THE BENDING THEORY OF SHELL AND ITS APPLICATION

壳体的有矩理论和实用计算方法

英文版

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PREFACE

This book is a systematic fundamental study for bending theory of shell structure on the basis of " THE GENERAL THEORY OF SHELL AND ITS APPLICATION IN ENGINEERING" by W.S.Wlassow. To start with the coordinate system, we derive the fundamental differential equations for shell surface of any curvature. It contains two parts :the first part is bending theory and expounds the physical hypotheses for derivation ; the second part is its application in engineering. First three chapters are in theory and the remainder integrate theory with practice. Three general differential equilibrium equations which are expressed in displacements u,v,w are derived in first chapter. The fundamental differential equations of several usual shells are derived in chapter 2. In chapter 3, differential equation of 8th order which only contains a normal displacement parameter w is derived.

For practical purposes, the focus of this book is on the calculating formulae for stress analysis in spherical dome with simply supported and fixed edges which can also be used as building foundation under soft soil and clamped edges with skylight opening. In rectangular flat shell with simply supported edges and in revolutionary one-sheeted hyperboloid thin shell with simply supported edges which is a new type shell of negative gaussian curvature may be available for use in roof structure. Let the curvature $k_1 = k_2 = 0$ in rectangular flat shell it will become the formulae for plates. This book also gives the tables of calculating results about several kinds of shell for design purpose. Also the TRUE BASIC source program for shell stress analysis are written. In addition, the introduction of moment theory in hyperbolic paraboloid shell according to Wilby are derived. Also the formulae for stress analysis of cylindrical shell are provided. An engineering solution for the stress function φ of spherical shallow shell is given and the formulae of normal stresses N_1, N_2 are derived in order to complement this research project in reference [1].

This may be a reference book for students in colleges and technical institutes . It is also hoped that it will be a handy reference book for the designer. The author wishes to acknowledge his indebtedness to the reader for their helpful advise and suggestions. Appreciation is also due to the publications division for producing this book.

Wuhan, Hubei
July, 2002 *Shen Jin-tao*

contents

CHAPTER I

THE FUNDAMENTAL DIFFERENTIAL EQUATIONS OF SHELL BY BENDING THEORY

1. The Lama Coefficient in Curved Orthogonal Coordinates

(1) Radius vector in Cartesian coordinate expressed as in Fig. 1

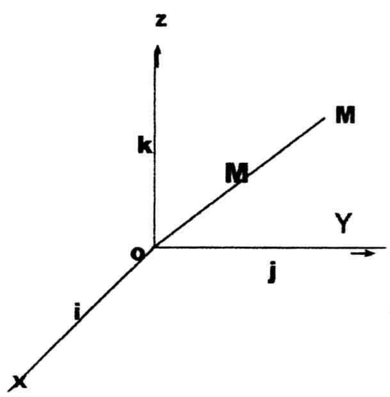


Fig. 1

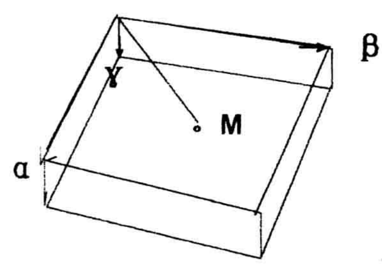


Fig. 2

$$\mathbf{OM} = \mathbf{M} = \mathbf{X}\mathbf{i} + \mathbf{Y}\mathbf{j} + \mathbf{Z}\mathbf{k} \qquad 1$$

Where **i, j, k** are unit vector along **ox, oy, oz axes.**

(2) A curved orthogonal coordinates is shown in Fig.2. When α, β, γ are constant , the so formed surfaces are called coordinate surfaces. The intersecting lines of these surfaces are defined as coordinate lines. Assume a point **M** is defined by the coordinates $\alpha = \beta = \gamma =$ const. A point **N** is defined by the coordinates $\alpha + d\alpha =$const., $\beta + d\beta =$const., $\gamma + d\gamma =$ const.. So the total differential of radius vector **M** may be defined as

$$d\mathbf{M} = \frac{\partial \mathbf{M}}{\partial \alpha} d\alpha + \frac{\partial \mathbf{M}}{\partial \beta} d\beta + \frac{\partial \mathbf{M}}{\partial \gamma} d\gamma \qquad 2$$

