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王宽诚教育基金会

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主编 钱伟长

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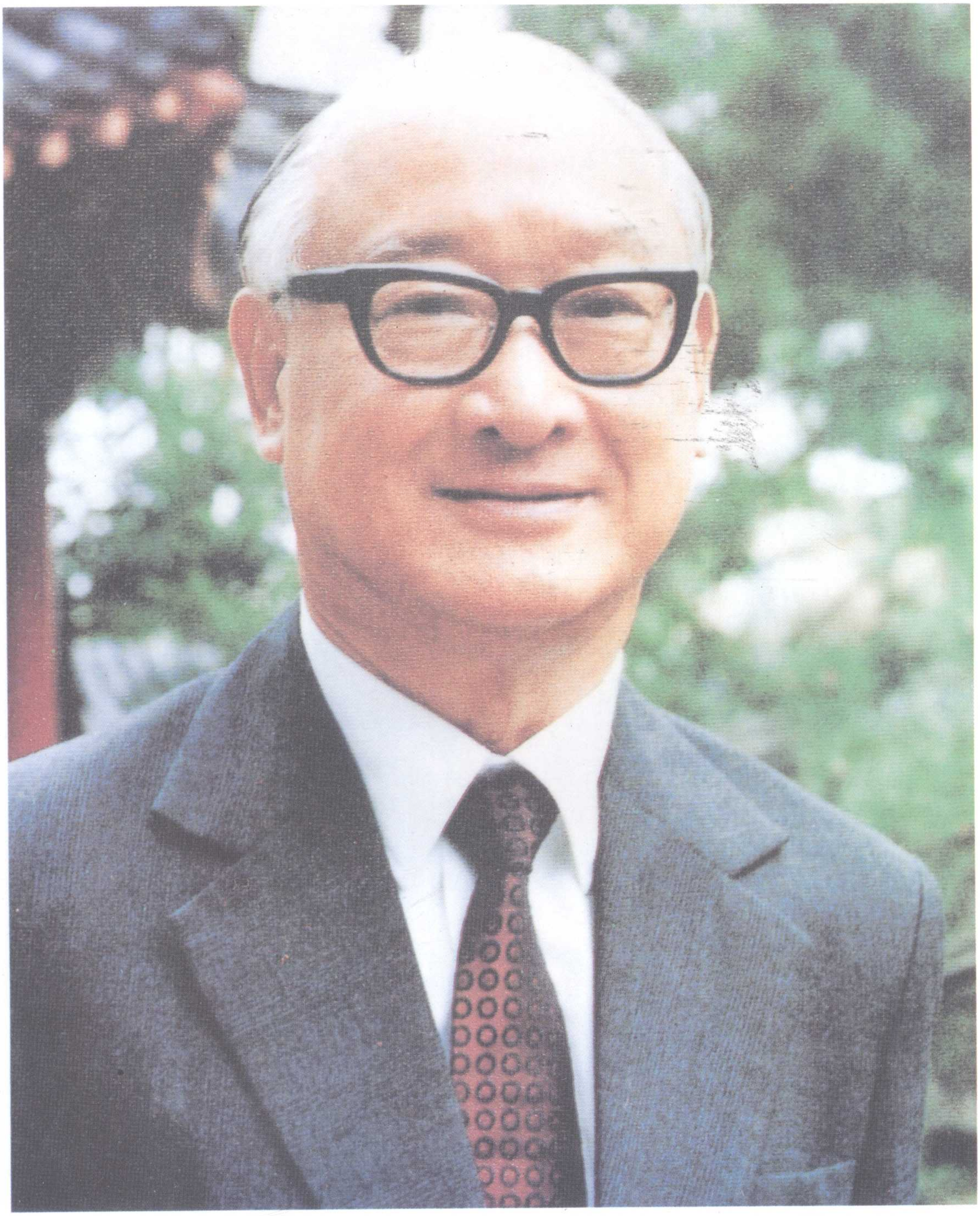

王寛诚先生

K. C. WONG (1907–1986)

王宽诚教育基金会简介

王宽诚先生(1907~1986)为香港著名爱国人士，热心祖国教育事业，生前为故乡宁波的教育事业做出积极贡献。1985 年独立捐巨资创建王宽诚教育基金会，其宗旨在于为国家培养高级技术人才，为祖国四个现代化效力。

王宽诚先生在世时聘请海内外著名学者担任基金会考选委员会和学务委员会委员，共商大计，确定采用“送出去”和“请进来”的方针，为国家培养各科专门人才，提高内地和港澳高等院校的教学水平，资助学术界人士互访以促进中外文化交流。在此方针指导下，1985、1986 两年，基金会在国家教委支持下，选派学生 85 名前往英、美、加拿大、德国、瑞士和澳大利亚各国攻读博士学位，并计划资助内地学者赴港澳讲学，资助港澳学者到内地讲学，资助美国学者来国内讲学。正当基金会事业初具规模、蓬勃发展之时，王宽诚先生一病不起，于 1986 年年底逝世。这是基金会的重大损失，共事同仁，无不深切怀念，不胜惋惜。

