THE CLASSIC WORK
NEWLY UPDATED AND REVISED

The Art of Computer Programming

VOLUME 1
Fundamental Algorithms
Third Edition

计算机程序设计艺术

第 1 卷 基本算法

(第 3 版)

[美] DONALD E. KNUTH 著

(英文影印版)

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The Art of Computer Programming Volume 1：Fundamental Algorithms (Third Edition).
Donald E.Knuth

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本套书由3卷组成

第1卷 基本算法

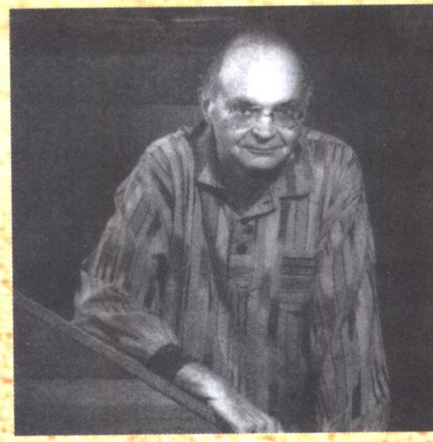
第1卷首先介绍编程的基本概念和技术，然后详细讲解信息结构方面的内容，包括信息在计算机内部的表示方法、数据元素之间的结构关系，以及有效的信息处理方法。此外，书中还描述了编程在模拟、数值方法、符号计算、软件与系统设计等方面的初级应用。新版本增加了数十项简单但重要的算法和技术，并根据当前研究发展趋势在数学预备知识方面做了大量修改。

第2卷 半数值算法

第2卷对半数值算法领域做了全面介绍，分“随机数”和“算术”两章。本卷总结了主要算法范例及这些算法的基本理论，广泛剖析了计算机程序设计与数值分析间的相互联系。第3版中最引人注目的是，Knuth对随机数生成器进行了重新处理，对形式幂级数计算作了深入讨论。

第3卷 排序和查找

这是对第3卷的头一次修订，不仅是对经典计算机排序和查找技术的最全面介绍，而且还对第1卷中的数据结构处理技术作了进一步的扩充，通盘考虑了大小型数据库和内外存储器。它遴选了一些经过反复检验的计算机方法，并对其效率做了定量分析。第3卷的突出特点是对“最优排序”一节作了修订，对排列论原理与通用散列法作了全新讨论。



Donald.E.Knuth（唐纳德.E.克努特，中文名高德纳）是算法和程序设计技术的先驱者，是计算机排版系统 TEX 和 METAFONT 的发明者，他因这些成就和大量创造性的影响深远的著作（19 部书和 160 篇论文）而誉满全球。作为斯坦福大学计算机程序设计艺术的荣誉退休教授，他当前正全神贯注于完成其关于计算机科学的史诗性的七卷集。这一伟大工程在 1962 年他还是加利福尼亚理工学院的研究生时就开始了。Knuth 教授获得了许多奖项和荣誉，包括美国计算机协会图灵奖（ACM Turing Award），美国前总统卡特授予的科学金奖（Medal of Science），美国数学学会斯蒂尔奖（AMS Steele Prize），以及 1996 年 11 月由于发明先进技术而荣获的备受推崇的京都奖（Kyoto Prize）。Knuth 教授现与其妻 Jill 生活于斯坦福校园内。

访问 Addison-Wesley 网站可以获得有关杰出的科学家和作者 Knuth 教授的更多信息：

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www-cs-faculty.stanford.edu/~knuth

This series of books is affectionately dedicated
to the Type 650 computer once installed at
Case Institute of Technology,
in remembrance of many pleasant evenings.

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PREFACE

Here is your book, the one your thousands of letters have asked us to publish. It has taken us years to do, checking and rechecking countless recipes to bring you only the best, only the interesting, only the perfect. Now we can say, without a shadow of a doubt, that every single one of them, if you follow the directions to the letter, will work for you exactly as well as it did for us, even if you have never cooked before.

— *McCall's Cookbook* (1963)

THE PROCESS of preparing programs for a digital computer is especially attractive, not only because it can be economically and scientifically rewarding, but also because it can be an aesthetic experience much like composing poetry or music. This book is the first volume of a multi-volume set of books that has been designed to train the reader in various skills that go into a programmer's craft.

