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环境科学与工程中的数学问题

Alexandre Ern 刘维屏 编

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Mathematical Problems in Environmental Science and Engineering

Alexandre Ern Liu Weiping

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前　言

过去十年，无论是在工业化国家还是发展中国家，对环境问题重视程度的不断上升是有目共睹的．随着越来越严格的法律和法规的出台，环境影响已成为工业和公众社会事业机构的关键问题．因此，为了正确评估和有效修复世界范围内环境资源的沉重负担，科学家和工程师们致力于环境科学与工程的研究．另外，环境科学中的数学工具在预测环境灾害 (如水灾，暴雨等) 中也起着越来越重要的作用．

从学术的观点看，环境问题的解决是非常困难的．这种复杂性源于各种原因，包括：需要考虑的生物、化学、物理等因素范围太广，可利用的满足模型的数据严重缺乏，数字模拟实验的费用极其昂贵．在这本论文集中，运用了数学这个有用的工具来帮助阐明数据与模型之间的内在关系和建立在离散方程或模型的随机性上的合适框架．

论文集中收集的材料来源于各种关于环境科学数学模型的评述报告和与工程、工业实践紧密相关的邀请会．这些稿件已在中法应用数学研究所主办的“环境科学与工程相关的数学问题研讨会”上讨论．作为 2000 年关于应用数学中法会议的主要活动之一，这次研讨会的意图是增进中法两国数学家、环境科学家和工程师之间的合作．其目的是着重介绍当前使用的一些数学工具，推广基础知识和探索数学在环境科学中的应用，以及推进中法两国在这些论题上的合作．

这次研讨会于 2000 年 8 月 20 至 26 日在中国杭州浙江大学召开，与会人员超过 100 位，其中 20 位来自法国，1 位来自希腊，8 位来自泰国．与会者来自高校和工业部门的专家，政府官员，在校研究生和毕业生．研讨会的活动包括 5 个关于在环境应用中起重要

作用数学论题的系列讲座，10个邀请大会报告和20个分组报告，集中讨论建模工具和法律法规的座谈会，以及最后与政府官员的一个圆桌讨论会议.

这本书中收录的系列讲座从简单介绍着手，然后详细阐述他们的材料，并通过在环境科学中的应用来例证近期理论的进展. 评述报告用最佳的控制仪器、偏微分方程的有限元离散、随机多功能场等对数据进行类比处理. 邀请报告会强调与大气污染、水处理和管理相关的技术和工艺. 在工业部门的这类工作中，这本书关于数学在环境科学中所起的作用将提出具有远见性的建议. 而在高校和科研机构中，人们将会从中发现新的具挑战性的问题. 本书中所收集的材料对数学或环境科学专业的毕业生和研究生同样有很大的帮助.

作为研讨会的组织者，我们借此机会向以下所有支持会议的机构和人员表示最真诚的感谢：中国教育部数学研究与高等人才培养中心、中国自然科学基金委员会、复旦大学、浙江大学、法国驻中国大使馆、法国驻上海总领事馆、法国电器和沸芬帝水处理公司. 感谢全体报告者和与会人员的热情和帮助. 将最深的感谢给予我们的同事陈叔平、严金海、蔡志杰、郑巍、童裳伦和 Daniel Zimmer，正是他们的工作使研讨会得到了圆满的成功. 还要感谢那些不具名的审稿人，他们的大力协作对于本书的编辑是非常有用的. 感谢复旦大学李大潜院士和 Pierre-Arnaud Raviart 教授对举办这次研讨会的信任和支持. 最后我们感谢高等教育出版社在本书编辑过程中所给予的帮助.

Alexandre Ern　刘维屏

2001 年 9 月

Preface

The last decade has witnessed an increasing awareness of environmental problems in both developing and industrialized countries. With policies and regulations getting more and more stringent over the years, environmental impact has become a key issue for industry and public institutions. As a result, scientists and engineers have devoted extensive efforts to environmental sciences and engineering with the goal to assess and hopefully remedy the heavy burden currently placed on worldwide environmental resources. Environmental sciences have also played an increasingly important role in the forecast of environmental cataclysms such as floods or heavy storms.