自 1987 年起，王宽诚教育基金会继承王宽诚先生为国家培养高级技术人才的遗愿，继续对中国内地、台湾及港澳学者出国攻读博士学位、博士后研究及学术交流提供资动。委请国家教育部、中国科学院和上海大学校长钱伟长教授等逐年安排资助学术交流的项目。相继与（英国）皇家学会、英国学术院、法国科研中心、德国学术交流中心等著名欧州学术机构合作，设立“王宽诚（英国）皇家学会奖学金”、“王宽诚英国学术院奖学金”、“王宽诚法国科研中心奖学金”、“王宽诚德国学术交流中心奖学金”，资助具有博士学位、副教授或同等学历职称的中国内地学者前往英国、法国、德国等地的高等学府及科研机构进行为期 3 至 12 个月之博士后研究。

王宽诚教育基金会过去和现在的工作态度一贯以王宽诚先生倡导的“公正”二字为守则，谅今后基金会亦将秉此行事，奉行不辍，借此王宽诚教育基金会《学术讲座汇编》出版之际，特简明介绍如上。王宽诚教育基金会日常工作繁忙，基金会各位董事均不辞劳累，做出积极贡献。

钱 伟 长

二〇〇二年十二月

前　言

王宽诚教育基金会是由已故全国政协常委、香港著名工商企业家王宽诚先生(1907~1986)出于爱国热忱，出资一亿美元于1985年在香港注册登记创立的。

1987年，基金会开设“学术讲座”项目，此项目由当时的全国政协委员、现任全国政协副主席、著名科学家、中国科学院院士、上海大学校长、王宽诚教育基金会贷款留学生考选委员会主任委员兼学务委员会主任委员钱伟长教授主持，由钱伟长教授亲自起草设立“学术讲座”的规定，资助内地学者前往香港、澳门讲学，资助美国学者来中国讲学，资助港澳学者前来内地讲学，用以促进中外学术交流，提高内地及港澳高等院校的教学质量。

本汇编收集的文章，均系各地学者在“学术讲座”活动中的讲稿。文章作者中，有年逾八旬的学术界硕彦，亦有由王宽诚教育基金会考选委员会委员推荐的学者和后起之秀。文章内容有科学技术，有历史文化，有经济专论，有文学，有宗教和中国古籍研究。本汇编涉及的学术领域颇为广泛，而每篇文章都有一定的深度和广度，分期分册以《王宽诚教育基金会学术讲座汇编》的名义出版，并无偿分送国内外部分高等院校、科研机构和图书馆，以广流传。

王宽诚教育基金会除资助“学术讲座”学者进行学术交流之外，在钱伟长教授主持的项目下，还资助由国内有关高等院校推荐的学者前往欧、美、亚、澳参加国际学术会议，出访的学者均向所出席的会议提交论文，这些论文亦颇有水平，本汇编亦将其收入，以供参考。

王宽诚教育基金会学务委员会

凡　例

（一）编排次序

本书所收集的王宽诚教育基金会学术讲座的讲稿及由王宽诚教育基金会资助学者赴欧、美、亚、澳参加国际学术会议的论文均按照文稿日期先后或文稿内容编排刊列，不分类别。

（二）分期分册出版并作简明介绍

因文稿较多，为求便于携带，有利阅读与检索，故分期分册出版，每册约 150 页至 200 页不等。为便于读者查考，每篇学术讲座的讲稿均注明作者姓名、学位、职务、讲学日期、地点、访问院校名称。内地及港、澳学者到欧、美、澳及亚洲的国家和地区参加国际学术会议的论文均注明学者姓名、参加会议的名称、时间、地点和推荐的单位。上述两类文章均注明由王宽诚教育基金会资助字样。

（三）文字种类

本书为学术性文章汇编，均以学术讲座学者之讲稿原稿或参加国际学术会议者向会议提交的论文原稿文字为准，原讲稿或论文是中文的，即以中文刊出，原讲稿或论文是外文的，仍以外文刊出。

目 录 CONTENTS

王宽诚教育基金会
《学术讲座汇编》第 22 集，1～27 (2002)

A Brief Survey of Block Toeplitz Iterative Solvers

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(Faculty of Science and Technology, University of Macau, Macau, P. R. China)

Abstract In this expository paper, we survey some of the latest developments and applications by using the preconditioned conjugate gradient method for solving block Toeplitz systems. One of the main results is that the complexity of solving a large class of mn-by-mn block Toeplitz systems can be reduced to $\mathcal{O}(mn \log mn)$ operations. Different preconditioners proposed for different block Toeplitz systems are reviewed. Some applications are studied. These applications include numerical differential equations and image processing.

Keywords block toeplitz matrices, preconditioners, preconditioned conjugate gradient method, multigrid method, differential equations, image processing

AMS(MOS) Subject Classifications. 45F10, 62F10, 65F10, 65N22, 65P05

1 Introduction

An n-by-n Toeplitz matrix is of the following form

$$\boldsymbol{T}_n = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{2-n} & t_{1-n} \\ t_1 & t_0 & t_{-1} & \cdots & t_{2-n} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & \cdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}, \tag{1}$$

i.e., $\boldsymbol{T}_n$ is constant along its diagonals. An mn-by-mn block Toeplitz matrix is of the following

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* 金小庆博士，澳门大学科技学院副教授，由王宽诚教育基金会资助，于 2000 年 12 月赴浙江大学、上海交通大学讲学，此为其讲稿。

form

$$
\boldsymbol{T}_{mn} = \begin{pmatrix} T_{(0)} & T_{(-1)} & \cdots & T_{(2-m)} & T_{(1-m)} \\ T_{(1)} & T_{(0)} & \ddots & \ddots & T_{(2-m)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ T_{(m-2)} & \ddots & \ddots & T_{(0)} & T_{(-1)} \\ T_{(m-1)} & T_{(m-2)} & \cdots & T_{(1)} & T_{(0)} \end{pmatrix}, \qquad (2)
$$

where $T_{(i)}$, $i = \pm 1, \cdots, \pm(m-1)$, are arbitrary n-by-n matrices. Particularly, if $\boldsymbol{T}_{mn}$ is a block Toeplitz matrix with the Toeplitz blocks $T_{(i)}$, for $i = \pm 1, \cdots, \pm(m-1)$, then $\boldsymbol{T}_{mn}$ is said to be a BTTB matrix.