Vector $\mathbf{dM^2}$ is equal to the scalar of $\mathbf{dS^2}$ which is the distance between **M** and **N.**

$$ds^2 = \left(\frac{\partial \mathbf{M}}{\partial \alpha}\right)^2 d\alpha^2 + \left(\frac{\partial \mathbf{M}}{\partial \beta}\right)^2 d\beta^2 + \left(\frac{\partial \mathbf{M}}{\partial \gamma}\right)^2 d\gamma^2 + 2\,\frac{\partial \mathbf{M}}{\partial \alpha}\frac{\partial \mathbf{M}}{\partial \beta}\,\partial\alpha\partial\beta + 2\,\frac{\partial \mathbf{M}}{\partial \beta}\frac{\partial \mathbf{M}}{\partial \gamma}\,d\beta\, d\gamma$$

$$+ 2\,\frac{\partial \mathbf{M}}{\partial \gamma}\frac{\partial \mathbf{M}}{\partial \alpha}\,d\gamma\, d\alpha \qquad 3$$

The scalar product of two vectors $\frac{\partial \mathbf{M}}{\partial \alpha}\ \frac{\partial \mathbf{M}}{\partial \beta}$, $\frac{\partial \mathbf{M}}{\partial \beta}\ \frac{\partial \mathbf{M}}{\partial \gamma}$, $\frac{\partial \mathbf{M}}{\partial \gamma}\ \frac{\partial \mathbf{M}}{\partial \alpha}$ is a scalar. In curved orthogonal coordinates, these products are all equal to zero. Hence Eq. (3) becomes

$$ds^2 = \left(\frac{\partial \mathbf{M}}{\partial \alpha}\right)^2 d\alpha^2 + \left(\frac{\partial \mathbf{M}}{\partial \beta}\right)^2 d\beta^2 + \left(\frac{\partial \mathbf{M}}{\partial \gamma}\right)^2 d\gamma^2 \qquad \mathbf{4}$$

From Eq. (1), we have

$$\frac{\partial \mathbf{M}}{\partial \alpha} = \frac{\partial \mathbf{X}}{\partial \alpha} i + \frac{\partial \mathbf{Y}}{\partial \alpha} j + \frac{\partial \mathbf{Z}}{\partial \alpha} k$$
$$\frac{\partial \mathbf{M}}{\partial \beta} = \frac{\partial \mathbf{X}}{\partial \beta} i + \frac{\partial \mathbf{Y}}{\partial \beta} j + \frac{\partial \mathbf{Z}}{\partial \beta} k \qquad \mathbf{5}$$
$$\frac{\partial \mathbf{M}}{\partial \gamma} = \frac{\partial \mathbf{X}}{\partial \gamma} i + \frac{\partial \mathbf{Y}}{\partial \gamma} j + \frac{\partial \mathbf{Z}}{\partial \gamma} k$$

Take square of Eq. (5) and let

$$H_1^2 = \left(\frac{\partial \mathbf{M}}{\partial \alpha}\right)^2 = \left(\frac{\partial \mathbf{X}}{\partial \alpha}\right)^2 + \left(\frac{\partial \mathbf{Y}}{\partial \alpha}\right)^2 + \left(\frac{\partial \mathbf{Z}}{\partial \alpha}\right)^2$$
$$H_2^2 = \left(\frac{\partial \mathbf{M}}{\partial \beta}\right)^2 = \left(\frac{\partial \mathbf{X}}{\partial \beta}\right)^2 + \left(\frac{\partial \mathbf{Y}}{\partial \beta}\right)^2 + \left(\frac{\partial \mathbf{Z}}{\partial \beta}\right)^2 \qquad \mathbf{6}$$
$$H_3^2 = \left(\frac{\partial \mathbf{M}}{\partial \gamma}\right)^2 = \left(\frac{\partial \mathbf{X}}{\partial \gamma}\right)^2 + \left(\frac{\partial \mathbf{Y}}{\partial \gamma}\right)^2 + \left(\frac{\partial \mathbf{Z}}{\partial \gamma}\right)^2$$

Therefore , Eq. (4) becomes

$$ds^2 = H_1^2 d\alpha^2 + H_2^2 d\beta^2 + H_3^2 d\gamma^2 \qquad \mathbf{7}$$

Where H_1, H_2, H_3 are Lama coefficients. According to Eq.(6), we get the relation between curved orthogonal coordinates (α ,β, γ) and Cartesian coordinates (x, y, z). The Lama coefficients can be calculated as follows:

