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第二册

Volume 2

主　编　程代展　李　川

副主编　陈　杰　段广仁　黄　捷　贾英民　李少远

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内容简介

本书共收入1026篇论文。这些论文经过中国自动化学会控制理论专业委员会组织评审，为第27届中国控制会议正式发表论文。论文内容包括系统理论与控制理论，非线性系统及其控制，复杂性与复杂系统理论，分布参数系统，混杂系统与DEDS，大系统，随机系统，稳定性与镇定，建模、辨识与信号处理，最优控制与优化，鲁棒控制与 H_∞ 控制，自适应控制与学习控制，变结构控制，神经网络，模糊系统与模糊控制，模式识别，控制设计方法，遗传算法与演化计算，运动控制，智能机器人，分布式控制系统，信息处理系统，故障诊断，通讯网络系统，CIMS与制造系统，交通系统，生物与生态系统，社会经济系统，工业系统等领域的应用研究成果。

本书可供从事自动控制理论及应用研究的高等院校教师和研究生、科研单位的研究人员以及工业部门的工程技术人员研究参考。

本书进入 IEEE 会议出版程序，论文可从 IEEE Xplore 下载。2006 年起，CCC 论文集被 EI(Enginnering Index)收录。

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目录 (Contents)

系统理论与控制理论 (System Theory and Control Theory)

非线性系统及其控制 (Nonlinear System and Control)

复杂性与复杂系统理论 (Complexity and Complex System Theory)

分布参数系统 (Distributed Parameter Systems)

混杂系统与 DEDS (Hybrid Systems and DEDS)

大系统 (Large Scale Systems)

随机系统 (Stochastic Systems)

系统理论与控制理论

System Theory and Control Theory

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Parameterized Solution to Generalized Sylvester Matrix Equation*

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Abstract: The paper considers the parameterized solution to generalized Sylvester matrix equation. A set of matrix computation formulas have been developed. Using them, formulas for converting generalized Sylvester matrix mappings into conventional linear mappings are obtained. Based on this equation, an easily computable numerical algorithm for complete parameterized solutions of generalized Sylvester matrix equation is provided. The standard Sylvester matrix equation $AX - EXF = BY$ and its dual equation $XA - FXE = YC$ are considered as its special cases and then the corresponding solutions can be produced easily. Some further properties are also investigated. Comparing with existing algorithms, new approach simplified the computation significantly.

Key Words: Sylvester matrix equation, Parameterized solution

1 INTRODUCTION

Sylvester matrix equation has many applications in control theory. Particularly, when a singular control system is considered, it is used for many design of controls. Say, pole placement, tracking, design of Luenberger observers etc. So it has been studied widely [6, 7, 8, 9, 11, 12] and the references therein for details.

A generalized Sylvester matrix equation considered in this paper is of the following form

$$AX+XB+CXD+EX^{\mathrm{T}}F=GY+MYN+PY^{\mathrm{T}}Q \quad (1)$$

where $A, C \in M_{n\times n}$, $B, D, N \in M_{p\times p}$, $E, F, P \in M_{n\times p}$, $M, G \in M_{n\times r}$, $Q \in M_{r\times p}$, with unknowns $X \in M_{n\times p}$ and $Y \in M_{r\times p}$. We use $M_{m\times n}$ for the set of $m \times n$ matrices. One can easily check that as the equation is liner with respect to X and Y, equation (1) is most general, as the dimension matching condition of matrix product is taking into consideration.

Several methods have been proposed for solving Sylvester equation and its dual equations [4, 5, 10]. Since equation (1) is linear with respect to X and Y, it is very natural to consider converting it into a standard linear algebraic equation. If it can be realized, finding parameterized solution is standard. Indeed, it can be achieved, but we need a set of notations and computation formulas for matrices. We briefly describe them here and refer to [2] for more details.