Block Toeplitz systems arise in a variety of applications in mathematics, scientific computing and engineering[16],[48]. These applications have motivated mathematicians, scientists and engineers to develop fast and specific algorithms for solving block Toeplitz systems. Such kind of algorithms are called the block Toeplitz solvers.

Most of the current research works on block Toeplitz solvers are focused on iterative method, especially on the preconditioned conjugate gradient (PCG) method. One of the main important results of this methodology is that the complexity of solving a large class of block Toeplitz systems can be reduced to $\mathcal{O}(mn \log mn)$ operations provided that suitable preconditioners are chosen under certain conditions defined on Toeplitz systems. In this paper, we will survey some recent results of these iterative block Toeplitz solvers. Applications to some practical problems will also be reviewed.

1.1 Background

Let us begin by introducing the background knowledge that will be used throughout the article. We assume that the diagonals $\{t_k\}_{k=-n+1}^{n-1}$ of $\boldsymbol{T}_n$ in (1) are the Fourier coefficients of a function f, i.e.,

$$
t_k \equiv t_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x, \qquad \mathrm{i} \equiv \sqrt{-1}.
$$

The f is called the generating function of $\boldsymbol{T}_n$ and assumed in certain class of functions such that all the $\boldsymbol{T}_n$ are invertible. In practical problems from industry and engineering, we are usually given f first, not the Toeplitz matrix $\boldsymbol{T}_n$, see [16].

In order to analyse the convergence rate of the conjugate gradient (CG) method, we need to introduce the following definition of clustered spectrum[16].

Definition 1.1 The eigenvalues of a sequence of matrices $\{\boldsymbol{H}_n\}_{n=1}^{\infty}$ are said to be clustered around a point $\gamma \in \mathbf{R}$ if for any $\varepsilon > 0$, there exist positive integers M and N such that for all $n > N$, at most M eigenvalues of $\boldsymbol{H}_n - \gamma \boldsymbol{I}_n$ have absolute values greater than ε, where $\boldsymbol{I}_n$ is the identity matrix.

If the eigenvalues of $\{\boldsymbol{H}_n\}_{n=1}^{\infty}$ are clustered around a point γ, then the CG method has a fast convergence rate by the following theorem given by Van der Vorst[70].

Theorem 1.1 Let $\boldsymbol{u}^k$ be the k-th iterant of the CG method applied to the system $\boldsymbol{H}_n\boldsymbol{u} = \boldsymbol{b}$ and let $\boldsymbol{x}$ be the true solution of the system. If the eigenvalues λ_j of $\boldsymbol{H}_n$ are ordered such that

$$0 < \lambda_1 \leq \dots \leq \lambda_p \leq b_1 \leq \lambda_{p+1} \leq \dots \leq \lambda_{n-q} \leq b_2 \leq \lambda_{n-q+1} \leq \dots \leq \lambda_n,$$

then

$$\frac{\|\boldsymbol{u}-\boldsymbol{u}^k\|}{\|\boldsymbol{u}-\boldsymbol{u}^0\|} \leq 2\left(\frac{\alpha-1}{\alpha+1}\right)^{k-p-q} \cdot \max_{\lambda\in[b_1,b_2]} \prod_{j=1}^{p} \left(\frac{\lambda-\lambda_j}{\lambda_j}\right).$$

Here $\|\cdot\|$ is the energy norm given by $\|\boldsymbol{v}\|^2 = \boldsymbol{v}^*\boldsymbol{H}_n\boldsymbol{v}$ ("$*$" denotes conjugate transposition) and $\alpha \equiv (\frac{b_2}{b_1})^{\frac{1}{2}} \geq 1$.

From Theorem 1.1, we know that the more clustered the eigenvalues are, the faster the convergence rate will be. Unfortunately, the spectra of matrices are not clustered around a certain point in general. Thus, the convergence rate of the CG method is slow usually. In order to accelerate the convergence rate, we need to precondition the system, i.e., instead of solving the original system $\boldsymbol{H}_n\boldsymbol{u} = \boldsymbol{b}$, we solve the following preconditioned system

$$\boldsymbol{M}_n^{-1}\boldsymbol{H}_n\boldsymbol{u} = \boldsymbol{M}_n^{-1}\boldsymbol{b}. \tag{3}$$

The preconditioner $\boldsymbol{M}_n$ is chosen with two criteria in mind, see [29]:

I. $\boldsymbol{M}_n\boldsymbol{r} = \boldsymbol{d}$ is easy to solve;

II. the spectrum of $\boldsymbol{M}_n^{-1}\boldsymbol{H}_n$ is clustered and (or) $\boldsymbol{M}_n^{-1}\boldsymbol{H}_n$ is well-conditioned compared to $\boldsymbol{H}_n$.