From a scientific viewpoint, environmental problems are extremely difficult to handle. Their complexity arises from various reasons, including the wide range of biological, chemical and physical phenomena that need to be accounted for, the scarce availability of data to feed the models and the extremely high costs often involved in numerical simulations. In this context, applied mathematics appear as a very valuable tool to help clarify the intimate relationship between data and models, the appropriate framework in which to discretize equations or the stochastic nature of the model fields.

The material collected in this book originates from survey lectures on various mathematical methods for environmental sciences and invited talks more closely related to engineering and industrial practice. These contributions were given during the ISFMA "Symposium on Environmental Science and Engineering with Related Mathematical Problems". As one of the main activities of the Chinese-French Institute for Applied Mathematics (ISFMA) in the year 2000, the symposium aimed at promoting the collaboration between mathematicians, environmental scientists and engineers from France and China. The purpose of the symposium was in particular to introduce the present state of the art for several mathematical tools, to popularize the basic knowledge and explore the potential applications of mathematics in environmental sciences and to push forward the cooperation between France and China on these topics.

The symposium, held at Zhejiang University, Hangzhou, China on August 20-26, 2000, was attended by over 100 participants including 20 from France, 1 from Greece and 8 from Thailand. Among the participants were experts from university and industry, government officers, postgraduate fellows and graduate students. The activities of the symposium included five survey lectures on mathematical topics which play an important role in environment applications, 10 invited and 20 contributed talks focusing on engineering issues, modeling tools and regulation policies and a final roundtable with government officers.

The survey lectures contained in this volume start at an introductory level and then elaborate their material to include recent theoretical advances with illustrations through applications in the environmental sciences. The survey lectures deal with data assimilation by optimal control techniques, finite element discretizations of partial differential equations, stochastic multifractal fields COMPLETE. The invited lectures emphasize engineering and industrial aspects related to air pollution, water treatment and management COMPLETE . To those working in industry, this book should provide a broad insight into the role mathematics can play in the environmental sciences. Those working in universities and research institutes will find here new challenging problems. The material collected in this book will also be beneficial to graduate and postgraduate students specializing in mathematics or in environmental sciences.

As organizers of the symposium, we would like to take this opportunity to express our gratitude to various institutions for their support: the Mathematical Center of the Education Ministry of China, the National Natural Science Foundation of China, Fudan University, Zhejiang University, the Embassy of France in China, the General Consulate of France at Shanghai, Electricité de France and Vivendi. We also thank all the lecturers, plenary speakers and participants to the symposium for their enthusiasm and contribution. Our deepest appreciation goes to many of our colleagues, among others Chen Shuping, Yan Jinhai, Cai Zhijie, Zheng Wei, Tong Changlun and Daniel Zimmer, whose work has been instrumental for the success of the symposium. The collaboration of several anonymous referees has been extremely useful in editing this volume and is gratefully acknowledged. We would like to thank Academician Li Ta-Tsien and Professor Pierre-Arnaud Raviart for their confidence and support in launching this symposium. Finally we address our sincere thanks to Higher Education Press for their help in editing this book.

Alexandre ERN   Weiping LIU

September, 2001

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邀请报告

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A Subgrid Viscosity Method for Solving Non-Coercive PDE's in Environmental Science

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ABSTRACT

The goal of this course is to present a stabilized Galerkin technique for approximating non-coercive PDE's. This technique is based on a two-level hierarchical decomposition of the approximation space. This space is broken up into resolved scales and subgrid scales. We show that in general the Galerkin formulation provides an a priori control on the resolved scales of the approximate solution, whereas it cannot control the subgrid scales. The missing stability is obtained by slightly modifying the Galerkin formulation by introducing an artificial diffusion on the subgrid scales. Numerical tests show that the method applies also to nonlinear problems.

1. Model problems

In this section we recall an abstract existence result and we show that the Galerkin formulation is not optimal for approximating PDE's dominated by first order differential operators.

