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(裏へつづく)

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***All communications relating to this Bulletin should be addressed to the Director of the Earthquake Research Institute, University of Tokyo, Japan.***


Earthquake Research Institute, University of Tokyo
Address:
1-1, Yayoi 1-chome, Bunkyo-ku,
Tokyo Metropolis, Japan.
Telephone: 812-2111-Extension 5669
Cable address: ZISINKEN, TOKYO.

東京大学地震研究所
所在地
東京都文京区弥生1丁目1番1号
電話 812-2111 (大代) 構内 5669
電略 ブンケウ トウダイ ヂシン

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BULLETIN OF THE EARTHQUAKE
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# 1. Asymptotic Behavior of Spheroidal Eigenfrequencies of a Multi-Layered Spherical Earth. — Modes of Very High Phase Velocity —

By Toshikazu ODAKA,

Earthquake Research Institute.

(Received Feb. 29, 1980)


## Abstract

Derivation of a frequency equation is made in terms of the matrix formulation for spheroidal oscillations of a multi-layered spherical Earth. Then, it is shown that the equation splits at very high frequency into three independent equations corresponding to three body-wave types, *PKIKP*, $(ScS)_r$ and *J* respectively.

The result is used to obtain asymptotic frequency equations in explicit forms for simple Earth models consisting of a homogeneous liquid core and a one- to three-layered mantle. Comparison of those formulas leads to the conclusion that the equation for *PKP*-type and that for $(ScS)_r$-type are similar in form to each other when the number of internal discontinuities effective to respective body waves are the same. The fundamental difference in their forms is that the former equation depends on the evenness and oddness of the Legendre order while the latter one does not. It is proved through numerical computations that the solutions of the above equations to the first order approximation are useful for explaining asymptotic patterns of distribution of eigenfrequencies.

Further computations are made for two Earth models with realistic mantle structure, one with two distinct discontinuities in the upper mantle and the other with a continuously varying structure. Then, it is proved that in general there exists a remarkable difference between the two patterns of distribution of their eigenfrequencies. However the difference falls off at low frequencies because the whole upper-mantles, where elastic parameters change sharply with depth, act as the same scale of discontinuities on long-period free oscillations. Their patterns of oscillatory features are explainable in terms of an additive effect of the individual "solotone effect" associated with each discontinuity in the Earth.


## 1. Introduction

Since 1974, many investigations have been made on asymptotic behavior of eigenfrequencies of free oscillations of the Earth (*e.g.*, ANDERSSEN and CLEARY, 1974; LAPWOOD, 1975; WANG *et al.*, 1977; SATO and LAPWOOD, 1977 a, b). These researches are mainly concerned with torsional oscillations and few papers refer to spheroidal oscillations (*e.g.*, ANDERSSEN *et al.*, 1975; GILBERT, 1975). There, especially, seems to be no quantitative discussion concerning spheroidal modes on the effect of discontinuities in the Earth on the distribution of the eigenfrequencies.

In this paper, we first derive a frequency equation for the spheroidal oscillations of a multi-layered spherical Earth in terms of the matrix method. Then, its asymptotic formula, valid at high frequency limit, is derived. Asymptotic frequency equations for simple Earth models are obtained in explicit forms and their solutions to the zero order and first order approximations are derived. Finally, numerical computation is made for two kinds of models, one with a very simple structure and the other with a rather realistic mantle structure, in order to confirm the validity of the above mentioned approximate solutions and to examine by experiments the effect of the discontinuities on the asymptotic patterns of the distribution of the eigenfrequencies.

The matrix method is equivalent to the so-called Thomson-Haskell method applied primarily to wave propagation in a plane stratified medium. Its principle is now familiar to us and we can find some applications to spherically stratified media (GILBERT and MACDONALD, 1960; BEN-MENAHEM, 1964b; PHINNEY and ALEXANDER, 1966; BHATTACHARYA, 1976). However, no expression of a spheroidal frequency equation for an Earth with a solid inner core seems to be directly available. Here, we will develop independent formulation to obtain the formal frequency equation in a form convenient for our present purpose. The effect of gravity is ignored since it is expected to be small for higher modes.

## 2. Frequency Equation for a Multi-Layered Earth

We assume that an Earth is formed of the crust/mantle, the liquid outer core and the solid inner core, each medium consisting of the stack of uniform spherical layers in welded contact. A realistic Earth model is obtained by increasing the number of uniform layers. The numbering of the layers and boundaries are shown in Fig. 1, where the numbers 1 through $K$ refer to the inner core, $K+1$ through $L$ to the outer core and $L+1$ through $M$ to the mantle/crust respectively.