The main work involved in implementing the CG method to the preconditioned system (3) is the matrix-vector product $\boldsymbol{M}_n^{-1}\boldsymbol{H}_n\boldsymbol{v}$ for some vector $\boldsymbol{v}$. We will show that for Toeplitz matrices with circulant preconditioners, the cost of this matrix-vector product can be reduced dramatically.

1.2 Circulant preconditioners

In 1986, Strang [62] and Olkin [55] proposed independently the use of the PCG method with circulant matrices as preconditioners for solving Toeplitz systems. The circulant matrix is defined as follows:

$$\boldsymbol{C}_n = \begin{pmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & \cdots & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{pmatrix}.$$

It is well-known that circulant matrices can be diagonalized by the Fourier matrix $\boldsymbol{F}_n$, see [26], i.e.,

$$\boldsymbol{C}_n = \boldsymbol{F}_n^*\Lambda_n\boldsymbol{F}_n. \tag{4}$$

Here the entries of $\boldsymbol{F}_n$ are given by

$$(\boldsymbol{F}_n)_{j,k} = \frac{1}{\sqrt{n}} \mathrm{e}^{2\pi \mathrm{i} jk/n}, \qquad \mathrm{i} \equiv \sqrt{-1},$$

with $0 \leq j, k \leq n-1$ and Λ_n is a diagonal matrix holding the eigenvalues of $\boldsymbol{C}_n$. In [62] and [55], Strang and Olkin noted that for any Toeplitz matrix $\boldsymbol{T}_n$ with a circulant preconditioner $\boldsymbol{C}_n$, the product $\boldsymbol{C}_n^{-1}\boldsymbol{T}_n\boldsymbol{v}$ can be computed in $\mathcal{O}(n \log n)$ operations for any vector $\boldsymbol{v}$ as circulant systems can be solved efficiently by the Fast Fourier Transform (FFT) and the multiplication $\boldsymbol{T}_n\boldsymbol{v}$ can also be computed by FFTs by first embedding $\boldsymbol{T}_n$ into a $2n$-by-$2n$ circulant matrix. More precisely, we have a $2n$-by-$2n$ circulant matrix with $\boldsymbol{T}_n$ embedded inside as follows,

$$\begin{pmatrix} \boldsymbol{T}_n & \times \\ \times & \boldsymbol{T}_n \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{T}_n\boldsymbol{v} \\ \dagger \end{pmatrix},$$

and then the multiplication can be carried out by using the decomposition as in (4). The operation cost is, therefore, $\mathcal{O}(2n \log(2n))$. Thus, the cost per iteration of the PCG method is still $\mathcal{O}(n \log n)$.

A lot of circulant preconditioners have been proposed for solving Toeplitz systems since 1986. We introduce some of them which have been proved to be good preconditioners.

1.2.1 Strang's circulant preconditioner

For Toeplitz matrix (1), Strang's preconditioner $s(\boldsymbol{T}_n)$ is defined to be the circulant matrix obtained by copying the central diagonals of $\boldsymbol{T}_n$ and bringing them around to complete the circulant. More precisely, the diagonals of $s(\boldsymbol{T}_n)$ are given by

$$s_k = \begin{cases} t_k, & 0 \leq k \leq \lfloor n/2 \rfloor, \\ t_{k-n}, & \lfloor n/2 \rfloor < k < n, \\ s_{n+k}, & 0 < -k < n. \end{cases} \tag{5}$$

1.2.2 T. Chan's circulant preconditioner

Let

$$\mathcal{M}_{\boldsymbol{U}} = \{\boldsymbol{U}^* \Lambda_n \boldsymbol{U} \mid \Lambda_n \text{ is an } n\text{-by-}n \text{ diagonal matrix}\}, \tag{6}$$

where $\boldsymbol{U}$ is an n-by-n unitary matrix. We note that when $\boldsymbol{U} = \boldsymbol{F}$, the Fourier matrix, $\mathcal{M}_{\boldsymbol{F}}$ is the set of all circulant matrices, see [26]. For an n-by-n Toeplitz matrix $\boldsymbol{T}_n$, T. Chan's circulant preconditioner $c_F(\boldsymbol{T}_n)$[23] is defined to be the minimizer of the Frobenius norm

$$\|\boldsymbol{T}_n - \boldsymbol{W}\|_{\boldsymbol{F}} \tag{7}$$

over all $\boldsymbol{W} \in \mathcal{M}_{\boldsymbol{F}}$. The matrix $c_F(\boldsymbol{T}_n)$ is called the optimal circulant preconditioner in [23]. The diagonals of $c_F(\boldsymbol{T}_n)$ are just the average of the diagonals of $\boldsymbol{T}_n$, with the diagonals being extended to length n by a wrap-around. More precisely, the diagonals of $c_F(\boldsymbol{T}_n)$ are given by

$$c_k = \begin{cases} \dfrac{(n-k)t_k + kt_{k-n}}{n}, & 0 \leq k < n, \\ c_{n+k}, & 0 < -k < n. \end{cases} \tag{8}$$

When $\boldsymbol{T}_n$ is a general matrix, the circulant minimizer $c_F(\boldsymbol{T}_n)$ of (7) can still be defined by taking the arithmetic average of the entries along the diagonal of $\boldsymbol{T}_n$.