The following chapters are *not* meant to serve as an introduction to computer programming; the reader is supposed to have had some previous experience. The prerequisites are actually very simple, but a beginner requires time and practice in order to understand the concept of a digital computer. The reader should possess:

a) Some idea of how a stored-program digital computer works; not necessarily the electronics, rather the manner in which instructions can be kept in the machine's memory and successively executed.

b) An ability to put the solutions to problems into such explicit terms that a computer can "understand" them. (These machines have no common sense; they do exactly as they are told, no more and no less. This fact is the hardest concept to grasp when one first tries to use a computer.)

c) Some knowledge of the most elementary computer techniques, such as looping (performing a set of instructions repeatedly), the use of subroutines, and the use of indexed variables.

d) A little knowledge of common computer jargon — "memory," "registers," "bits," "floating point," "overflow," "software." Most words not defined in the text are given brief definitions in the index at the close of each volume.

These four prerequisites can perhaps be summed up into the single requirement that the reader should have already written and tested at least, say, four programs for at least one computer.

I have tried to write this set of books in such a way that it will fill several needs. In the first place, these books are reference works that summarize the knowledge that has been acquired in several important fields. In the second place, they can be used as textbooks for self-study or for college courses in the computer and information sciences. To meet both of these objectives, I have incorporated a large number of exercises into the text and have furnished answers for most of them. I have also made an effort to fill the pages with facts rather than with vague, general commentary.

This set of books is intended for people who will be more than just casually interested in computers, yet it is by no means only for the computer specialist. Indeed, one of my main goals has been to make these programming techniques more accessible to the many people working in other fields who can make fruitful use of computers, yet who cannot afford the time to locate all of the necessary information that is buried in technical journals.

We might call the subject of these books "nonnumerical analysis." Computers have traditionally been associated with the solution of numerical problems such as the calculation of the roots of an equation, numerical interpolation and integration, etc., but such topics are not treated here except in passing. Numerical computer programming is an extremely interesting and rapidly expanding field, and many books have been written about it. Since the early 1960s, however, computers have been used even more often for problems in which numbers occur only by coincidence; the computer's decision-making capabilities are being used, rather than its ability to do arithmetic. We have some use for addition and subtraction in nonnumerical problems, but we rarely feel any need for multiplication and division. Of course, even a person who is primarily concerned with numerical computer programming will benefit from a study of the nonnumerical techniques, for they are present in the background of numerical programs as well.

The results of research in nonnumerical analysis are scattered throughout numerous technical journals. My approach has been to try to distill this vast literature by studying the techniques that are most basic, in the sense that they can be applied to many types of programming situations. I have attempted to coordinate the ideas into more or less of a "theory," as well as to show how the theory applies to a wide variety of practical problems.

Of course, "nonnumerical analysis" is a terribly negative name for this field of study; it is much better to have a positive, descriptive term that characterizes the subject. "Information processing" is too broad a designation for the material I am considering, and "programming techniques" is too narrow. Therefore I wish to propose *analysis of algorithms* as an appropriate name for the subject matter covered in these books. This name is meant to imply "the theory of the properties of particular computer algorithms."

The complete set of books, entitled *The Art of Computer Programming*, has the following general outline:

Volume 1. Fundamental Algorithms

Chapter 1. Basic Concepts
Chapter 2. Information Structures

Volume 2. Seminumerical Algorithms
Chapter 3. Random Numbers
Chapter 4. Arithmetic

Volume 3. Sorting and Searching
Chapter 5. Sorting
Chapter 6. Searching

Volume 4. Combinatorial Algorithms
Chapter 7. Combinatorial Searching
Chapter 8. Recursion

Volume 5. Syntactical Algorithms
Chapter 9. Lexical Scanning
Chapter 10. Parsing

Volume 4 deals with such a large topic, it actually represents three separate books (Volumes 4A, 4B, and 4C). Two additional volumes on more specialized topics are also planned: Volume 6, *The Theory of Languages* (Chapter 11); Volume 7, *Compilers* (Chapter 12).