Let $A = (a_{ij}) \in M_{m\times n}$. Its column stacking form is expressed as

$$V_c(A) = (a_{11}, a_{21}, \cdots, a_{m1}, \cdots, a_{1n}, a_{2n}, \cdots, a_{mn})^{\mathrm{T}} \quad (2)$$

Its row stacking form is

$$V_r(A) = (a_{11}, a_{12}, \cdots, a_{1n}, \cdots, a_{m1}, a_{m2}, \cdots, a_{mn})^{\mathrm{T}} \quad (3)$$

Let $x = (x_i) \in \mathbb{R}^{mn}$. Then

1.

$$V_c^{-1}(x, m) = \begin{bmatrix} x_1 & x_{m+1} & \cdots & x_{(n-1)m+1} \\ x_2 & x_{m+2} & \cdots & x_{(n-1)m+2} \\ \vdots & & & \\ x_m & x_{2m} & \cdots & x_{nm} \end{bmatrix} \quad (4)$$

2.

$$V_r^{-1}(x, n) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_{n+1} & x_{n+2} & \cdots & x_{2n} \\ \vdots & & & \\ x_{(m-1)n+1} & x_{(m-1)n+2} & \cdots & x_{mn} \end{bmatrix} \quad (5)$$

Next, we convert a linear matrix mapping into a conventional linear mapping. Given a mapping $\rho : X \mapsto \rho(X)$, where $X \in M_{n\times p}$. Say, we use column stacking form. Denote $x = V_c(X) \in \mathbb{R}^{np}$, $y = V_c(\rho(X))$, and assume the matrix mapping ρ is a linear mapping, with its matrix form M_ρ^c, which means

$$y = V_c(\rho(X)) = M_\rho^c x \quad (6)$$

For various linear matrix mappings we can construct their matrix forms respectively. The followings are some typical ones.

Theorem 1 Assume $A \in M_{m\times n}$, $B \in M_{p\times q}$, $C \in M_{m\times p}$, $C \in M_{n\times q}$, and $X \in M_{n\times p}$.

1. If $\rho : X \mapsto AX$, then
$$M_\rho^c = I_p \otimes A \quad (7)$$
2. If $\rho : X \mapsto XB$, then
$$M_\rho^c = B^{\mathrm{T}} \otimes I_n \quad (8)$$
3. If $\rho : X \mapsto CX^{\mathrm{T}}$, then
$$M_\rho^c = (I_n \otimes C)W_{[p,n]} \quad (9)$$
4. If $\rho : X \mapsto X^{\mathrm{T}}D$, then
$$M_\rho^c = (D^{\mathrm{T}} \otimes I_p)W_{[p,n]} \quad (10)$$
5. If $\rho : X \mapsto AXB + CX^{\mathrm{T}}D$, then
$$M_\rho^c = (B^{\mathrm{T}} \otimes A) + (D^{\mathrm{T}} \otimes C)W_{[p,n]} \quad (11)$$

Note that in the above $\otimes$ is the Kronecker product, and $W_{[m,n]}$ is a swap matrix. Let $A \in M_{m\times n}$, then

$$\begin{cases} W_{[m,n]}V_r(A) = V_c(A) \\ W_{[n,m]}V_c(A) = V_r(A) \end{cases} \quad (12)$$

The swap property (12) uniquely determin $W_{[n,m]}$. We refer to [2] or [1] for details.

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Next, we use row stacking form. That is, for $\rho : X \mapsto \rho(X)$, we denote $x = V_r(X)$, $y = V_r(AX)$, and express the matrix form of ρ by M_ρ^r, which means

$$y = V_r(\rho(X)) = M_\rho^r x \tag{13}$$

Similar to Theorem 1, under row stacking form we have

Theorem 2 Assume $A \in M_{m\times n}$, $B \in M_{p\times q}$, $C \in M_{m\times p}$, $C \in M_{n\times q}$, and $X \in M_{n\times p}$.