1.2.3 R. Chan's circulant preconditioner

R. Chan's preconditioner $r(\boldsymbol{T}_n)$ proposed in [7] is defined as follows. For $\boldsymbol{T}_n$ given by (1), the preconditioner $r(\boldsymbol{T}_n)$ is the circulant matrix with diagonals:

$$r_k = \begin{cases} t_{k-n} + t_k, & 0 \le k < n, \\ r_{n+k}, & 0 < -k < n, \end{cases} \tag{9}$$

where t_{-n} is taken to be 0.

1.2.4 Huckle's circulant preconditioner

For $\boldsymbol{T}_n$ given by (1), Huckle's preconditioner $\boldsymbol{H}_n^p$ is defined to be the circulant matrix with eigenvalues

$$\lambda_k(\boldsymbol{H}_n^p) = \sum_{j=-p+1}^{p-1} t_j \left(1 - \frac{|j|}{p}\right) \mathrm{e}^{2\pi \mathrm{i} jk/n}, \quad k = 0, \cdots, n-1.$$

When $p = n$, it is nothing new but T. Chan's preconditioner.

If the generating function f of $\boldsymbol{T}_n$ is positive and smooth, then the PCG method with these circulant preconditioners has been proved to be a successful method which converges superlinearly, see [16]. Therefore, the operation cost of the algorithm will remain $\mathcal{O}(n \log n)$.

1.3 Non-circulant optimal preconditioners

Besides FFT, there are a lot of fast transforms used in scientific computing and engineering. With the U in (6) taking other matrices based on different fast transforms, we then have new classes of optimal preconditioners for solving Toeplitz systems.

1.3.1 Optimal preconditioner based on fast transform

Let $\mathcal{M}_{\boldsymbol{\Phi}^\alpha}$ in (6) with $\boldsymbol{U} = \boldsymbol{\Phi}^\alpha$ be the set of all n-by-n matrices that can be diagonalized by the matrix $\boldsymbol{\Phi}^\alpha$, i.e.,

$$\mathcal{M}_{\boldsymbol{\Phi}^\alpha} = \{(\boldsymbol{\Phi}^\alpha)^{\mathrm{T}} \boldsymbol{\Lambda}_n \boldsymbol{\Phi}^\alpha \mid \boldsymbol{\Lambda}_n \text{ is an } n\text{-by-}n \text{ diagonal matrix}\}.$$

For $\alpha = s$, the (j,k)-th entry of the sine transform matrix $\boldsymbol{\Phi}^s$ is defined by

$$\sqrt{\frac{2}{n+1}} \sin\left(\frac{\pi jk}{n+1}\right),$$

for $1 \le j, k \le n$. For $\alpha = c$, the (j,k)-th entry of the cosine transform matrix $\boldsymbol{\Phi}^c$ is defined by

$$\sqrt{\frac{2 - \delta_{j1}}{n}} \cos\left(\frac{(j-1)(2k-1)\pi}{2n}\right),$$

for $1 \le j, k \le n$, where δ_{jk} is Kronecker delta. Moreover, for $\alpha = h$, the (j,k)-th entry of the Hartley transform matrix $\boldsymbol{\Phi}^h$ is defined by

$$\frac{1}{\sqrt{n}} \cos\left(\frac{2\pi jk}{n}\right) + \frac{1}{\sqrt{n}} \sin\left(\frac{2\pi jk}{n}\right),$$

for $0 \le j, k \le n-1$.

Given any arbitrary n-by-n matrix $\boldsymbol{A}_n$, we define an operator Ψ_α which maps $\boldsymbol{A}_n$ to the matrix $\Psi_\alpha(\boldsymbol{A}_n)$ that minimizes $\|\boldsymbol{A}_n - \boldsymbol{B}_n\|_F$ over all matrices $\boldsymbol{B}_n \in \mathcal{M}_{\Phi^\alpha}$. For the construction of matrix $\Psi_\alpha(\boldsymbol{A}_n)$ with $\alpha = s, c, h$, we refer to [4, 11, 18].

1.3.2 Convergence result and operation cost

Let $\mathcal{C}_{2\pi}$ be the set of all 2π-periodic continuous real-valued functions defined on $[-\pi, \pi]$ and f be the generating function of $\boldsymbol{T}_n$. For the convergence rate of the PCG method with the optimal preconditioners based on fast transforms, we have the following theorem, see [18, 11, 41].

Theorem 1.2 Let $f \in \mathcal{C}_{2\pi}$ be an even positive function. Then the spectra of

$$(\Psi_\alpha(\boldsymbol{T}_n))^{-1}\boldsymbol{T}_n$$

are clustered around 1 for large n, where $\alpha = s, c, h$. Thus, the convergence rate of the PCG method is superlinear.

In each iteration of the PCG method, we have to compute matrix-vector multiplication $\boldsymbol{T}_n\boldsymbol{v}$ and solve the system $\Psi_\alpha(\boldsymbol{T}_n)\boldsymbol{y} = \boldsymbol{u}$. The $\boldsymbol{T}_n\boldsymbol{v}$ can be computed in $\mathcal{O}(n \log n)$ operations. The system $\Psi_\alpha(\boldsymbol{T}_n)\boldsymbol{y} = \boldsymbol{u}$ can also be solved in $\mathcal{O}(n \log n)$ operations by using the fast sine transform for $\Psi_s(\boldsymbol{T}_n) \in \mathcal{M}_{\Phi^s}$; by using the fast cosine transform for $\Psi_c(\boldsymbol{T}_n) \in \mathcal{M}_{\Phi^c}$; or by using the fast Hartley transform for $\Psi_h(\boldsymbol{T}_n) \in \mathcal{M}_{\Phi^h}$.