I started out in 1962 to write a single book with this sequence of chapters, but I soon found that it was more important to treat the subjects in depth rather than to skim over them lightly. The resulting length of the text has meant that each chapter by itself contains more than enough material for a one-semester college course; so it has become sensible to publish the series in separate volumes. I know that it is strange to have only one or two chapters in an entire book, but I have decided to retain the original chapter numbering in order to facilitate cross-references. A shorter version of Volumes 1 through 5 is planned, intended specifically to serve as a more general reference and/or text for undergraduate computer courses; its contents will be a subset of the material in these books, with the more specialized information omitted. The same chapter numbering will be used in the abridged edition as in the complete work.

The present volume may be considered as the "intersection" of the entire set, in the sense that it contains basic material that is used in all the other books. Volumes 2 through 5, on the other hand, may be read independently of each other. Volume 1 is not only a reference book to be used in connection with the remaining volumes; it may also be used in college courses or for self-study as a text on the subject of *data structures* (emphasizing the material of Chapter 2), or as a text on the subject of *discrete mathematics* (emphasizing the material of Sections 1.1, 1.2, 1.3.3, and 2.3.4), or as a text on the subject of *machine-language programming* (emphasizing the material of Sections 1.3 and 1.4).

The point of view I have adopted while writing these chapters differs from that taken in most contemporary books about computer programming in that I am not trying to teach the reader how to use somebody else's software. I am concerned rather with teaching people how to write better software themselves.

My original goal was to bring readers to the frontiers of knowledge in every subject that was treated. But it is extremely difficult to keep up with a field

a) A programmer is greatly influenced by the language in which programs are written; there is an overwhelming tendency to prefer constructions that are simplest in that language, rather than those that are best for the machine. By understanding a machine-oriented language, the programmer will tend to use a much more efficient method; it is much closer to reality.

b) The programs we require are, with a few exceptions, all rather short, so with a suitable computer there will be no trouble understanding the programs.

c) High-level languages are inadequate for discussing important low-level details such as coroutine linkage, random number generation, multi-precision arithmetic, and many problems involving the efficient usage of memory.

d) A person who is more than casually interested in computers should be well schooled in machine language, since it is a fundamental part of a computer.

e) Some machine language would be necessary anyway as output of the software programs described in many of the examples.

f) New algebraic languages go in and out of fashion every five years or so, while I am trying to emphasize concepts that are timeless.

From the other point of view, I admit that it is somewhat easier to write programs in higher-level programming languages, and it is considerably easier to debug the programs. Indeed, I have rarely used low-level machine language for my own programs since 1970, now that computers are so large and so fast. Many of the problems of interest to us in this book, however, are those for which the programmer's art is most important. For example, some combinatorial calculations need to be repeated a trillion times, and we save about 11.6 days of computation for every microsecond we can squeeze out of their inner loop. Similarly, it is worthwhile to put an additional effort into the writing of software that will be used many times each day in many computer installations, since the software needs to be written only once.

Given the decision to use a machine-oriented language, which language should be used? I could have chosen the language of a particular machine X, but then those people who do not possess machine X would think this book is only for X-people. Furthermore, machine X probably has a lot of idiosyncrasies that are completely irrelevant to the material in this book yet which must be explained; and in two years the manufacturer of machine X will put out machine $X+1$ or machine $10X$, and machine X will no longer be of interest to anyone.

To avoid this dilemma, I have attempted to design an "ideal" computer with very simple rules of operation (requiring, say, only an hour to learn), which also resembles actual machines very closely. There is no reason why a student should be afraid of learning the characteristics of more than one computer; once one machine language has been mastered, others are easily assimilated. Indeed, serious programmers may expect to meet many different machine languages in the course of their careers. So the only remaining disadvantage of a mythical machine is the difficulty of executing any programs written for it. Fortunately, that is not really a problem, because many volunteers have come forward to that is economically profitable, and the rapid rise of computer science has made such a dream impossible. The subject has become a vast tapestry with tens of thousands of subtle results contributed by tens of thousands of talented people all over the world. Therefore my new goal has been to concentrate on "classic" techniques that are likely to remain important for many more decades, and to describe them as well as I can. In particular, I have tried to trace the history of each subject, and to provide a solid foundation for future progress. I have attempted to choose terminology that is concise and consistent with current usage. I have tried to include all of the known ideas about sequential computer programming that are both beautiful and easy to state.

