag{1.2.2}$$

Other higher moments of X can be found analogously.

EXAMPLE 1.2.1 **The Binomial Random Variable** Let X be a binomial random variable with parameters n and p and

$$P\{X = j\} = a_j = \binom{n}{j} p^j q^{n-j} \qquad j = 0, 1, \ldots, n,$$

where $q = 1 - p$. The probability generating function is given by

$$P_X(z) = \sum_{j=0}^{n} a_j z^j = \sum_{j=0}^{n} \binom{n}{j} p^j q^{n-j} z^j = \sum_{j=0}^{n} \binom{n}{j} (pz)^j q^{n-j} = (pz+q)^n.$$

With $P_X^{(1)}(z) = n(pz+q)^{n-1}p$ and $P_X^{(2)}(z) = n(n-1)(pz+q)^{n-2}p^2$, we obtain $E[X] = P_X^{(1)}(1) = np$ and $E[X^2] = n(n-1)p^2 + np$ by Equation 1.2.2. This gives $Var[X] = npq$. ▪

EXAMPLE 1.2.2 **The Poisson Random Variable** Let X be a Poisson random variable with parameter $\lambda > 0$ and

$$P\{X = n\} = a_n = e^{-\lambda}\frac{\lambda^n}{n!} \qquad n = 0, 1, \ldots.$$

The probability generating function is given by

$$P_X(z) = \sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!} z^n = e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda z)^n}{n!} = e^{\lambda(z-1)}.$$