We denote radial factors of displacements and stresses for the spheroidal modes in a vector form as

$$\boldsymbol{y}_i(r)=(rU_i(r)\ ,\quad rV_i(r)\ ,\quad r^2S_i(r)\ ,\quad r^2T_i(r))^T \quad (r_{i-1}\leq r\leq r_i) \qquad (2.1)$$

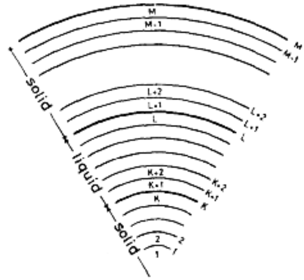


Fig. 1. Multi-layered Earth model consisting of the solid inner core ($K$ layers), liquid outer core ($L-K$ layers) and mantle/crust ($M-L$ layers).

where $U_i$ and $V_i$ are the radial and tangential displacement components in the $i$-th layer, $S_i$ and $T_i$ the radial and tangential stress components acting on the plane normal to the radial direction. $r$ means the radial distance and $r_i$ that of the $i$-th interface. By the superscript $T$ (transpose) we define $\boldsymbol{y}_i(r)$ as a column vector, which is, in a homogeneous and isotropic medium, given by

$$\boldsymbol{y}_i(r)=\boldsymbol{E}_i(r)\boldsymbol{c}_i \quad (r_{i-1}\leq r\leq r_i)\ , \tag{2.2}$$

where

$$\begin{aligned} &\boldsymbol{E}_i(r)=(e^i_{jk}) \quad (j,\ k=1,\ 2,\ 3,\ 4)\ , \\ &\boldsymbol{c}_i=(A_i,\ B_i,\ C_i,\ D_i)^T\ . \end{aligned} \tag{2.3}$$

$\boldsymbol{E}_i$ is the $4\times 4$ matrix and its elements $e^i_{jk}(r)$ are, referring to the solutions of equations of motion obtained by SEZAWA (1932), given by

$$(e^i_{jk})=\begin{pmatrix} h_i r j'_n(h_i r) & N^2 j_n(k_i r) & h_i r n'_n(h_i r) & N^2 n_n(k_i r) \\ j_n(h_i r) & k_i r j'_n(k_i r)+j_n(k_i r) & n_n(h_i r) & k_i r n'_n(k_i r)+n_n(k_i r) \\ \mu_i g(j_n, h_i r) & \mu_i N^2 f(j_n, k_i r) & \mu_i g(n_n, h_i r) & \mu_i N^2 f(n_n, k_i r) \\ \mu_i f(j_n, h_i r) & \mu_i h(j_n, k_i r) & \mu_i f(n_n, h_i r) & \mu_i h(n_n, k_i r) \end{pmatrix} \tag{2.4}$$

where $j_n(\zeta_i r)$ and $n_n(\zeta_i r)$ are the spherical Bessel and the spherical Neumann function of the order $n$ respectively, $h_i$ and $k_i$ the wave numbers of $P$ and $S$ waves in the $i$-th layer, $\mu_i$ the rigidity, and

$$N^2=n(n+1)\ , \quad z'_n(\zeta_i r)=dz_n(\zeta_i r)/d(\zeta_i r) \quad (\zeta_i=h_i \text{ or } k_i)\ ,$$
$$f(z_n,\ \zeta_i r)=2\zeta_i r z'_n(\zeta_i r)-2z_n(\zeta_i r)\ ,$$

$$g(z_n,\,\zeta_i r)=-4\zeta_i r z_n'(\zeta_i r)-\{(k_i r)^2-2N^2\}z_n(\zeta_i r)\ ,$$
$$h(z_n,\,k_i r)=f(z_n,\,k_i r)+g(z_n,\,k_i r)\ . \tag{2.5}$$

The elements of the vector $c_i$ are unknown constants in the $i$-th layer, which are to be determined from boundary conditions and source conditions.

From the physical requirement that displacement components have to be finite at the center of the Earth, we put, for the innermost layer, all the terms that the spherical Neumann function is concerned with to be zero. Hence, we have

$$\begin{aligned} &e^1_{j3}=e^1_{j4}=0 \quad (j=1,\,2,\,3,\,4)\ ,\\ &C_1=D_1=0\ . \end{aligned} \tag{2.6}$$

In the liquid medium, the rigidity is zero and a shear stress vanishes. Hence we put, for $i=K+1,\,K+2,\,\cdots,\,L$,

$$\begin{aligned} &B_i=D_i=0\ ,\\ &e^i_{j2}=e^i_{j4}=0 \quad (j=1,\,2,\,3,\,4)\ ,\quad e^i_{4k}=0 \quad (k=1,\,3)\ ,\\ &e^i_{31}=-\lambda_i(h_i r)^2 j_n(h_i r)\ ,\quad e^i_{33}=-\lambda_i(h_i r)^2 n_n(h_i r)\ ,\quad (r_{i-1}\leq r\leq r_i)\ . \end{aligned} \tag{2.7}$$

$e^i_{31}$ and $e^i_{33}$ are rewritten by use of $\lambda_i$ (Lamé elastic parameter). Here we introduce the following notations

$$\begin{aligned} &E^{jk}_i(r)=\begin{pmatrix} e^i_{j1} & e^i_{j2} & e^i_{j3} & e^i_{j4}\\ e^i_{k1} & e^i_{k2} & e^i_{k3} & e^i_{k4}\end{pmatrix}\ ,\quad \hat{E}_i(r)=\begin{pmatrix} e^i_{11} & e^i_{13}\\ e^i_{31} & e^i_{33}\end{pmatrix}\ ,\\ &E^j_i(r)=(e^i_{j1},\,e^i_{j2},\,e^i_{j3},\,e^i_{j4})\ ,\qquad \hat{\mathbf{c}}_i=\begin{pmatrix} A_i\\ C_i\end{pmatrix}\ , \end{aligned} \tag{2.8}$$