A few words are in order about the mathematical content of this set of books. The material has been organized so that persons with no more than a knowledge of high-school algebra may read it, skimming briefly over the more mathematical portions; yet a reader who is mathematically inclined will learn about many interesting mathematical techniques related to discrete mathematics. This dual level of presentation has been achieved in part by assigning ratings to each of the exercises so that the primarily mathematical ones are marked specifically as such, and also by arranging most sections so that the main mathematical results are stated *before* their proofs. The proofs are either left as exercises (with answers to be found in a separate section) or they are given at the end of a section.

A reader who is interested primarily in programming rather than in the associated mathematics may stop reading most sections as soon as the mathematics becomes recognizably difficult. On the other hand, a mathematically oriented reader will find a wealth of interesting material collected here. Much of the published mathematics about computer programming has been faulty, and one of the purposes of this book is to instruct readers in proper mathematical approaches to this subject. Since I profess to be a mathematician, it is my duty to maintain mathematical integrity as well as I can.

A knowledge of elementary calculus will suffice for most of the mathematics in these books, since most of the other theory that is needed is developed herein. However, I do need to use deeper theorems of complex variable theory, probability theory, number theory, etc., at times, and in such cases I refer to appropriate textbooks where those subjects are developed.

The hardest decision that I had to make while preparing these books concerned the manner in which to present the various techniques. The advantages of flow charts and of an informal step-by-step description of an algorithm are well known; for a discussion of this, see the article "Computer-Drawn Flowcharts" in the ACM *Communications*, Vol. 6 (September 1963), pages 555–563. Yet a formal, precise language is also necessary to specify any computer algorithm, and I needed to decide whether to use an algebraic language, such as ALGOL or FORTRAN, or to use a machine-oriented language for this purpose. Perhaps many of today's computer experts will disagree with my decision to use a machine-oriented language, but I have become convinced that it was definitely the correct choice, for the following reasons:

write simulators for the hypothetical machine. Such simulators are ideal for instructional purposes, since they are even easier to use than a real computer would be.

I have attempted to cite the best early papers in each subject, together with a sampling of more recent work. When referring to the literature, I use standard abbreviations for the names of periodicals, except that the most commonly cited journals are abbreviated as follows:

CACM = Communications of the Association for Computing Machinery

JACM = Journal of the Association for Computing Machinery

Comp. J. = The Computer Journal (British Computer Society)

Math. Comp. = Mathematics of Computation

AMM = American Mathematical Monthly

SICOMP = SIAM Journal on Computing

FOCS = IEEE Symposium on Foundations of Computer Science

SODA = ACM–SIAM Symposium on Discrete Algorithms

STOC = ACM Symposium on Theory of Computing

Crelle = Journal für die reine und angewandte Mathematik

As an example, "*CACM* **6** (1963), 555–563" stands for the reference given in a preceding paragraph of this preface. I also use "*CMath*" to stand for the book *Concrete Mathematics*, which is cited in the introduction to Section 1.2.

Much of the technical content of these books appears in the exercises. When the idea behind a nontrivial exercise is not my own, I have attempted to give credit to the person who originated that idea. Corresponding references to the literature are usually given in the accompanying text of that section, or in the answer to that exercise, but in many cases the exercises are based on unpublished material for which no further reference can be given.

I have, of course, received assistance from a great many people during the years I have been preparing these books, and for this I am extremely thankful. Acknowledgments are due, first, to my wife, Jill, for her infinite patience, for preparing several of the illustrations, and for untold further assistance of all kinds; secondly, to Robert W. Floyd, who contributed a great deal of his time towards the enhancement of this material during the 1960s. Thousands of other people have also provided significant help — it would take another book just to list their names! Many of them have kindly allowed me to make use of hitherto unpublished work. My research at Caltech and Stanford was generously supported for many years by the National Science Foundation and the Office of Naval Research. Addison–Wesley has provided excellent assistance and cooperation ever since I began this project in 1962. The best way I know how to thank everyone is to demonstrate by this publication that their input has led to books that resemble what I think they wanted me to write.