2 Block Preconditioners

Let us consider a general block system $\boldsymbol{H}_{mn}\boldsymbol{u} = \boldsymbol{b}$ first, where $\boldsymbol{H}_{mn}$ is an mn-by-mn matrix partitioned as

$$\boldsymbol{H}_{mn} = \begin{pmatrix} \boldsymbol{H}_{1,1} & \boldsymbol{H}_{1,2} & \cdots & \boldsymbol{H}_{1,m} \\ \boldsymbol{H}_{2,1} & \boldsymbol{H}_{2,2} & \cdots & \boldsymbol{H}_{2,m} \\ \vdots & \ddots & \ddots & \vdots \\ \boldsymbol{H}_{m,1} & \boldsymbol{H}_{m,2} & \cdots & \boldsymbol{H}_{m,m} \end{pmatrix}$$

and the blocks $\boldsymbol{H}_{i,j}$ with $1 \leq i, j \leq m$ are square matrices of order n.

2.1 Block preconditioners for block systems

Several preconditioners that preserve the block structure of $\boldsymbol{H}_{mn}$ are constructed in [13]. In view of the point case in Section 1.2.2, it is natural to define the circulant-block preconditioner of $\boldsymbol{H}_{mn}$ as follows,

$$c_F^{(1)}(\boldsymbol{H}_{mn}) \equiv \begin{pmatrix} c_F(\boldsymbol{H}_{1,1}) & c_F(\boldsymbol{H}_{1,2}) & \cdots & c_F(\boldsymbol{H}_{1,m}) \\ c_F(\boldsymbol{H}_{2,1}) & c_F(\boldsymbol{H}_{2,2}) & \cdots & c_F(\boldsymbol{H}_{2,m}) \\ \vdots & \ddots & \ddots & \vdots \\ c_F(\boldsymbol{H}_{m,1}) & c_F(\boldsymbol{H}_{m,2}) & \cdots & c_F(\boldsymbol{H}_{m,m}) \end{pmatrix},$$

where the blocks $c_F(\boldsymbol{H}_{i,j})$ defined as in (8) are just T. Chan's circulant preconditioners of $\boldsymbol{H}_{i,j}$, $1 \le i,j \le m$. Actually, the matrix $c_F^{(1)}(\boldsymbol{H}_{mn})$ is the minimizer of

$$\|\boldsymbol{H}_{mn} - \boldsymbol{W}_{mn}\|_F$$

over all matrices $\boldsymbol{W}_{mn}$ that are m-by-m block matrices with n-by-n circulant blocks. It can be viewed as an approximation of $\boldsymbol{H}_{mn}$ long one specific direction.

It is therefore natural to consider the preconditioner that results from an approximation of $\boldsymbol{H}_{mn}$ along the other direction. Let $(\boldsymbol{H}_{mn})_{i,j;k,l}$ denote the (i,j)-th entry of the (k,l)-th block of $\boldsymbol{H}_{mn}$ and let $\boldsymbol{P}$ be the permutation matrix that satisfies

$$(\boldsymbol{P}^*\boldsymbol{H}_{mn}\boldsymbol{P})_{k,l;i,j} = (\boldsymbol{H}_{mn})_{i,j;k,l}, \tag{10}$$

for $1 \le i,j \le n, 1 \le k,l \le m$. The preconditioner $\tilde{c}_F^{(1)}(\boldsymbol{H}_{mn})$ is a block-circulant matrix which is defined as

$$\tilde{c}_F^{(1)}(\boldsymbol{H}_{mn}) \equiv \boldsymbol{P}^* c_F^{(1)}(\boldsymbol{P}\boldsymbol{H}_{mn}\boldsymbol{P}^*)\boldsymbol{P}.$$

With the composite of operators $c_F^{(1)}$ and $\tilde{c}_F^{(1)}$, one can obtain a preconditioner

$$c_{F,F}^{(2)}(\boldsymbol{H}_{mn}) \equiv \tilde{c}_F^{(1)} \circ c_F^{(1)}(\boldsymbol{H}_{mn})$$

which is based on circulant approximations within each block and also on block level. The preconditioner $c_{F,F}^{(2)}(\boldsymbol{H}_{mn})$ is said to be a BCCB matrix. Actually, it is the minimizer of $\|\boldsymbol{H}_{mn} - \boldsymbol{W}_{mn}\|_F$ over all $\boldsymbol{W}_{mn} \in \mathcal{M}_{F\otimes F}$. Here

$$\mathcal{M}_{F\otimes F} \equiv \{(\boldsymbol{F}_m \otimes \boldsymbol{F}_n)^* \Lambda_{mn}(\boldsymbol{F}_m \otimes \boldsymbol{F}_n) \mid \Lambda_{mn} \text{ is a diagonal matrix}\}$$

where $\otimes$ denotes the tensor product and $\boldsymbol{F}_m$, $\boldsymbol{F}_n$ are Fourier matrices. We note that $\mathcal{M}_{F\otimes F}$ is the set of all BCCB matrices, see [26]. The BCCB preconditioners for solving BTTB matrices have been investigated in [13, 24, 52, 67].