(a) Spheroidal coordinate

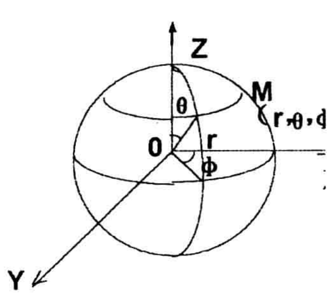
Let $\alpha = \theta$, $\beta = \varphi$, $\gamma = r$

$X = r \operatorname{Sin}\theta \ \cos\varphi$

$Y = r \sin\theta \ \sin\varphi$

$Z = R \cos\theta$

Fig. 3



$$H_1^2 = r^2 \cos^2\theta \cos^2\varphi + r^2 \cos^2\theta \sin^2\varphi + r^2 \sin^2\theta = r^2$$

$$H_2^2 = r^2 \sin^2\theta \sin^2\varphi + r^2 \sin^2\theta \cos^2\varphi = r^2 \sin^2\theta$$
$$H_3^2 = \sin^2\theta \cos^2\varphi + \sin^2\theta \sin^2\varphi + \cos^2\theta = 1$$

So, $ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 + dr^2$ **8**

(b) Cylindrical coordinate

Let $\alpha = r, \quad \beta = \varphi, \quad \gamma = z$

$X = r\cos\varphi$

$Y = r\sin\varphi$

$Z = z$

$H_1^2 = \cos^2\varphi + \sin^2\varphi = 1$

$H_2^2 = r^2\sin^2\varphi + r^2\cos^2\varphi = r^2$

$H_3^2 = 1$

$ds^2 = dr^2 + r^2 d\varphi^2 + dz^2$ 9

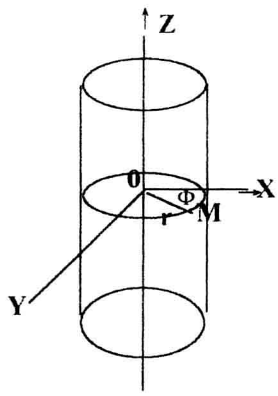
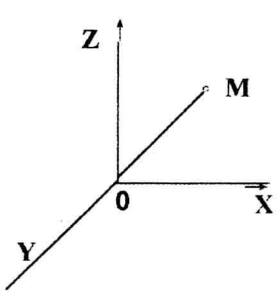


Fig. 4

(c) Cartesian coordinate



Let $\alpha = X, \quad \beta = Y, \quad \gamma = Z$

$H_1^2 = 1$

$H_2^2 = 1$

$H_3^2 = 1$

$ds^2 = dx^2 + dy^2 + dz^2$ **10**

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(d) For spheroidal shell where $\theta = 90^0$ in plane projection, Eq.(8) becomes

$ds^2 = r^2 d\varphi^2 + dr^2$ **11**

alternately set $d\beta = d\gamma = 0$, $d\alpha = d\gamma = 0$, and $d\alpha = d\beta = 0$, we immediately obtain $ds_1 = H_1 d\alpha$, $ds_2 = H_2 d\beta$, $ds_3 = H_3 d\gamma$ and $H_1 = ds_1/d\alpha$, $H_2 = ds_2/d\beta$, $H_3 = ds_3/d\gamma$, illustrating the geometrical interpretation of the Lama coefficients H_1, H_2, H_3.

2. The Codazzi-Gauss Criteria For Curved Surface

Selecting the Z-axis as an axis of revolution , a point on the surface generated by rotating the curve r=f(z) is defined by two coordinates viz. Z and β. Here Z is selected to be positive in the downward direction and β is positive for a clockwise rotation as viewed from 0, which is the center of the coordinate system. Hence the point M on the surface is uniquely determined by the coordinates Z, r, β.