Preface to the Third Edition

After having spent ten years developing the TeX and METAFONT systems for computer typesetting, I am now able to fulfill the dream that I had when I began that work, by applying those systems to *The Art of Computer Programming*. At last the entire text of this book has been captured inside my personal computer, in an electronic form that will make it readily adaptable to future changes in printing and display technology. The new setup has allowed me to make literally thousands of improvements that I have been wanting to incorporate for a long time.

In this new edition I have gone over every word of the text, trying to retain the youthful exuberance of my original sentences while perhaps adding some more mature judgment. Dozens of new exercises have been added; dozens of old exercises have been given new and improved answers.

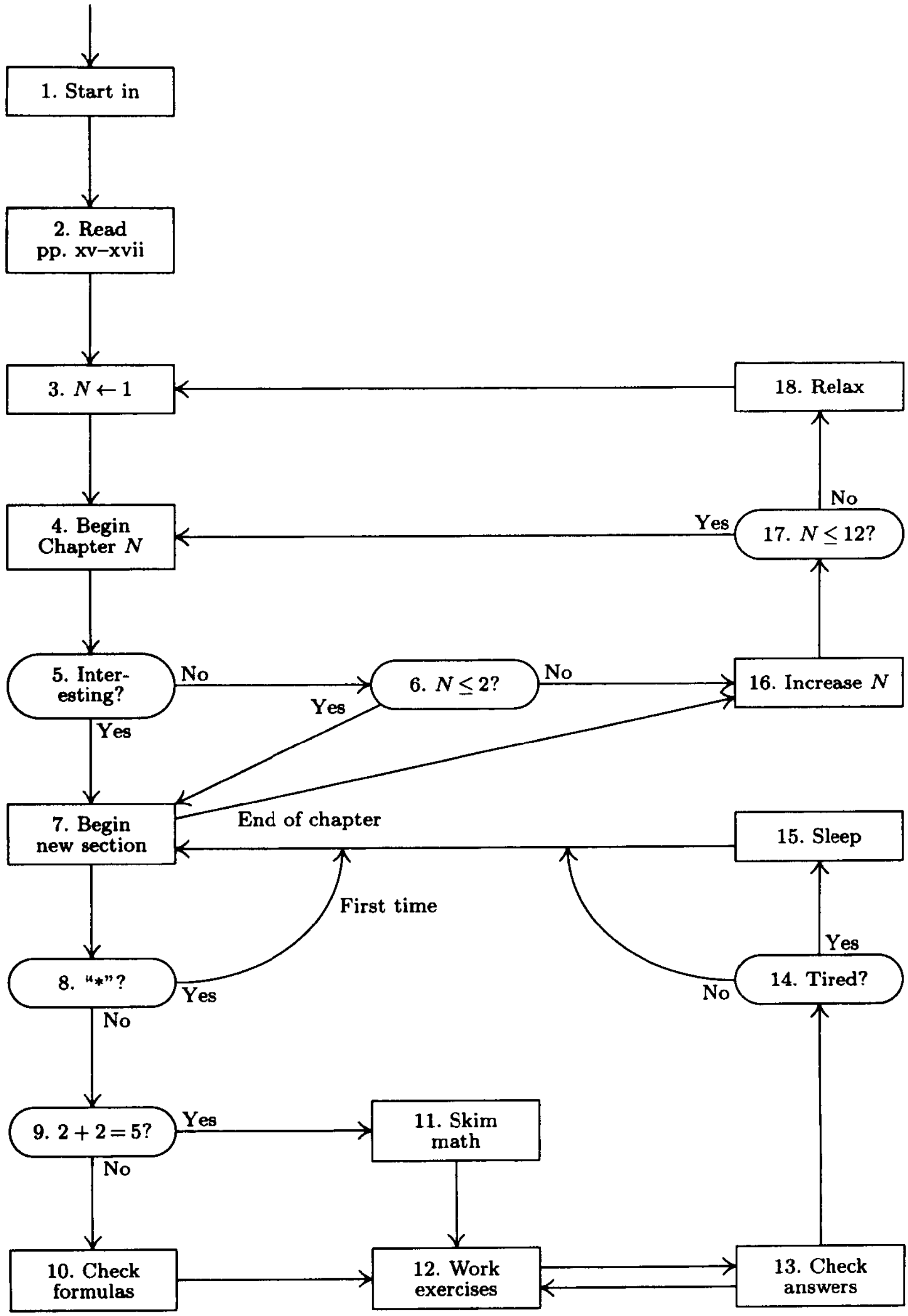
The Art of Computer Programming is, however, still a work in progress. Therefore some parts of this book are headed by an "under construction" icon, to apologize for the fact that the material is not up-to-date. My files are bursting with important material that I plan to include in the final, glorious, fourth edition of Volume 1, perhaps 15 years from now; but I must finish Volumes 4 and 5 first, and I do not want to delay their publication any more than absolutely necessary.