Then, the boundary conditions that displacement and stress components are continuous at each interface lead to

$$\begin{aligned} &\boldsymbol{E}_i(r_i)\boldsymbol{c}_i=\boldsymbol{E}_{i+1}(r_i)\boldsymbol{c}_{i+1} \quad (i=1,\,2,\,\cdots,\,K-1,\,L+1,\,L+2,\,\cdots,\,M-1)\ ,\\ &\hat{\boldsymbol{E}}_i(r_i)\hat{\mathbf{c}}_i=\hat{\boldsymbol{E}}_{i+1}(r_i)\hat{\mathbf{c}}_{i+1} \quad (i=K+1,\,K+2,\,\cdots,\,L-1)\ ,\\ &\boldsymbol{E}^{13}_K(r_K)\boldsymbol{c}_K=\hat{\boldsymbol{E}}_{K+1}(r_K)\hat{\mathbf{c}}_{K+1}\ ,\quad \boldsymbol{E}^4_K(r_K)\boldsymbol{c}_K=0\ ,\\ &\hat{\boldsymbol{E}}_L(r_L)\hat{\mathbf{c}}_L=\boldsymbol{E}^{13}_{L+1}(r_L)\boldsymbol{c}_{L+1}\ ,\quad \boldsymbol{E}^4_{L+1}(r_L)\boldsymbol{c}_{L+1}=0\ . \end{aligned} \tag{2.9}$$

With the aid of the first relation of Eq. (2.9), it is possible to connect the vector $\boldsymbol{c}_M$ with $\boldsymbol{c}_{L+1}$ and the vector $\boldsymbol{c}_K$ with $\boldsymbol{c}_1$. Then, putting stress components on the free surface ($r=r_M=a$) to be zero, we get

$$\boldsymbol{y}_M(a)=(a\,U_M(a),\,a\,V_M(a),\,0,\,0)^T=\boldsymbol{F}_M\boldsymbol{c}_{L+1}\ , \tag{2.10}$$

where

$$\boldsymbol{F}_M=\boldsymbol{D}_M\boldsymbol{D}_{M-1}\cdot\cdots\cdot\boldsymbol{D}_{L+2}\boldsymbol{E}_{L+1}(r_{L+1})\ , \tag{2.11}$$

and

$$D_i = E_i(r_i)E_i^{-1}(r_{i-1}) . \tag{2.12}$$

$E_i^{-1}$ is the inverse matrix of $E_i$. The other relation is

$$c_K = F_K c_1 , \tag{2.13}$$

where

$$F_K = E_K^{-1}(r_{K-1})D_{K-1}D_{K-2} \cdot \cdots \cdot D_2 E_1(r_1) . \tag{2.14}$$

In a similar manner, from the second relation of Eq. (2.9), we get

$$\hat{c}_L = \hat{F}_L \hat{c}_{K+1} , \tag{2.15}$$

where

$$\hat{F}_L = \hat{E}_L^{-1}(r_{L-1})\hat{D}_{L-1}\hat{D}_{L-2} \cdot \cdots \cdot \hat{D}_{K+2}\hat{E}_{K+1}(r_{K+1}) . \tag{2.16}$$

A matrix with a hat means a $2\times 2$ matrix. From the latter four equations of Eq. (2.9) and Eqs. (2.10), (2.13), (2.15), we obtain

$$\begin{aligned} &E_K^4(r_K)F_K c_1 = 0 , \quad E_K^{13}(r_K)F_K c_1 - \hat{E}_{K+1}(r_K)\hat{c}_{K+1} = \hat{0} , \\ &\hat{E}_L(r_L)\hat{F}_L\hat{c}_{K+1} - E_{L+1}^{13}(r_L)c_{L+1} = \hat{0} , \quad E_{L+1}^4(r_L)c_{L+1} = 0 , \\ &F_M^{34}c_{L+1} = \hat{0} , \end{aligned} \tag{2.17}$$

where $\hat{0}$ denotes the zero vector in two dimensions, and $F_M^{34}$ is the $2\times 4$ matrix consisting of the third and fourth rows of the matrix $F_M$, defined in a similar manner as $E_i^{jk}$ in Eq. (2.8). These equations can be arranged in one equational form as

$$Ac = 0 \tag{2.18}$$

where

$$\begin{aligned} &A = (a_{jk}) \quad (j, k = 1, 2, \cdots, 8) , \\ &c = (A_1, B_1, A_{K+1}, C_{K+1}, A_{L+1}, B_{L+1}, C_{L+1}, D_{L+1})^T \end{aligned} \tag{2.19}$$

and $0$ means the zero vector in eight dimensions. The elements of the matrix $A$ are given by

$$\begin{aligned} &(a_{11}, a_{12}, 0, 0) = E_K^4(r_K)F_K , \quad \begin{pmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{pmatrix} = -\hat{E}_{K+1}(r_K) \\ &\begin{pmatrix} a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \end{pmatrix} = E_K^{13}(r_K)F_K , \quad \begin{pmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{pmatrix} = \hat{E}_L(r_L)\hat{F}_L , \\ &\begin{pmatrix} a_{45} & a_{46} & a_{47} & a_{48} \\ a_{55} & a_{56} & a_{57} & a_{58} \end{pmatrix} = -E_{L+1}^{13}(r_L) , \quad (a_{65}, a_{66}, a_{67}, a_{68}) = E_{L+1}^4(r_L) \\ &\begin{pmatrix} a_{75} & a_{76} & a_{77} & a_{78} \\ a_{85} & a_{86} & a_{87} & a_{88} \end{pmatrix} = F_M^{34} . \end{aligned} \tag{2.20}$$

Other elements are all identically zero.