1.1. An abstract existence and stability result. Let $V \subset L$

be two real Hilbert spaces with dense and continuous embedding. For any Hilbert spaces H we denote by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ the scalar product and the norm in H respectively. For any Banach space, we denote by $B' = \mathcal{L}(B;\mathbb{R})$ the dual of B. Hereafter we make the usual identifications $V \subset L \equiv L' \subset V'$. Let $a \in \mathcal{L}(V \times L;\mathbb{R})$ and consider the following problem.

$$\begin{cases} \text{For all } f \in L, \text{ find } u \in V \text{ s.t.} \\ a(u,v) = (f,v)_L, \quad \forall v \in L. \end{cases} \tag{1.1}$$

Sufficient and necessary conditions for this problem to be well-posed are stated in the following theorem due to Nečas[Neč62]:

Theorem 1.1 (Nečas). *Problem 1.1 is well posed if and only if*

$$\exists \alpha > 0, \qquad \inf_{u\in V} \sup_{v\in L} \frac{a(u,v)}{\|u\|_V \|v\|_L} \geq \alpha, \tag{1.2}$$

$$\forall v \in L, \qquad (v \neq 0) \Rightarrow \left(\sup_{u\in V} \frac{a(u,v)}{\|u\|_V} \neq 0 \right). \tag{1.3}$$

To interpret this theorem, let us define the operator $A : D(A) = V \subset L \to L$ such that $(Au,v)_L = a(u,v)$ for all $(u,v) \in V \times L$. Condition (1.2) is equivalent to assuming that A is injective and its range is closed, whereas (1.3) states that A^t is injective. As a result, these two conditions are equivalent to assuming that A is bijective [Bre91].

Now let us look at the approximation of (1.1). Let $V_h \subset V$ and $L_h \subset L$ be two finite-dimensional vectors spaces and consider the following discrete problem.

$$\begin{cases} \text{For } u_h \in V_h \text{ s.t.} \\ a(u_h,v_h) = (f,v_h)_L, \quad \forall v_h \in L_h. \end{cases} \tag{1.4}$$

Proposition 1.1. *Assume that* $\dim V_h = \dim L_h$ *and*

$$\exists \alpha_h > 0, \quad \forall w_h \in V_h, \quad \sup_{v_h \in L_h} \frac{a(w_h, v_h)}{\|v_h\|_L} \geq \alpha_h \|w_h\|_V. \tag{1.5}$$

Then, problem (1.4) has a unique solution and $\|u_h\|_V \leq \dfrac{1}{\alpha_h}\|f\|_L$.

Lemma 1.1 (Céa). *Under the hypotheses of throrem 1.1 and proposition 1.1 we have*

$$\|u - u_h\|_V \leq \left(1 + \frac{\|a\|}{\alpha_h}\right) \inf_{w_h \in V_h} \|u - w_h\|_V. \tag{1.6}$$

1.2. Example 1: advection/reaction. Let us consider an advection/reaction problem. Let β be a smooth vector field in $\mathbb{R}^d$, say $\beta \in L^\infty(\Omega)^d$ and $\nabla \cdot \beta \in L^\infty(\Omega)$, and set

$$\Gamma^- = \{x \in \Gamma \mid \beta(x) \cdot n(x) < 0\}, \quad \Gamma^+ = \{x \in \Gamma \mid \beta(x) \cdot n(x) > 0\}.$$

Γ^- is the inflow boundary and Γ^+ is the outflow boundary. It may happen that these two subsets of are empty if β is such that $\beta \cdot n(x) = 0$ for all $x \in \Gamma$. Let μ be a function in $L^\infty(\Omega)$. We introduce the following differential operator

$$A(u) = \mu u + \beta \cdot \nabla u.$$

To give a precise meaning to A, we introduce its domain

$$V = D(A) = \{w \in L^2(\Omega); \beta \cdot \nabla w \in L^2(\Omega)\} \subset L^2(\Omega).$$

When equipped with the norm $\|w\|_V = (\|w\|_{0,\Omega}^2 + \|\beta \cdot \nabla w\|_{0,\Omega}^2)^{1/2}$, it is clear that V is a Hilbert space and $A \in \mathcal{L}(V; L)$. In general A is not an isomorphism if we don't assume any other hypotheses on μ and β. Hereafter we assume that there is $\mu_0 > 0$ so that

$$\mu(x) - \frac{1}{2}\nabla \cdot \beta(x) \geq \mu_0 > 0 \quad \text{a.e. } x \text{ in } \Omega. \tag{1.7}$$