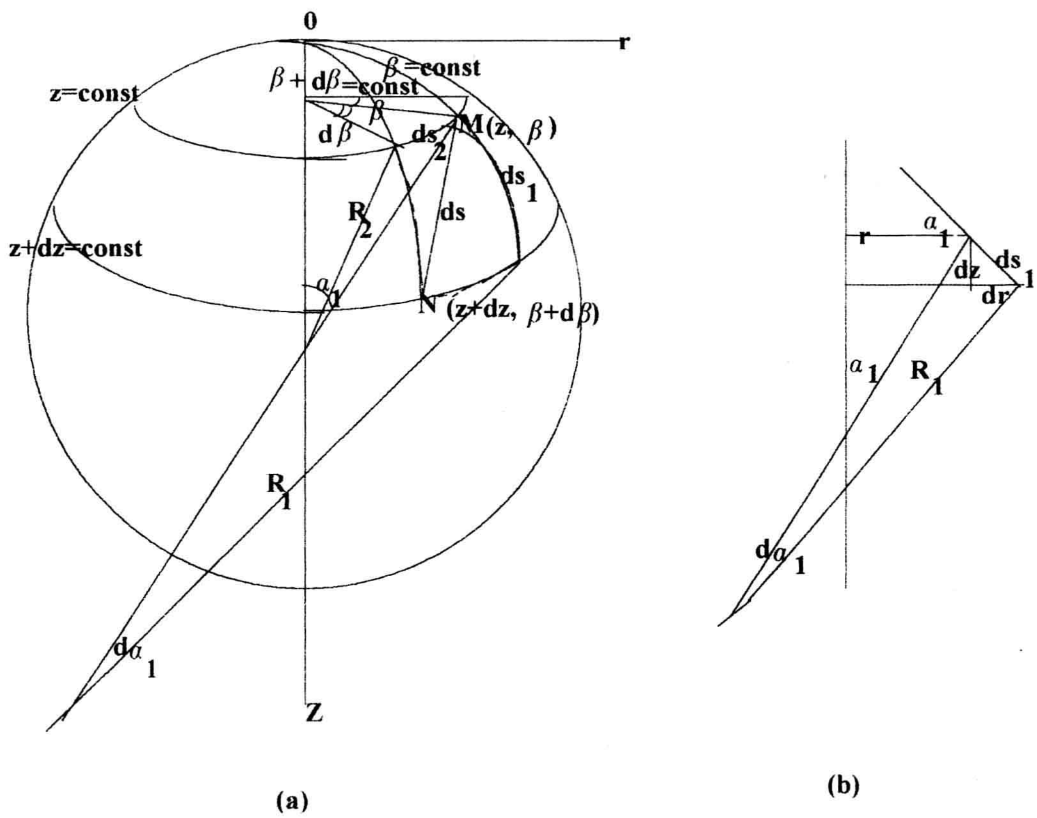

Fig. 7

There are two independent variables, namely **Z** and β. Any arbitrary constant value of **Z, Z**=const. represents a horizontal plane which intersects the surface of revolution in a parallel of latitude which diameter is a function of the value of **Z**. The plane β = const intersects the surface in a meridian passing through 0 and a point **M** on the surface of revolution. The meridian for any other value of β will have precisely the same shape. Point **M** , the interception of a meridian and a circle of latitude on the surface of revolution is thus defined by coordinates **Z** and β. **N**, another point on the surface, adjacent to **M**, may be defined by the coordinates **Z+dz, β + dβ.** Now $ds^2 = ds_1{}^2 + ds_2{}^2$ where ds_1 is the displacement along the meridian and ds_2 the displacement along the parallel of latitude. From Fig.7b , it is seen that

$$ds_1^2 = dz^2 + dr^2 = \frac{(dz^2+dr^2)}{dz^2}\, dz^2 = (1+r'^2)dz^2$$

$$ds_1 = (1+r'^2)^{1/2}\, dz$$

$$ds_2 = r d\beta$$

$$ds^2 = ds_1^2 + ds_2^2 = (1+r'^2)dz^2 + r^2 d\beta^2 = A^2 dz^2 + B^2 d\beta^2 \qquad \textbf{12}$$

Where $\mathbf{A} = (1 + r'^2)^{1/2}$

$\mathbf{B} = r$ **13**

This is the first of the generalized forms of equations in curved surface theory, in which **A** and **B** are parameters. If in Eq. (12) we set $d\beta = 0$ then $ds_1 = Adz$, $dz = 0$ then $ds_2 = Bd\beta$. It is evident that **A** is the arc length along a meridian for $dz=1$ and **B** is the arc length along the parallel of latitude for $d\beta=1$. For a generalized curved surface with an arbitrarily selected orthogonal coordinate system defined by the coordinates α and β, Eq.(12) assumes the generalized form

$$ds^2 = \mathbf{A}d\alpha^2 + \mathbf{B}d\beta^2 \qquad \textbf{14}$$

Thus, the coordinate Z=const. in Fig.(7a) corresponds to α and the coordinate β=const. corresponds to β. The coefficients will now be functions of α and β. We may again write

$ds_1 = \mathbf{A}d\alpha$

$ds_2 = \mathbf{B}d\beta$

Eq.(12) and (14) are of great importance in the theory of curved surfaces and hence in comprehending shell theory. Compare to Eq.(7) we have