1. If $\rho : X \mapsto AX$, then
$$M_\rho^r = A \otimes I_p \tag{14}$$
2. If $\rho : X \mapsto XB$, then
$$M_\rho^r = I_n \otimes B^{\mathrm{T}} \tag{15}$$
3. If $\rho : X \mapsto CX^{\mathrm{T}}$, then
$$M_\rho^r = (C \otimes I_n)W_{[n,p]} \tag{16}$$
4. If $\rho : X \mapsto X^{\mathrm{T}}D$, then
$$M_\rho^r = (I_p \otimes D^{\mathrm{T}})W_{[n,p]} \tag{17}$$
5. If $\rho : X \mapsto AXB + CX^{\mathrm{T}}D$, then
$$M_\rho^r = (A \otimes B^{\mathrm{T}}) + (C \otimes D^{\mathrm{T}})W_{[n,p]} \tag{18}$$

2 PARAMETERIZED SOLUTIONS

Using Theorem 1 we can convert equation (1) into a system of linear equations as

$$\begin{bmatrix} R_1 & R_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \tag{19}$$

where

$$\begin{aligned} R_1 &= I_p \otimes A + B^{\mathrm{T}} \otimes I_n + D^{\mathrm{T}} \otimes C + (F^{\mathrm{T}} \otimes E)W_{[p,n]}, \\ R_2 &= -I_p \otimes G - N^{\mathrm{T}} \otimes M - (Q^{\mathrm{T}} \otimes P)W_{[p,r]}, \\ x &= V_c(X),\ y = V_c(Y). \end{aligned}$$

Proposition 1 Equation (1) has solutions of rp degree of freedom, if and only if the coefficient matrix of equation (19) has full row rank. That is

$$\operatorname{rank}[R_1\ R_2] = pn \tag{20}$$

Assume (20) holds, then we have rp linearly independent solutions of (19)

$$\begin{bmatrix} x^1 \\ y^1 \end{bmatrix}, \begin{bmatrix} x^2 \\ y^2 \end{bmatrix}, \cdots, \begin{bmatrix} x^{rp} \\ y^{rp} \end{bmatrix} \tag{21}$$

Then the set of rp linearly independent solutions of (1) are

$$\begin{cases} X^i = V_c^{-1}(x^i, n) \\ Y^i = V_c^{-1}(y^i, r) \end{cases} \quad i = 1, 2, \cdots, rp \tag{22}$$

It follows that the parameterized solution is

$$\begin{cases} X = \sum_{i=1}^{rp} \mu_i V_c^{-1}(x^i, n) \\ Y = \sum_{i=1}^{rp} \mu_i V_c^{-1}(y^i, r) \end{cases} \tag{23}$$

where $\mu = (\mu_1, \cdots, \mu_{rp})^{\mathrm{T}}$ are parameters. Each $\mu \neq 0$ corresponds to a non-zero solution.

Equation (20) is important in finding independent solutions. An easy way to find the solutions is to choose any rp rows, equivalently, an $rp \times (r+n)p$ matrix Φ, such that

$$\Psi := \begin{bmatrix} R_1 & R_2 \\ \multicolumn{2}{c}{\Phi} \end{bmatrix}$$

is non-singular. Then the last rp columns of Ψ^{-1} form (21), the set of rp linearly independent solutions of equation (19).

We can also use Theorems 2 to solve (1). First, convert equation (1) into a system of linear equations as

$$\begin{bmatrix} R_3 & R_4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \tag{24}$$

where

$$\begin{aligned} R_3 &= A \otimes I_p + I_n \otimes B^{\mathrm{T}} + C \otimes D^{\mathrm{T}} + (E \otimes F^{\mathrm{T}})W_{[n,p]}, \\ R_4 &= -G \otimes I_p - M \otimes N^{\mathrm{T}} - (P \otimes Q^{\mathrm{T}})W_{[r,p]}, \\ x &= V_r(X),\ y = V_r(Y). \end{aligned}$$