Hence, the frequency equation of the spheroidal oscillations of the spherically symmetric, multi-layered (solid-liquid-solid) Earth is formally given as

$$\det \boldsymbol{A}=0 \ . \tag{2.21}$$

Among $8\times8$ elements of the matrix $\boldsymbol{A}$, thirty components are identically zero and thus it is easy to reduce its dimension to a lower one, say, $4\times4$.

In obtaining the eigenfunctions, $U(r)$, $V(r)$, we have to get values of the constants $\boldsymbol{c}_i$ for all layers. This can be done as follows. If we standardize, in a conventional way, the radial component of surface displacements to be unity, that is, $U_M(a)=1$, we get another equation, from Eq. (2.10),

$$\boldsymbol{F}_M^1 \boldsymbol{c}_{L+1}=a \ , \tag{2.22}$$

where $\boldsymbol{F}_M^1$ is the row vector consisting of the first row of the matrix $\boldsymbol{F}_M$. Then we can solve the equations, (2.18) and (2.22), for $\boldsymbol{c}_1$, $\hat{\boldsymbol{c}}_{K+1}$ and $\boldsymbol{c}_{L+1}$, and subsequently Eq. (2.9) for all $\boldsymbol{c}_i$. Hence, from Eq. (2.2), we can obtain the eigenfunction $\boldsymbol{y}(r)$ for the whole space in the Earth.

A similar treatment is possible for the problem of excitation of free oscillations of the Earth due to an external force (say, a double couple point source) in it. Then, we introduce an equivalent source function (USAMI *et al.*, 1970), which is defined as a discontinuity of $\boldsymbol{y}_m(r)$ across the source surface ($r=r_s$) situated in the $m$-th layer. This imposes another boundary condition on $\boldsymbol{y}_i(r)$ besides Eq. (2.9). Hence, the problem has to be solved so that $\boldsymbol{y}_i(r)$ may have a jump by an amount $\delta\boldsymbol{y}_s$ (equivalent source function) at $r=r_s$ in the $m$-th layer. Then, it is found that Eq. (2.10) is modified to

$$(aU_M(a),\ aV_M(a),\ 0,\ 0)^T=\boldsymbol{F}_M\boldsymbol{c}_{L+1}+\boldsymbol{F}_s\boldsymbol{E}_m^{-1}(r_s)\delta\boldsymbol{y}_s \ , \tag{2.23}$$

where

$$\boldsymbol{F}_s=\boldsymbol{D}_M\boldsymbol{D}_{M-1}\cdots\boldsymbol{D}_{m+1}\boldsymbol{E}_m(r_m) \ . \tag{2.24}$$

If we put the source term as

$$\boldsymbol{F}_s\boldsymbol{E}_m^{-1}(r_s)\delta\boldsymbol{y}_s=\boldsymbol{f}^s=(f_1^s,\ f_2^s,\ f_3^s,\ f_4^s)^T \ , \tag{2.25}$$

we get, in place of the last relation of Eq. (2.17),

$$\boldsymbol{F}_M^{34}\boldsymbol{c}_{L+1}=-(f_3^s,\ f_4^s)^T \ , \tag{2.26}$$

and thus, in place of Eq. (2.18),

$$Ac=(0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -f_3^s,\ -f_4^s)^T\ . \tag{2.27}$$

By solving these simultaneous linear equations we can get the constants $c_1$, $\hat{c}_{K+1}$ and $c_{L+1}$. Hence, the formal solution for surface displacements is readily obtained from Eq. (2.23).

When an Earth consists of solid (crust/mantle) and liquid (core) media, we have only to remove the inner solid layers from the preceding model. Then, the $(K+1)st$ layer is shifted to the lowest layer which includes the center of the Earth, and we put $e_{13}^{K+1}=e_{33}^{K+1}=0$, $C_{K+1}=0$ in the same manner as Eq. (2.6). Slight modification of the preceding formulation leads, instead of Eq. (2.18), to

$$\begin{pmatrix} a_{43} & a_{45} & a_{46} & a_{47} & a_{48} \\ a_{53} & a_{55} & a_{56} & a_{57} & a_{58} \\ 0 & a_{65} & a_{66} & a_{67} & a_{68} \\ 0 & a_{75} & a_{76} & a_{77} & a_{78} \\ 0 & a_{85} & a_{86} & a_{87} & a_{88} \end{pmatrix} \begin{pmatrix} A_{K+1} \\ A_{L+1} \\ B_{L+1} \\ C_{L+1} \\ D_{L+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.28}$$

where the elements $a_{jk}$ are the same as those defined in Eq. (2.20). The frequency equation is given by the determinant of the above matrix $\widetilde{A}$, that is,

$$\det \widetilde{A} = 0\ , \tag{2.29}$$

where

$$\widetilde{A}=(a_{jk}) \quad \begin{pmatrix} j=4,\ 5,\ \cdots,\ 8 \\ k=3,\ 5,\ \cdots,\ 8 \end{pmatrix}\ . \tag{2.30}$$

