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Stokes方程和Navier-Stokes方程谱方法，抛物型和双曲型方程时空谱方法

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SHANGHAI UNIVERSITY DOCTORAL DISSERTATION (2003)

Spectral Methods for the Stokes Equations and the Navier–Stokes Equations, Time-Space Spectral Methods for Parabolic Equations and Hyperbolic Equations

Candidate: Tang Jianguo
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本论文经答辩委员会全体委员审查，确认符合上海大学博士学位论文质量要求.

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答辩委员会对论文的评语

Navier-Stokes 方程是流体力学中经典的模型问题，其数值方法是计算数学中的研究热点之一。论文分析了该方程现有的数值方法，研究了一类新的 Legendre 谱方法，构造了满足弱不可压缩条件的基底，证明了数值方法的误差估计，所得的结果是系统的，具有理论及实际应用价值，所采用的技巧也有新意。论文还研究了某些发展型方程时间方向谱方法，得到一些有意义的结果。论文的研究成果表明作者基础理论扎实，具有科学研究和创新能力。答辩委员会认为这是一篇优秀的博士论文。

答辩委员会表决结果

经答辩委员会表决，5票全票同意通过该生的博士论文答辩，建议授予理学博士学位。

答辩委员会主席：郭本瑜

2003年元月3日

摘 要

本文主要研究偏微分方程三大数值方法之一的谱方法. 我们选取了两个比较典型的模型，即流体力学中描述流体状态 (速度场和压力场) 的著名经典方程 —— 广义 Stokes 方程和 Navier–Stokes 方程来阐述我们所提出的新的谱方法. 这两个模型已成为检验一种数值方法优越与否的试金石. 其背景是，自从这两个方程一提出，数学物理工作者在探求如何构造计算简单、实施方便、又能保持高精度的一类格式方面所做的努力从来就没有停止过，每年在国际著名的计算数学杂志上有成百上千篇这方面的文献资料. 它们分别从不同的侧面分析研究了这两个方程的数值方法，提出了大量的算法. 但遗憾的是，由于问题本身的高难度和所采用方法的某些局限性，到目前为止，对广义 Stokes(或 Stokes) 方程我们还没有见到其多区域谱方法对速度的 L^2- 最优收敛阶. 我们在深刻分析了这两问题本质的基础上，通过散度 - 自由函数提出了新的速度与压力分离的单区域和多区域谱格式，较好地解决了理论分析与算法之间的矛盾.

本文研究的另一类问题是时空谱方法. 对于发展型偏微分

方程，经典的谱方法的做法是：在空间方向采用谱格式，而在时间方向采用差分格式. 然而当精确解在时空方向都有较好的光滑性时，数值逼近解的整体精度将受到时间方向有限差分格式的限制. 近年来，已有一些学者关注这一问题，正在探求一种整体高精度的逼近方法，从而提出了在时间方向也采用谱方法，即时空谱方法. 本文中我们在分析了大量这方面文献资料的基础上，以抛物型和双曲型偏微分方程为模型提出了一种新的时空谱方法.

首先，我们对广义 Stokes 方程和 Navier–Stokes 方程考虑了速度与压力分离的弱形式，以此为基础构造了广义 Stokes 方程和 Navier–Stokes 方程的单区域和多区域 Legendre 谱格式. 在误差分析中，我们首次引入了对误差估计起重要作用的一类投影算子 —— 单区域散度 - 自由投影算子 P_N^{div} 和多区域散度 - 自由投影算子 P_N^{div}. 通过对空间插值和方程对偶技术的巧妙运用，我们得到了单区域和多区域的这两个投影算子的 H^1- 和 L^2- 最优估计. 由此构造了两个分析具有散度 - 自由特征问题的通用的、强有力的工具. 将它们运用于广义 Stokes 方程，得到了速度的单区域和多区域 Legendre 谱格式的 H^1- 和 L^2- 最优收敛速率. 将它们运用于 Navier–Stokes 方程，虽然没有得到最优收敛速率 (仅相差半个阶)，但是改进了以往在这一问题的收敛阶.

其次，我们以抛物型和双曲型偏微分方程为模型，对周期和非周期边界条件情形，建立了相应的单区间和多区间时空 Legendre

谱方法. 考虑到一阶微分算子的不对称性，在构造时空谱格式时，时间方向的试探函数空间的选取将不同于检验函数空间. 与单区间方法相比，时间方向多区间谱方法具有以下优点：第一，可以减少问题的规模，并能进行并行计算，特别对于时间区间较长的情形需要对时间区间进行剖分；第二，在计算中具有更大的灵活性，即根据解的性质子区间可以取不同的长度以及在每个子区间上选取不同次数的多项式. 我们分别分析了时间单区间和多区间方法，并给出了误差估计，数值结果进一步印证了理论分析的结果. 从理论分析、算法描述 (包括运算量的分析) 和数值实验这三个方面综合表明该方法具有计算简单、高效可靠、实施方便、又能保持高精度的优点. 该方法对变系数问题同样适用，具有很好的推广应用价值.