Since any BCCB matrix $\boldsymbol{C}_{mn}$ can be diagonalized by the 2-dimensional Fourier matrix,

$$\boldsymbol{C}_{mn} = (\boldsymbol{F}_m^* \otimes \boldsymbol{F}_n^*)\Lambda_{mn}(\boldsymbol{F}_m \otimes \boldsymbol{F}_n)$$

where Λ_{mn} is a diagonal matrix holding the eigenvalues of $\boldsymbol{C}_{mn}$, therefore, the matrix-vector multiplication $\boldsymbol{C}_{mn}\boldsymbol{v}$ can be computed in $\mathcal{O}(mn\log mn)$ operations by using the 2-dimensional FFT. We note that the relation between the first column and the eigenvalues of $\boldsymbol{C}_{mn}$ is given by

$$\sqrt{mn}\boldsymbol{C}_{mn}\boldsymbol{e}_1 = (\boldsymbol{F}_m^* \otimes \boldsymbol{F}_n^*)\Lambda_{mn}\mathit{1}_{mn}, \tag{11}$$

where $\boldsymbol{e}_1$ and $\mathit{1}_{mn}$ denote the first unit vector and the vector of all 1's respectively. Thus, Λ_{mn} can be obtained in $\mathcal{O}(mn\log mn)$ operations by taking 2-dimensional FFTs of the first column of $\boldsymbol{C}_{mn}$. Moreover, an $mn \times mn$ BTTB matrix can be enlarged into a $4mn \times 4mn$ BCCB matrix, so it can also be multiplied by a vector in $\mathcal{O}(mn\log mn)$ operations, see [13].

2.2 Convergence rate and operation cost

We analyse the convergence rate and operation cost of the PCG method when applied to solving BTTB systems $\boldsymbol{T}_{mn}\boldsymbol{u} = \boldsymbol{b}$. Let the entries of $\boldsymbol{T}_{mn}$ be denoted by

$$(\boldsymbol{T}_{mn})_{p,q;r,s} = t_{p-q}^{(r-s)},$$

for $1 \le p, q \le n, 1 \le r, s \le m$. The $\boldsymbol{T}_{mn}$ is associated with a generating function $f(x, y)$ as follows,

$$t_k^{(j)}(f) \equiv \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\mathrm{e}^{-\mathrm{i}(jx+ky)}\mathrm{d}x\mathrm{d}y, \qquad \mathrm{i} \equiv \sqrt{-1}.$$

We note that for any m and n, $\boldsymbol{T}_{mn}$'s have the following important properties:

(1) When f is real-valued, then $\boldsymbol{T}_{mn}(f)$ are Hermitian, i.e., $t_k^{(j)}(f) = \bar{t}_{-k}^{(-j)}(f)$.

(2) When f is real-valued with $f(x, y) = f(-x, -y)$, then $\boldsymbol{T}_{mn}(f)$ are real symmetric, i.e., $t_k^{(j)}(f) = t_{-k}^{(-j)}(f)$.

(3) When f is real-valued and even, i.e., $f(x, y) = f(|x|, |y|)$, then $\boldsymbol{T}_{mn}(f)$ are level-2 symmetric, i.e., $t_k^{(j)}(f) = t_{|k|}^{(|j|)}(f)$.

Let $\mathcal{C}_{2\pi\times 2\pi}$ denote the Banach space of all 2π-periodic (in each direction) continuous real-valued functions equipped with the supremum norm $|||\cdot|||_\infty$. The following theorem gives the relation between the values of $f(x, y)$ and the eigenvalues of $T_{mn}(f)$, see [42, 60].

Theorem 2.1 If $f \in \mathcal{C}_{2\pi\times 2\pi}$ with $f_{\min} < f_{\max}$ where $f_{\min}$ and $f_{\max}$ denote the minimum and maximum values of f respectively, then for all positive integers m and n, we have

$$f_{\min} < \lambda_i(\boldsymbol{T}_{mn}) < f_{\max}, \qquad \text{for} \quad i = 1, \cdots, mn,$$

where $\lambda_i(\boldsymbol{T}_{mn})$ is the i-th eigenvalue of $\boldsymbol{T}_{mn}$. Moreover,

$$\lim_{m,n\to\infty} \lambda_{\max}(\boldsymbol{T}_{mn}) = f_{\max} \quad \text{and} \quad \lim_{m,n\to\infty} \lambda_{\min}(\boldsymbol{T}_{mn}) = f_{\min}.$$

From theorem 2.1, we know that if $f \ge 0$, then $\boldsymbol{T}_{mn}(f)$ is always positive definite. When f vanishes at some points $(x_0, y_0) \in [-\pi, \pi] \times [-\pi, \pi]$, then the condition number $\kappa(\boldsymbol{T}_{mn})$ of $\boldsymbol{T}_{mn}$ is unbounded as m or n tend to infinity, i.e., $\boldsymbol{T}_{mn}$ is ill-conditioned.