When an Earth is constructed by only solid layers, we remove the liquid and inner solid layers from the first model. Then, the $(L+1)$st layer is shifted to the lowest one and we have to put $e_{j3}^{L+1}=e_{j4}^{L+1}=0$ $(j=1,\ 2,\ 3,\ 4)$, $C_{L+1}=D_{L+1}=0$. In this case, the last equation of (2.17) can be rewritten as

$$\begin{pmatrix} f_{31}^M & f_{32}^M \\ f_{41}^M & f_{42}^M \end{pmatrix} \begin{pmatrix} A_{L+1} \\ B_{L+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.31}$$

where $f_{jk}^M$ is an element of the matrix $F_M$. Hence, the frequency equation is simply given as

$$f_{31}^M f_{42}^M - f_{32}^M f_{41}^M = 0\ . \tag{2.32}$$

## 3. Asymptotic Frequency Equation

When an argument of the spherical Bessel (or Neumann) function is very large compared with its order, $j_n(z)$ and $n_n(z)$ are asymptotically approximated as (Watson 1952, p. 199)

$$j_n(z)\simeq(1/z)\sin(z-n\pi/2)\ ,\quad n_n(z)\simeq-(1/z)\cos(z-n\pi/2)\ .$$

Hence, if we assume $h_ir\gg n$ ($r\geq r_1$, $i=1, 2, \cdots, M$), we have the following approximations for the functions in $\boldsymbol{E}_i(r)$

$$\begin{aligned}
&j_n(z)\simeq(1/z)\sin Z\ ,\quad n_n(z)\simeq-(1/z)\cos Z\ ,\\
&j_n'(z)\simeq(1/z)\cos Z\ ,\quad n_n'(z)\simeq(1/z)\sin Z\ ,\\
&f(j_n, z)\simeq2\cos Z\ ,\quad g(j_n, z)\simeq-(z_k^2/z)\sin Z\ ,\quad h(j_n, z_k)\simeq-z_k\sin Z_k\ ,\\
&f(n_n, z)\simeq2\sin Z\ ,\quad g(n_n, z)\simeq(z_k^2/z)\cos Z\ ,\quad h(n_n, z_k)\simeq z_k\cos Z_k\ ,
\end{aligned}\tag{3.1}$$

where

$$z=h_ir\quad\text{or}\quad k_ir\ ,\quad z_k=k_ir\ ,\quad Z=z-n\pi/2\ ,\quad Z_k=z_k-n\pi/2\ .$$

Substituting the above formulas into Eq. (2.4), and keeping the most predominant terms, we get

$$\boldsymbol{E}_i(r)\simeq\begin{pmatrix}\cos H_r^i & 0 & \sin H_r^i & 0\\ 0 & \cos K_r^i & 0 & \sin K_r^i\\ -\omega\rho_i\alpha_ir\sin H_r^i & 0 & \omega\rho_i\alpha_ir\cos H_r^i & 0\\ 0 & -\omega\rho_i\beta_ir\sin K_r^i & 0 & \omega\rho_i\beta_ir\cos K_r^i\end{pmatrix}\tag{3.2}$$

where

$$H_r^i=h_ir-n\pi/2,\ K_r^i=k_ir-n\pi/2\ ,\tag{3.3}$$

and $\rho_i$, $\alpha_i$ and $\beta_i$ mean the density, $P$ and $S$ wave velocities in the $i$-th layer respectively, and $\omega$ the angular frequency. During reduction, the relations $\mu_i(k_ir)^2/(h_ir)=\omega\rho_i\alpha_ir$, $\mu_ik_ir=\omega\rho_i\beta_ir$ are employed. Thus, the matrix $\boldsymbol{E}_i(r)$ is reduced to a very simple form, and its inverse matrix is immediately obtained as

$$\boldsymbol{E}_i^{-1}(r)\simeq\begin{pmatrix}\cos H_r^i & 0 & -(1/\omega\rho_i\alpha_ir)\sin H_r^i & 0\\ 0 & \cos K_r^i & 0 & -(1/\omega\rho_i\beta_ir)\sin K_r^i\\ \sin H_r^i & 0 & (1/\omega\rho_i\alpha_ir)\cos H_r^i & 0\\ 0 & \sin K_r^i & 0 & (1/\omega\rho_i\beta_ir)\cos K_r^i\end{pmatrix}\tag{3.4}$$

Hence, we have, from Eq. (2.12),

$$\boldsymbol{D}_i\simeq\begin{pmatrix}\cos h_id_i & 0 & (1/\omega\rho_i\alpha_ir_{i-1})\sin h_id_i & 0\\ 0 & \cos k_id_i & 0 & (1/\omega\rho_i\beta_ir_{i-1})\sin k_id_i\\ -\omega\rho_i\alpha_ir_i\sin h_id_i & 0 & (r_i/r_{i-1})\cos h_id_i & 0\\ 0 & -\omega\rho_i\beta_ir_i\sin k_id_i & 0 & (r_i/r_{i-1})\cos k_id_i\end{pmatrix}\tag{3.5}$$

where $d_i = r_i - r_{i-1}$ (thickness of the $i$-th layer).