最后，本文在对所研究的方程得到较好收敛阶的基础上，详尽地描述了格式的算法实施. 首先对于广义 Stokes 方程单区域格式，通过合理选取速度逼近空间的基函数，所得方程组的系数矩阵为一个多对角矩阵，求解十分方便. 而求压力则不需要解任何方程组，仅需计算内积，运算量相对于速度的运算量可以忽略不计. 详细给出了计算速度和压力的运算量. 对于广义 Stokes 方程多区域格式，我们构造了多区域格式的一个并行算法，也给出了此种情形计算速度和压力的运算量. 数值实验的结果与理论分析相互映衬，表明了该方法的高精度和优越性. 对于 Navier–Stokes

方程我们采用了显式格式，在每一时间步， Navier–Stokes 方程的单区域和多区域格式即成为广义 Stokes 方程的速度与压力分离的 Legendre 谱格式，因此可十分方便地求解并可构造多区域格式的并行算法．其次对于时空谱方法构造了实施方便的算法，并对多区间格式构造了并行算法，详细地给出了算法的运算量．

关键词：广义 Stokes 方程，Navier–Stokes 方程，区域 (间) 分解，散度 - 自由函数空间，散度 - 自由投影算子， Legendre 谱方法，H^1- 和 L^2- 最优误差估计，抛物型方程，双曲型方程，时空谱方法

Abstract

In this thesis the spectral methods which are one of three kinds of numerical methods for partial differential equations are investigated. We take two typical model problems, that is, the famous, classical equations for describing the velocity and pressure of fluids in fluid dynamics——the generalized Stokes equation and the Navier–Stokes equation, to demonstrate our new spectral methods. These two models have become touchstones of measuring the advantage and the disadvantage of a kind of numerical methods. The background is that mathematicians and physicists have been ceaselessly searching for how to construct a kind of schemes with characteristics such as simpilcity computation, convenient performance and keeping on high accuracy since these two problems were proposed. Hundreds even thousands of references in this field are appeared in the international famous computational mathematics journals every year. A great deal numerical methods of these two models have been studied in various aspects and a lot of algorithms

have been proposed. But it is a pity that up to now, the L^2-optimal convergence order of spectral methods in multi-domain cases for the generalized Stokes (or Stokes) equations can not be found since the high difficulty of the model and limitations of the adopted methods. On the basis of analyzing the essence of these two problems profoundly, we proposed the single and multi-domain Legendre spectral schemes via divergence-free functions, which decoupled the velocity and the pressure, and thereby the clash between theoretical analysis and algorithms are eliminated.

In this thesis the other kind of objects we studied are the time-space spectral methods. For time-dependent partial differential equations, if the spectral scheme is used in spatial, the difference scheme is usually adopted in time. However, when the exact solution is very smooth both in spatial and in time, the accuracy of the approximate solution would be limited by the finite difference scheme in time. In recent years, some scholars have noted this problem, and they have been seeking for a kind of integrant high accuracy methods of approximation, and then the spectral methods in time have been proposed. On the basis of analyzing a great deal references in this field, we take parabolic and hyperbolic equations as models to present the new time-space spectral methods.

Firstly, we consider two decoupled weak formulations with respect to the velocity and the pressure for the generalized Stokes equation and the Navier–Stokes equation. On the basis of these, we construct decoupled single and multi-domain Legendre spectral schemes with respect to the velocity and the pressure. We introduce a kind of projection operators which play the important roles in the error analysis ——the single domain divergence-free projection operator P_N^{div} and the multi-domain divergence-free projection operator $P_{\mathcal{N}}^{\mathrm{div}}$. By ingeniously using the interpolation of spaces and the technique of duality, we obtain the H^1- and L^2-optimal estimates for these two projection operators. Thereby two common, strongly useful tools for analyzing the problems with divergence-free characteristics are constructed. Applying them to the generalized Stokes equation, the H^1- and L^2-optimal error estimates are achieved for the velocity for the single and multi-domain Legendre spectral schemes; applying them to the Navier–Stokes equation, although the optimal error estimates are not achieved (only a half order is lost), the convergence order of this problem is improved.

Secondly, we take the parabolic and hyperbolic equations with the periodic and non-periodic boundary conditions as models to establish the single and multi-interval Legendre spectral methods in time. Based on the nature of nonsymetry of the first-order partial differential operator in time, we let the trial function spaces differ from the test function spaces so that better convergence results are achieved. The multi-interval spectral method in time has some merits compared with the single interval one. It can reduce the scale of the problems and allow for the best use of parallel computers. Also, there is more flexibility in computation, i.e., one may vary mesh sizes of subintervals or choose polynomials of different degrees in each subinterval according to properties of the solutions. We analyze the single and multi-interval schemes in time and obtain better error estimates. Numerical results are coincide well with the theoretical analysis. The theoretical analysis, algorithm descriptions (including the analysis of arithmetic operations) and numerical experiments show that the methods have some merits, such as simplicity computation, reliable efficiency, convenient performance and high order accuracy etc.. The methods are suitable for variable coefficient problems. We believe that they are useful for more time-dependent partial differential equations.

Finally, after better convergence order of the problems is achieved, we describe the implementations of algorithms exhaustively. For the single domain scheme of the generalized Stokes equation, the obtained systems have multi-diagonal forms after choosing resonable basis functions of the velocity approximation spaces, and it is very convenience for solving them. It suffices to compute inner products for computing pressure and need not solve any system. The operations for computing presssure relative to those for computing velocity can be ignored. We give the operations for computing velocity and pressure with detail. For the multi-domain scheme of the generalized Stokes equation, we construct a parallel algorithm, and give also the operations for computing velocity and pressure in this case. Numerical results are coincide well with theoretical analysis, which show the high order accuracy and advantage of the methods. For the Navier–Stokes equation, we use the explicit scheme, and then in each step of time, the single and multi-domain Legendre spectral schemes for it become the the single and multi-domain Legendre spectral schemes for the generalized Stokes equation. Therefore it is very convenience for solving them and can construct a parallel algorithm for the multi-domain scheme. Then, we construct convenient performance algorithms for the time-space spectral methods, and construct parallel algorithms for the multi-interval scheme, and give the operations of the algorithms with detail.