Similar argument as for Proposition 1 yields the following corollary

Corollary 1 Equation (24) has solutions of degree of freedom rp, if and only if the coefficient matrix of equation (24) has full row rank. That is

$$\operatorname{rank}[R_3\ R_4] = pn \tag{25}$$

Now assume (25) holds, and the linearly independent solutions of (1) have the form of (21). Then the set of rp linearly independent solutions of (1) are

$$\begin{cases} X^i = V_r^{-1}(x^i, p) \\ Y^i = V_r^{-1}(y^i, p) \end{cases} \quad i = 1, 2, \cdots, rp \tag{26}$$

The parameterized solution is

$$\begin{cases} X = \sum_{i=1}^{rp} \mu_i V_r^{-1}(x^i, p) \\ Y = \sum_{i=1}^{rp} \mu_i V_r^{-1}(y^i, p) \end{cases} \tag{27}$$

where $\mu = (\mu_1, \cdots, \mu_{rp})^{\mathrm{T}}$ are parameters. $\mu \neq 0$ corresponds to non-zero solution.

3 APPLICATION ON THE SYLVESTER MATRIX EQUATION AND ITS DUAL EQUATION

In this section we consider the Sylvester matrix equation as a special case of (1). It has the form

$$AX - EXF = BY \tag{28}$$

where $A, E \in M_{n\times n}$, $B \in M_{n\times r}$, $F \in M_{p\times p}$, with unknowns $X \in M_{n\times p}$ and $Y \in M_{r\times p}$. The Sylvester matrix equation (28) and its dual equation play an important role in linear system analysis and control design.

A basic assumption for the solution of (28) is the so called R-controllable. (E, A, B) is called R-controllable if

$$\operatorname{rank}\begin{bmatrix} sE - A & B \end{bmatrix} = n, \quad \forall s \in \mathbb{C}, \operatorname{rank}(B) = r \tag{29}$$

The following lemma is proved in [5].

Lemma 1 If E, A, B satisfy R-controllable condition (29) then equation (28) has rp degree of freedom. In other words, equation (28) has rp linearly independent solutions.

Recently, in [10] a complete general parametric expression for the solution (X, Y) is obtained under the assumption that (E, A, B) is R-controllable. In this note it can be obtained in a easy way.

For system (28), the equation (19) becomes

$$\begin{bmatrix} I_p \otimes A - F^{\mathrm{T}} \otimes E & -I_p \otimes B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad (30)$$

where $x = V_c(X)$ and $y = V_c(Y)$. Then we have the following corollary:

Corollary 2 Equation (28) has solutions of rp degree of freedom, if and only if the coefficient matrix of equation (30) has full row rank. That is

$$\operatorname{rank}\begin{pmatrix} I_p \otimes A - F^{\mathrm{T}} \otimes E & -I_p \otimes B \end{pmatrix} = pn \qquad (31)$$

Assume U is a nonsingular matrix such that $U^{-1}FU := J$ is the Jordan canonical form of F. We define $\tilde{X} = XU$ and $\tilde{Y} = YU$, then equation (28) can be expressed equivalently as

$$A\tilde{X} - E\tilde{X}J = B\tilde{Y} \qquad (32)$$

Correspondingly, (30) becomes

$$\begin{bmatrix} I_p \otimes A - J^{\mathrm{T}} \otimes E & -I_p \otimes B \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = 0 \qquad (33)$$

where $\tilde{x} = V_c(\tilde{X})$ and $\tilde{y} = V_c(\tilde{Y})$. Now equation (33) can be written as

$$\Gamma \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = 0 \qquad (34)$$

where Γ has block lower triangular form as

$$\Gamma = \begin{bmatrix} A - \lambda_1 E & -B & 0 & \cdots & 0 & \\ * & & A - \lambda_2 E & -B \cdots & 0 & \\ \vdots & & & & & \\ * & & * & & \cdots A - \lambda_p E & -B \end{bmatrix}$$

and λ_i, $i = 1, \cdots, p$ are eigenvalues of F. From (34) we have the following

Proposition 2 Equation (28) has solutions of (minimum) degree of freedom rp, if and only if

$$\operatorname{rank}(\lambda E - A \;\; B) = n, \quad \forall \lambda \in \sigma(F) \qquad (35)$$

Obviously, Lemma 1 is a particular case of Proposition 2, because (29) assures (35).