If we write a term associated with $P$ wave as "$P$" and that with $S$ wave as "$S$", we find that $\boldsymbol{E}_i$, $\boldsymbol{E}_i^{-1}$ and $\boldsymbol{D}_i$ are all denoted formally as

$$\begin{pmatrix} P & 0 & P & 0 \\ 0 & S & 0 & S \\ P & 0 & P & 0 \\ 0 & S & 0 & S \end{pmatrix} \tag{3.6}$$

Then, it is readily proved that $\boldsymbol{F}_M$ and $\boldsymbol{F}_K$ also retain a similar matrix form as (3.6). All the elements of $\hat{\boldsymbol{F}}_L$ are naturally identified as "$P$". In the result, each element $a_{jk}$ in Eq. (2.20) is expressed in its asymptotic form as

$$\begin{aligned} &a_{11} \simeq 0,\ a_{12} \simeq S_1^K,\ a_{21} \simeq P_1^K,\ a_{22} \simeq 0,\ a_{31} \simeq P_2^K,\ a_{32} \simeq 0,\ a_{23} \simeq P_1^L,\ a_{24} \simeq P_2^L\ , \\ &a_{33} \simeq P_3^L,\ a_{34} \simeq P_4^L,\ a_{43} \simeq P_5^L,\ a_{44} \simeq P_6^L,\ a_{53} \simeq P_7^L,\ a_{54} \simeq P_8^L\ , \\ &a_{45} \simeq P_1^M,\ a_{46} \simeq 0,\ a_{47} \simeq P_2^M,\ a_{48} \simeq 0,\ a_{55} \simeq P_3^M,\ a_{56} \simeq 0,\ a_{57} \simeq P_4^M,\ a_{58} \simeq 0\ , \\ &a_{65} \simeq 0,\ a_{66} \simeq S_1^M,\ a_{67} \simeq 0,\ a_{68} \simeq S_2^M,\ a_{75} \simeq P_5^M,\ a_{76} \simeq 0,\ a_{77} \simeq P_6^M,\ a_{78} \simeq 0\ , \\ &a_{85} \simeq 0,\ a_{86} \simeq S_3^M,\ a_{87} \simeq 0,\ a_{88} \simeq S_4^M\ , \end{aligned} \tag{3.7}$$

where the symbols "$P$" and "$S$" mean that an element is connected with $P$ and $S$ waves respectively and the superscripts $K$, $L$ and $M$ discriminate elements which are associated with the inner core, outer core and crust/mantle respectively. The numerical subscripts are merely put in order of appearance.

Now, Eq. (2.21) is formally reduced to

$$\det\begin{pmatrix} 0 & S_1^K & 0 & 0 & & & & \\ P_1^K & 0 & P_1^L & P_2^L & & 0 & & \\ P_2^K & 0 & P_3^L & P_4^L & & & & \\ 0 & 0 & P_5^L & P_6^L & P_1^M & 0 & P_2^M & 0 \\ 0 & 0 & P_7^L & P_8^L & P_3^M & 0 & P_4^M & 0 \\ & & & & 0 & S_1^M & 0 & S_2^M \\ & 0 & & & P_5^M & 0 & P_6^M & 0 \\ & & & & 0 & S_3^M & 0 & S_4^M \end{pmatrix} = 0\ . \tag{3.8}$$

Further reduction yields three independent equations,

$$S_1^K = 0\ , \quad \det\begin{pmatrix} S_1^M & S_2^M \\ S_3^M & S_4^M \end{pmatrix} = 0\ , \quad \det\begin{pmatrix} P_1^K & P_1^L & P_2^L & 0 & 0 \\ P_2^K & P_3^L & P_4^L & 0 & 0 \\ 0 & P_5^L & P_6^L & P_1^M & P_2^M \\ 0 & P_7^L & P_8^L & P_3^M & P_4^M \\ 0 & 0 & 0 & P_5^M & P_6^M \end{pmatrix} = 0\ . \tag{3.9}$$

The first, second and third equations give eigenfrequencies for shear oscillations of the inner solid sphere (inner core), shear oscillations of the outer solid shell (crust/mantle) and compressional oscillations of the whole Earth respectively. These three modes are called $J$, $(ScS)_v$ and *PKIKP* type respectively, corresponding to three different body-wave types (ANDERRSEN *et al.*, 1975; GILBERT, 1975). This decoupling of the rays is possible only for their radial propagation in the Earth because no conversion of wave types occurs at a boundary in the Earth for their normal incidence on it and they behave independently there. In view of the mode-ray duality (BEN-MENAHEM, 1964a), it is found that the basic assumption in this section that $h_i r \gg n$ (*i.e.*, the phase velocity is very high) just fits this ray-geometrical condition.

For the Earth consisting of solid (crust/mantle) and liquid (core) media, Eq. (2.29) is available instead of (2.21). Hence, the asymptotic frequency equation (3.9) reduces to

$$\det\begin{pmatrix} S_1^M & S_2^M \\ S_3^M & S_4^M \end{pmatrix}=0\ , \quad \det\begin{pmatrix} P_5^L & P_1^M & P_2^M \\ P_7^L & P_3^M & P_4^M \\ 0 & P_5^M & P_6^M \end{pmatrix}=0\ . \tag{3.10}$$

GILBERT (1975) has proved the decoupling of a frequency equation at high frequency directly from decomposition of basic differential equations for elastic material. His paper does not, however, include investigation on the effect of discontinuities in the medium on eigenfrequencies. In the following part, we derive the asymptotic frequency equations in explicit forms for simple Earth models, consisting of a uniform liquid core and a small number of solid spherical layers overlying it. Hereafter, we will call the $(ScS)_v$ type modes simply as "*ScS*-type" and *PKIKP* type as "*PKP*-type" (due to nonexistence of inner core phase), corresponding to the two equations of Eq. (3.10) respectively.