Most of the hard work of preparing the new edition was accomplished by Phyllis Winkler and Silvio Levy, who expertly keyboarded and edited the text of the second edition, and by Jeffrey Oldham, who converted nearly all of the original illustrations to METAPOST format. I have corrected every error that alert readers detected in the second edition (as well as some mistakes that, alas, nobody noticed); and I have tried to avoid introducing new errors in the new material. However, I suppose some defects still remain, and I want to fix them as soon as possible. Therefore I will cheerfully pay $2.56 to the first finder of each technical, typographical, or historical error. The webpage cited on page iv contains a current listing of all corrections that have been reported to me.

Stanford, California
April 1997

D. E. K.

Things have changed in the past two decades.
— BILL GATES (1995)



Flow chart for reading this set of books.

Procedure for Reading This Set of Books

1. Begin reading this procedure, unless you have already begun to read it. *Continue to follow the steps faithfully.* (The general form of this procedure and its accompanying flow chart will be used throughout this book.)
2. Read the Notes on the Exercises, on pages xv–xvii.
3. Set N equal to 1.
4. Begin reading Chapter N. Do *not* read the quotations that appear at the beginning of the chapter.
5. Is the subject of the chapter interesting to you? If so, go to step 7; if not, go to step 6.
6. Is $N \leq 2$? If not, go to step 16; if so, scan through the chapter anyway. (Chapters 1 and 2 contain important introductory material and also a review of basic programming techniques. You should at least skim over the sections on notation and about MIX.)
7. Begin reading the next section of the chapter; if you have already reached the end of the chapter, however, go to step 16.
8. Is section number marked with "*"? If so, you may omit this section on first reading (it covers a rather specialized topic that is interesting but not essential); go back to step 7.
9. Are you mathematically inclined? If math is all Greek to you, go to step 11; otherwise proceed to step 10.
10. Check the mathematical derivations made in this section (and report errors to the author). Go to step 12.
11. If the current section is full of mathematical computations, you had better omit reading the derivations. However, you should become familiar with the basic results of the section; they are usually stated near the beginning, or in *slanted type* right at the very end of the hard parts.
12. Work the recommended exercises in this section in accordance with the hints given in the Notes on the Exercises (which you read in step 2).
13. After you have worked on the exercises to your satisfaction, check your answers with the answer printed in the corresponding answer section at the

rear of the book (if any answer appears for that problem). Also read the answers to the exercises you did not have time to work. *Note:* In most cases it is reasonable to read the answer to exercise n before working on exercise $n+1$, so steps 12–13 are usually done simultaneously.

14. Are you tired? If not, go back to step 7.

15. Go to sleep. Then, wake up, and go back to step 7.

16. Increase N by one. If $N = 3, 5, 7, 9, 11$, or 12, begin the next volume of this set of books.

17. If N is less than or equal to 12, go back to step 4.

18. Congratulations. Now try to get your friends to purchase a copy of Volume 1 and to start reading it. Also, go back to step 3.

Woe be to him that reads but one book.
— GEORGE HERBERT, *Jacula Prudentum*, 1144 (1640)

Le défaut unique de tous les ouvrages
c'est d'être trop longs.
— VAUVENARGUES, *Réflexions*, 628 (1746)

Books are a triviality. Life alone is great.
— THOMAS CARLYLE, *Journal* (1839)