Now, let us consider the class of level-2 symmetric BTTB systems

$$\boldsymbol{T}_{mn}\boldsymbol{u} = \boldsymbol{b} \tag{12}$$

where

$$\boldsymbol{T}_{mn} = \begin{pmatrix} \boldsymbol{T}_{(0)} & \boldsymbol{T}_{(1)} & \cdots & \boldsymbol{T}_{(m-2)} & \boldsymbol{T}_{(m-1)} \\ \boldsymbol{T}_{(1)} & \boldsymbol{T}_{(0)} & \ddots & \ddots & \boldsymbol{T}_{(m-2)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{T}_{(m-2)} & \ddots & \ddots & \boldsymbol{T}_{(0)} & \boldsymbol{T}_{(1)} \\ \boldsymbol{T}_{(m-1)} & \boldsymbol{T}_{(m-2)} & \cdots & \boldsymbol{T}_{(1)} & \boldsymbol{T}_{(0)} \end{pmatrix}$$

and the blocks $\boldsymbol{T}_{(i)}$, for $i = 0, \cdots, m-1$, are themselves symmetric Toeplitz matrices of order n. For solving (12), by using the PCG method with the preconditioners $c_F^{(1)}(\boldsymbol{T}_{mn})$, $\tilde{c}_F^{(1)}(\boldsymbol{T}_{mn})$

and $c^{(2)}_{F,F}(\boldsymbol{T}_{mn})$ respectively, the cost per iteration requires $\mathcal{O}(mn\log^2 m + mn\log n)$, $\mathcal{O}(nm\log^2 n + nm\log m)$ and $\mathcal{O}(mn\log mn)$ operations accordingly, see [13].

The convergence rate of the PCG method was also analysed by R. Chan and Jin[13, 48] for solving (12). It was shown that if the generating function $f > 0$ is in $\mathcal{C}_{2\pi\times 2\pi}$, then the spectra of preconditioned matrices $\left(c^{(1)}_F(\boldsymbol{T}_{mn})\right)^{-1}\boldsymbol{T}_{mn}$, $\left(\tilde{c}^{(1)}_F(\boldsymbol{T}_{mn})\right)^{-1}\boldsymbol{T}_{mn}$ and $\left(c^{(2)}_{F,F}(\boldsymbol{T}_{mn})\right)^{-1}\boldsymbol{T}_{mn}$ are clustered and the method converges linearly. Thus, the total operation cost for solving (12) only requires $\mathcal{O}(mn\log^2 m + mn\log n)$, $\mathcal{O}(nm\log^2 n + nm\log m)$ and $\mathcal{O}(mn\log mn)$ operations accordingly, see [13]. This is one of main important results of the PCG algorithm. However, Serra and Tyrtyshnikov[61] proved theoretically that any multilevel circulant preconditioners for multilevel Toeplitz matrices can not produce a superlinearly convergence rate by using the PCG algorithms. The works on other types of block preconditioners can be found in [16, 48].

3 BCCB Preconditioners from Kernels

A unified treatment of constructing circulant preconditioners from kernel functions was first introduced by R. Chan and Yeung[22]. This idea was then generalized to the block matrix case by Jin in [47]. Let $\boldsymbol{T}_{mn}(f)$ be the BTTB matrix generated by a function f.

3.1 BCCB preconditioners constructed from kernels

As analogous to $c^{(2)}_{F,F}(\boldsymbol{T}_{mn})$, we can define Strang's, R. Chan's BCCB preconditioners for the BTTB matrix $\boldsymbol{T}_{mn}$. For instance, we define the preconditioner $s^{(1)}_F(\boldsymbol{T}_{mn})$ as follows,

$$s^{(1)}_F(\boldsymbol{T}_{mn}) \equiv \begin{pmatrix} s(\boldsymbol{T}_{(0)}) & s(\boldsymbol{T}_{(-1)}) & \cdots & s(\boldsymbol{T}_{(1-m)}) \\ s(\boldsymbol{T}_{(1)}) & s(\boldsymbol{T}_{(0)}) & \cdots & s(\boldsymbol{T}_{(2-m)}) \\ \vdots & \ddots & \ddots & \vdots \\ s(\boldsymbol{T}_{(m-1)}) & s(\boldsymbol{T}_{(m-2)}) & \cdots & s(\boldsymbol{T}_{(0)}) \end{pmatrix}$$

where the blocks $s(\boldsymbol{T}_{(i)})$ defined in (5) are just Strang's circulant preconditioners of $\boldsymbol{T}_{(i)}$, for $i = 0, \pm 1, \cdots, \pm(m-1)$. Let $\boldsymbol{P}$ be the permutation matrix given by (10). The preconditioner $\tilde{s}^{(1)}_F(\boldsymbol{T}_{mn})$ is defined as

$$\tilde{s}^{(1)}_F(\boldsymbol{T}_{mn}) \equiv \boldsymbol{P}^* s^{(1)}_F(\boldsymbol{P}\boldsymbol{T}_{mn}\boldsymbol{P}^*)\boldsymbol{P}.$$

With the composite of operators $s^{(1)}_F$ and $\tilde{s}^{(1)}_F$, one can obtain Strang's BCCB preconditioner

$$s^{(2)}_{F,F}(\boldsymbol{T}_{mn}) \equiv \tilde{s}^{(1)}_F \circ s^{(1)}_F(\boldsymbol{T}_{mn}).$$

R. Chan's BCCB preconditioner $r^{(2)}_{F,F}(\boldsymbol{T}_{mn})$ can be defined similarly by using (9). Since any BCCB matrix can be determined by its first column, from (11), we know that once the eigenvalues of the BCCB matrix are obtained, one can get the first column of the BCCB matrix easily by using 2-dimensional FFTs. In the following, we will construct the eigenvalues of some well-known BCCB preconditioners from the viewpoint of convolution of the generating function with some famous kernels.