Since we assume the uniform liquid core, the $L$-th layer in Fig. 1 is reduced to the first layer. Hence, we put $L=1$ in the previous equations. Then, the layer 1 and $r_1$ indicate the uniform liquid core and its radius. Each element in Eq. (3.10) is obtained from Eqs. (2.15), (2.16), (2.20), (3.2) and (3.7), so we have

$$\begin{aligned}
a_{43}&=e_{11}^1(r_1)\simeq\cos H_{r_1}^1=P_5^L\ ,\\
a_{58}&=e_{31}^1(r_1)\simeq-\omega\rho_1\alpha_1 r_1\sin H_{r_1}^1=P_7^L\ ,\\
a_{45}&=-e_{11}^2(r_1)\simeq-\cos H_{r_1}^2=P_1^M\ ,\\
a_{47}&=-e_{13}^2(r_1)\simeq-\sin H_{r_1}^2=P_2^M\ ,\\
a_{56}&=-e_{31}^2(r_1)\simeq\omega\rho_2\alpha_2 r_1\sin H_{r_1}^2=P_3^M\ ,
\end{aligned}$$

$$
\begin{aligned}
a_{57} &= -e^2_{33}(r_1) \simeq -\omega\rho_2\alpha_2 r_1 \cos H^2_{r_1} = P^M_4 \ , \\
a_{66} &= e^2_{42}(r_1) \simeq -\omega\rho_2\beta_2 r_1 \sin K^2_{r_1} = S^M_1 \ , \\
a_{68} &= e^2_{44}(r_1) \simeq \omega\rho_2\beta_2 r_1 \cos K^2_{r_1} = S^M_2 \ ,
\end{aligned} \tag{3.11}
$$

where

$$
H^i_{r_j} = h_i r_j - n\pi/2 \ , \quad K^i_{r_j} = k_i r_j - n\pi/2 \quad (j = i \text{ or } i-1) \tag{3.12}
$$

These elements are common to any model with a uniform liquid core. The other elements, $P^M_5$, $P^M_6$, $S^M_3$, $S^M_4$, are obtained from an asymptotic formula for $F^M_M$, which depends on the number of layers overlying the core. For brevity's sake, we introduce the notations

$$
\begin{aligned}
R^P_i &= (\rho_{i+1}\alpha_{i+1} - \rho_i\alpha_i)/(\rho_{i+1}\alpha_{i+1} + \rho_i\alpha_i) \ , \\
R^S_i &= (\rho_{i+1}\beta_{i+1} - \rho_i\beta_i)/(\rho_{i+1}\beta_{i+1} + \rho_i\beta_i) \ ,
\end{aligned} \tag{3.13}
$$

which are the reflection coefficients for normal incidence of $P$ and $S$ waves on the $i$-th interface respectively.

( i ) Two-Layered Model (a homogeneous mantle and liquid core) From Eqs. (2.11), (2.20), (3.2) and (3.7), we get

$$
\begin{aligned}
a_{75} &= e^2_{31}(r_2) \simeq -\omega\rho_2\alpha_2 r_2 \sin H^2_{r_2} = P^M_5 \ , \\
a_{77} &= e^2_{33}(r_2) \simeq \omega\rho_2\alpha_2 r_2 \cos H^2_{r_2} = P^M_6 \ , \\
a_{86} &= e^2_{42}(r_2) \simeq -\omega\rho_2\beta_2 r_2 \sin K^2_{r_2} = S^M_3 \ , \\
a_{88} &= e^2_{44}(r_2) \simeq \omega\rho_2\beta_2 r_2 \cos K^2_{r_2} = S^M_4 \ .
\end{aligned} \tag{3.14}
$$

Inserting Eqs. (3.11) and (3.14) into (3.10) and arranging it, we get

$$
\begin{aligned}
&\sin k_2 d_2 = 0 \ , \\
&\sin(h_2 d_2 + h_1 r_1 - n\pi/2) + R^P_1 \sin(h_2 d_2 - h_1 r_1 + n\pi/2) = 0 \ .
\end{aligned} \tag{3.15}
$$

The first equation is the asymptotic frequency equation for the $ScS$-type modes and the second one is for the $PKP$-type modes.

( ii ) Three-Layered Model (a two-layered mantle and a liquid core) From Eq. (2.11), we have

$$
\boldsymbol{F}_M = \boldsymbol{D}_3 \boldsymbol{E}_2(r_2) \ . \tag{3.16}
$$

Then, with the aid of Eqs. (2.20), (3.2), (3.5) and (3.7), we obtain

$$
\begin{aligned}
a_{75} &\simeq -\omega\rho_3\alpha_3 r_3 \sin h_3 d_3 \cos H^2_{r_2} + (r_3/r_2)(P^M_5)_1 \cos h_3 d_3 = P^M_5 \ , \\
a_{77} &\simeq -\omega\rho_3\alpha_3 r_3 \sin h_3 d_3 \sin H^2_{r_2} + (r_3/r_2)(P^M_6)_1 \cos h_3 d_3 = P^M_6 \ , \\
a_{86} &\simeq -\omega\rho_3\beta_3 r_3 \sin k_3 d_3 \cos K^2_{r_2} + (r_3/r_2)(S^M_3)_1 \cos k_3 d_3 = S^M_3 \ , \\
a_{88} &\simeq -\omega\rho_3\beta_3 r_3 \sin k_3 d_3 \sin K^2_{r_2} + (r_3/r_2)(S^M_4)_1 \cos k_3 d_3 = S^M_4 \ ,
\end{aligned} \tag{3.17}
$$

where $(P^M_5)_1$, $(P^M_6)_1$, $(S^M_3)_1$ and $(S^M_4)_1$ are the coefficients defined for the preceding case and are identical with $P^M_5$, $P^M_6$, $S^M_3$ and $S^M_4$ in Eq. (3.14) respectively.