We define $V_0 = \{w \in V; w|_{\Gamma^-} = 0\}$. We introduce the bilinear form $a \in \mathcal{L}(V_0 \times L^2(\Omega); \mathbb{R})$ associated with the restriction of A to V_0:

$$a(u,v) = (\mu u + \beta \cdot \nabla u, v)_{0,\Omega}, \quad \forall u \in V_0, \quad \forall v \in L^2(\Omega). \tag{1.8}$$

Lemma 1.2. *The bilinear form defined in (1.8) satisfies the two condition of the Nečas theorem.*

The consequence of this lemma is that for all $f \in L^2(\Omega)$, the following problem

$$\begin{cases} \text{For } u \text{ in } V_0 \text{ s.t.} \\ a(u,v) = (f,v)_{0,\Omega}, \quad \forall v \in L^2(\Omega) \end{cases} \tag{1.9}$$

has a unique solution. Equivalently, it means that is an isomorphism.

Remark 1.1. If $\mu = 0$ and $\nabla \cdot \beta = 0$, the hypothesis (1.7) is not satisfied. Nevertheless, the conclusions of lemma 1.2 still hold if β is a filling field: i.e., if for almost every x in Ω, there is a characteristics of β that stars from x and reaches Γ^- in finite time. The reader is referred to Azerad and Pousin [Aze95] for other details on this problem.

1.3. Example 2: The Darcy equation. Let Ω be a porous medium characterized by the permeability tensor $K(x)$. This tensor is assumed to be symmetric positive definite and its smallest and largest eigen values are assumed to be bounded from below and from above uniformly in Ω. Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be a partition of Γ. We consider the following problem:

$$\begin{cases} K^{-1} \cdot u + \nabla p = f, \\ \nabla \cdot u = g, \\ u \cdot n|_{\Gamma_1} = 0, \quad p|_{\Gamma_2} = 0. \end{cases} \tag{1.10}$$

This problem is known as the Darcy problem. In nonlinear form, it plays an important role in underground storage problems, hydro-geology, and in the petroleum industry. It is very often coupled to a transport equation for the concentration of a chemical specie or a phase fraction.

To formulate (1.10) in weak form, we introduce some definitions.

$$\begin{aligned} X &= \{v \in L^2(\Omega)^d; \nabla \cdot v \in L^2(\Omega), v \cdot n|_{\Gamma_1} = 0\}, \\ \|v\|_X &= (\|v\|_{0,\Omega}^2 + \|\nabla \cdot v\|_{0,\Omega}^2)^{1/2}, \\ Y &= \{q \in L^2(\Omega); \nabla q \in L^2(\Omega), q|_{\Gamma_2} = 0\}, \\ \|q\|_Y &= \|q\|_{1,\Omega}. \end{aligned}$$

X and Y are Hilbert spaces. We set $V = X \times Y$ and $L = L^2(\Omega)^d \times L^2(\Omega)$ that we equip with the norms $\|(v,q)\|_V = (\|v\|_X^2 + \|q\|_Y^2)^{1/2}$ and $\|(v,q)\|_L = (\|v\|_{0,\Omega}^2 + \|q\|_{0,\Omega}^2)^{1/2}$ respectively. We now define the operator

$$\begin{aligned} &A : V \to L \\ &(v,q) \mapsto (K^{-1}v + \nabla q, \nabla \cdot v). \end{aligned}$$

A is clearly continuous. Finally, we introduce the bilinear form $a \in \mathcal{L}(V \times L; \mathbb{R})$ such that $a((u,p),(v,q)) = (A(u,p),(v,q))_L$.

Lemma 1.3. *The bilinear form a satisfies the two conditions of the Nečas theorem.*

The direct consequence of this lemma is that for all $f \in L^2(\Omega)^d$ and $q \in L^2(\Omega)$, the following problem

$$\begin{cases} \text{For } (u,p) \in V \text{ s.t.} \\ a((u,p),(v,q)) = ((f,g),(v,q))_L, \quad \forall (v,q) \in L, \end{cases} \tag{1.11}$$

has a unique solution.

1.4. A 1D model problem. Let us simplify the advection problem (1.9). Let $\Omega =]0,1[$ and set $\beta = 1$, $\mu = 0$. We define the Hilbert