Assume (35) holds, then for system (28) the formerly defined Ψ becomes

$$\Psi := \begin{bmatrix} I_p \otimes A - F^{\mathrm{T}} \otimes E & -I_p \otimes B \\ \Phi & \end{bmatrix}$$

then the set of rp linearly independent solution of equation (28) can be obtained as

$$\begin{cases} X = \sum\limits_{i=1}^{rp} \mu_i V_c^{-1}(x^i, n) \\ Y = \sum\limits_{i=1}^{rp} \mu_i V_c^{-1}(y^i, r) \end{cases}$$

where $\mu = (\mu_1, \cdots, \mu_{rp})^{\mathrm{T}}$ are parameters. Each $\mu \neq 0$ corresponds to a non-zero solution.

In the design of Luenberger observer, we have to solve the dual equation of equation (28) [10]. Precisely, it is

$$XA - FXE = YC \qquad (36)$$

where $A, E \in M_{n\times n}$, $C \in M_{m\times n}$, $F \in M_{p\times p}$, with unknowns $X \in M_{p\times n}$ and $Y \in M_{p\times m}$.

Using Theorems 2, we can convert equation (36) into the form

$$\begin{bmatrix} I_p \otimes A^{\mathrm{T}} - F \otimes E^{\mathrm{T}} & -I_p \otimes C^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad (37)$$

where $x = V_r(X)$ and $y = V_r(Y)$.

Define $\tilde{X} = U^{-1}X$ and $\tilde{Y} = U^{-1}Y$. Similar argument as for Proposition 2 yields the following corollary:

Corollary 3 Equation (36) has solutions of (minimum) degree of freedom rp, if and only if

$$\operatorname{rank}\begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n, \quad \forall \lambda \in \sigma(F) \qquad (38)$$

Now assume (38) holds, and the linearly independent solutions of (37) have the form of (21). Then the set of rp linearly independent solutions of (36) are

$$\begin{cases} X^i = V_r^{-1}(x^i, n) \\ Y^i = V_r^{-1}(y^i, m) \end{cases} \quad i = 1, 2, \cdots, rp \qquad (39)$$

The parameterized solution is

$$\begin{cases} X = \sum\limits_{i=1}^{rp} \mu_i V_r^{-1}(x^i, n) \\ Y = \sum\limits_{i=1}^{rp} \mu_i V_r^{-1}(y^i, m) \end{cases} \qquad (40)$$

4 AN ILLUSTRATIVE EXAMPLE

As an application example, we consider a singular linear system [10]

$$\begin{cases} E\dot{x} = Ax + Bu, & x \in \mathbb{R}^n,\ u \in \mathbb{R}^r \\ y = Cx, & y \in \mathbb{R}^m \end{cases} \qquad (41)$$

To assure the uniqueness of the solution, we assume (E, A) is a normal pair (or the system (41) is normal), if there exists $s \in \mathbb{C}$ such that

$$\det(sE - A) \neq 0 \qquad (42)$$

The system is called R-observable if

$$\operatorname{rank}\begin{bmatrix} sE - A \\ C \end{bmatrix} = n, \quad \forall s \in \mathbb{C} \qquad (43)$$

The Luenberger observer has the following form

$$\begin{cases} \dot{z} = Fz + Gy + Su, & z \in \mathbb{R}^p \\ \omega = Mz + Ny, & \omega \in \mathbb{R}^r \end{cases} \qquad (44)$$

The design purpose is finding parameter matrices $F \in M_{p\times p}$, $G \in M_{p\times m}$, $S \in M_{p\times r}$, $M \in M_{r\times p}$, $N \in M_{r\times m}$, such that for a certain $K \in M_{r\times n}$, any initial $x(0)$, $z(0)$ and arbitrary input $u(t)$ we have

$$\lim_{t\to\infty}(Kx(t) - \omega(t)) = 0 \qquad (45)$$

We refer to [3] or [10] for the following result.