Substitution of Eqs. (3.11) and (3.17) into (3.10) leads to, corresponding to the *ScS*-type and *PKP*-type respectively,

$$\sin(k_3d_3+k_2d_2)+R_2^S\sin(k_3d_3-k_2d_2)=0\ ,$$
$$\sin(h_3d_3+h_2d_2+h_1r_1-n\pi/2)+R_1^P\sin(h_3d_3+h_2d_2-h_1r_1+n\pi/2)$$
$$+R_2^P\sin(h_3d_3-h_2d_2-h_1r_1+n\pi/2)+R_2^PR_1^P\sin(h_3d_3-h_2d_2+h_1r_1-n\pi/2)$$
$$=0\ , \tag{3.18}$$

where $R_i^P$ and $R_i^S$ are the reflection coefficients defined by Eq. (3.13), and $d_i$ is the thickness of the $i$-th layer.

(iii) Four-Layered Model (a three-layered mantle and a liquid core) From Eq. (2.11), we have

$$\boldsymbol{F}_M=\boldsymbol{D}_4\boldsymbol{D}_3\boldsymbol{E}_2(r_2)\ , \tag{3.19}$$

which yields, through Eqs. (2.20), (3.2), (3.5) and (3.7),

$$a_{75}\simeq-\omega\rho_4\alpha_4r_4\sin h_4d_4\{\cos h_3d_3\cos H_{r2}^2-(\rho_2\alpha_2/\rho_3\alpha_3)\sin h_3d_3\sin H_{r2}^2\}$$
$$+(r_4/r_3)(P_5^M)_{11}\cos h_4d_4=P_5^M\ ,$$
$$a_{77}\simeq-\omega\rho_4\alpha_4r_4\sin h_4d_4\{\cos h_3d_3\sin H_{r2}^2+(\rho_2\alpha_2/\rho_3\alpha_3)\sin h_3d_3\cos H_{r2}^2\}$$
$$+(r_4/r_3)(P_6^M)_{11}\cos h_4d_4=P_6^M\ ,$$
$$a_{46}\simeq-\omega\rho_4\beta_4r_4\sin k_4d_4\{\cos k_3d_3\cos K_{r2}^2-(\rho_2\beta_2/\rho_3\beta_3)\sin k_3d_3\sin K_{r2}^2\}$$
$$+(r_4/r_3)(S_3^M)_{11}\cos k_4d_4=S_3^M\ ,$$
$$a_{35}\simeq-\omega\rho_4\beta_4r_4\sin k_4d_4\{\cos k_3d_3\sin K_{r2}^2+(\rho_2\beta_2/\rho_3\beta_3)\sin k_3d_3\cos K_{r2}^2\}$$
$$+(r_4/r_3)(S_4^M)_{11}\cos k_4d_4=S_4^M\ , \tag{3.20}$$

where $(P_5^M)_{11}$, $(P_6^M)_{11}$, $(S_3^M)_{11}$ and $(S_4^M)_{11}$ stand for the coefficients $P_5^M$, $P_6^M$, $S_3^M$ and $S_4^M$ in Eq. (3.17).

After substitution of Eqs. (3.11) and (3.20) into (3.10) and some algebraic manipulations, we get

$$\sin(\eta_4+\eta_3+\eta_2)+R_2^S\sin(\eta_4+\eta_3-\eta_2)+R_3^S\sin(\eta_4-\eta_3-\eta_2)$$
$$+R_3^SR_2^S\sin(\eta_4-\eta_3+\eta_2)=0$$
$$\sin(\xi_4+\xi_3+\xi_2+H_{r1}^1)+R_1^P\sin(\xi_4+\xi_3+\xi_2-H_{r1}^1)+R_2^P\sin(\xi_4+\xi_3-\xi_2-H_{r1}^1)$$
$$+R_3^P\sin(\xi_4-\xi_3-\xi_2-H_{r1}^1)+R_2^PR_1^P\sin(\xi_4+\xi_3-\xi_2+H_{r1}^1)$$
$$+R_3^PR_2^P\sin(\xi_4-\xi_3+\xi_2+H_{r1}^1)+R_3^PR_1^P\sin(\xi_4-\xi_3-\xi_2+H_{r1}^1)$$
$$+R_3^PR_2^PR_1^P\sin(\xi_4-\xi_3+\xi_2-H_{r1}^1)=0\ , \tag{3.21}$$

where $\eta_i$ and $\xi_i$ are short for $k_id_i$ and $h_id_i$ respectively and $H_{r1}^1$ for $h_1r_1-n\pi/2$. The first and second equations correspond to the *ScS-type* and *PKP*-type modes respectively.