Theorem 3 Assume system (41) is normal and R-observable. Then system (44) is a Kx observer, if and only if, there exist matrices F, T, G, S, M, N satisfying

$$\begin{cases} S = TB \\ TA - FTE = GC \\ K = MTE + NC \\ Re[\sigma(F)] < 0 \text{ (i.e., } F \text{ is Hurwitz)} \end{cases} \tag{46}$$

Next, we use the same example in [10] to show how convenient our approach is.

Example 1 Consider system (41). Assume

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

As in [10], we want to design a Luenberger observer to track Kx, where

$$K = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

F can be any stable matrix. Now following [10], we choose

$$F = \begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix}$$

Consider the second equation of (46) first. Using (37), a straightforward computation shows that it can be write as

$$\begin{bmatrix} -5 & 0 & 0 & 2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t \\ g \end{bmatrix} = 0 \tag{47}$$

where $t = V_r(T)$ and $g = V_r(G)$. We can choose a Φ and construct the $(n+r)p \times (n+r)p$ matrix Ψ as

$$\Psi = \begin{bmatrix} -5 & 0 & 0 & 2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the last four columns of Ψ^{-1} form the linearly independent set of solutions of equation (47):

$$\begin{bmatrix} t^1 & t^2 & t^3 & t^4 \\ g^1 & g^2 & g^3 & g^4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & -5 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & -3 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$T_1 = \begin{bmatrix} 0 & 2 & -2 \\ 0 & 1 & 0 \end{bmatrix} \quad T_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad T_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad G_2 = \begin{bmatrix} 2 & 0 \\ -3 & 0 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} -5 & 0 \\ -1 & 0 \end{bmatrix} \quad G_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Using (40), the parameterized solutions are

$$T = \begin{bmatrix} \mu_3 & 2\mu_1 + \mu_4 & -2\mu_1 \\ \mu_2 & \mu_1 & \mu_4 \end{bmatrix}$$

$$G = \begin{bmatrix} 2\mu_2 - 5\mu_3 & 2\mu_1 + \mu_4 \\ -3\mu_2 - \mu_3 & \mu_1 \end{bmatrix}$$

It follows that

$$S = TB = \begin{bmatrix} \mu_3 & 2\mu_1 + \mu_4 \\ \mu_2 & \mu_1 \end{bmatrix}$$

Then we solve the third equation of (46). Denote $\alpha = V_c(M)$ and $\beta = V_c(N)$, we have

$$\begin{bmatrix} E^{\mathrm{T}} T^{\mathrm{T}} \otimes I_3 & C^{\mathrm{T}} \otimes I_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = V_c(K) \tag{48}$$

And (48) can be rewrite as

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \tag{49}$$

Its general solution is

$$\begin{bmatrix} 0 & -1 & \alpha_1 & \alpha_2 & 0 & 1 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ are parameters.

Using (4), we have

$$M = \begin{bmatrix} 0 & \alpha_1 \\ -1 & \alpha_2 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Let $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = 1$, $\alpha_1 = \alpha_2 = 0$, we have the particular solution given in [10] as

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

5 CONCLUSION

This paper considered the parameterized solutions of generalized Sylvester matrix equation (1). Formulas for parameterized numerical solutions were obtained. Comparing it with the methods provided in [10], where the general solutions are parameterized by matrices, the conditions and algorithms developed in this paper are neater and simpler.