$\mathbf{H_1 = A}$, $\mathbf{H_2 = B}$ and $\mathbf{d\gamma = 0}$. **15**

Still a second set of relationships plays a role in curved surface theory and hence also in shell theory. These are related to the principal radii of curvature. From Fig.(7), we have

$ds_1 = R_1 d\alpha_1$

$$k_1 = \frac{1}{R_1} = \frac{d\alpha_1}{ds_1} = \frac{1}{(1+r'^2)^{1/2}} \frac{d\alpha_1}{dz}$$

and $r = R_2 \sin\alpha_1$

$$\sin\alpha_1 = \frac{dz}{ds_1} = \frac{1}{(1+r'^2)^{1/2}} = \frac{1}{A}$$

$$\cos\alpha_1 = \frac{dr}{ds_1} = \frac{1}{(1+r'^2)^{1/2}} \frac{dr}{dz} = \frac{1}{A}\frac{dr}{dz} = \frac{1}{A} r'$$

$$\frac{d}{dz}(\sin\alpha_1) = \frac{d(\sin\alpha_1)}{d\alpha_1}\frac{d\alpha_1}{dz} = \cos\alpha_1 \frac{d\alpha_1}{dz}$$

$$\frac{d}{dz}(\sin\alpha_1) = \frac{d}{dz}(A^{-1}) = -A^{-2}\frac{dA}{dz} = -\frac{1}{A^2}\frac{d}{dz}(1+r'^2)^{1/2} = -\frac{1}{A^2}\frac{1}{2}(1+r'^2)^{-1/2}\,2r'r'' = -\frac{1}{A^3}r'r''$$

$$= -\frac{r'r''}{(1+r'^2)^{3/2}}$$

Thus the curvature of the meridian is found to be

$$k_1 = \frac{1}{(1+r'^2)^{1/2}}\frac{d\alpha_1}{dz} = \frac{1}{(1+r'^2)^{1/2}}\frac{1}{\cos\alpha_1}\frac{d(\sin\alpha)}{dz}$$

$$= \frac{1}{(1+r'^2)^{1/2}}\frac{1}{\cos\alpha_1}\left(-\frac{r'r''}{(1+r'^2)^{3/2}}\right) = -\frac{r''}{(1+r'^2)^{3/2}} = -\frac{1}{A^3}\frac{d^2r}{dz^2} \qquad 16$$

The curvature of the parallel of latitude is

$$k_2 = \frac{1}{R_2} = \frac{\sin\alpha_1}{r} = \frac{1}{r(1+r'^2)^{1/2}} = \frac{1}{AB} \qquad 17$$

Let the thickness of parallelepiped be γ, we can derive the radii of curvature of coordinate surfaces of parallelepiped in curved orthogonal coordinate system as shown in Fig.(8). The intersecting points of coordinate surfaces of parallelepiped are defined as follows:

M: α, β, γ	**N:** $\alpha+d\alpha, \beta+d\beta, \gamma+d\gamma$
M_1: $\alpha+d\alpha, \beta, \gamma$	**N_1:** $\alpha, \beta+d\beta, \gamma+d\gamma$
M_2: $\alpha, \beta+d\beta, \gamma$	**N_2:** $\alpha+d\alpha, \beta, \gamma+d\gamma$
M_3: $\alpha, \beta, \gamma+d\gamma$	**N_3:** $\alpha+d\alpha, \beta+d\beta, \gamma$

Thus the surface area and solid volum of parallelepiped are defined as follows:

$$dF_1 = H_2H_3d\beta d\gamma \qquad (MM_3N_1M_2)$$
$$dF_2 = H_1H_3d\alpha d\gamma \qquad (MM_3N_2M_1)$$
$$dF_3 = H_1H_2d\alpha d\beta \qquad (MM_2N_3M_1) \qquad 18$$
$$dV = H_1H_2H_3d\alpha d\beta d\gamma$$

Now we can determine the radii of curvature of coordinate surfaces. The angles shown in Fig. (8) are defined as follows:

$$d\varphi_{1\gamma} = \frac{M_1N_2 - MM_3}{MM_1} = \frac{\left(H_3d\gamma + \frac{\partial H_3 d\gamma}{\partial\alpha}d\alpha\right) - H_3d\gamma}{H_1d\alpha} = \frac{1}{H_1}\frac{\partial H_3}{\partial\alpha}d\gamma$$

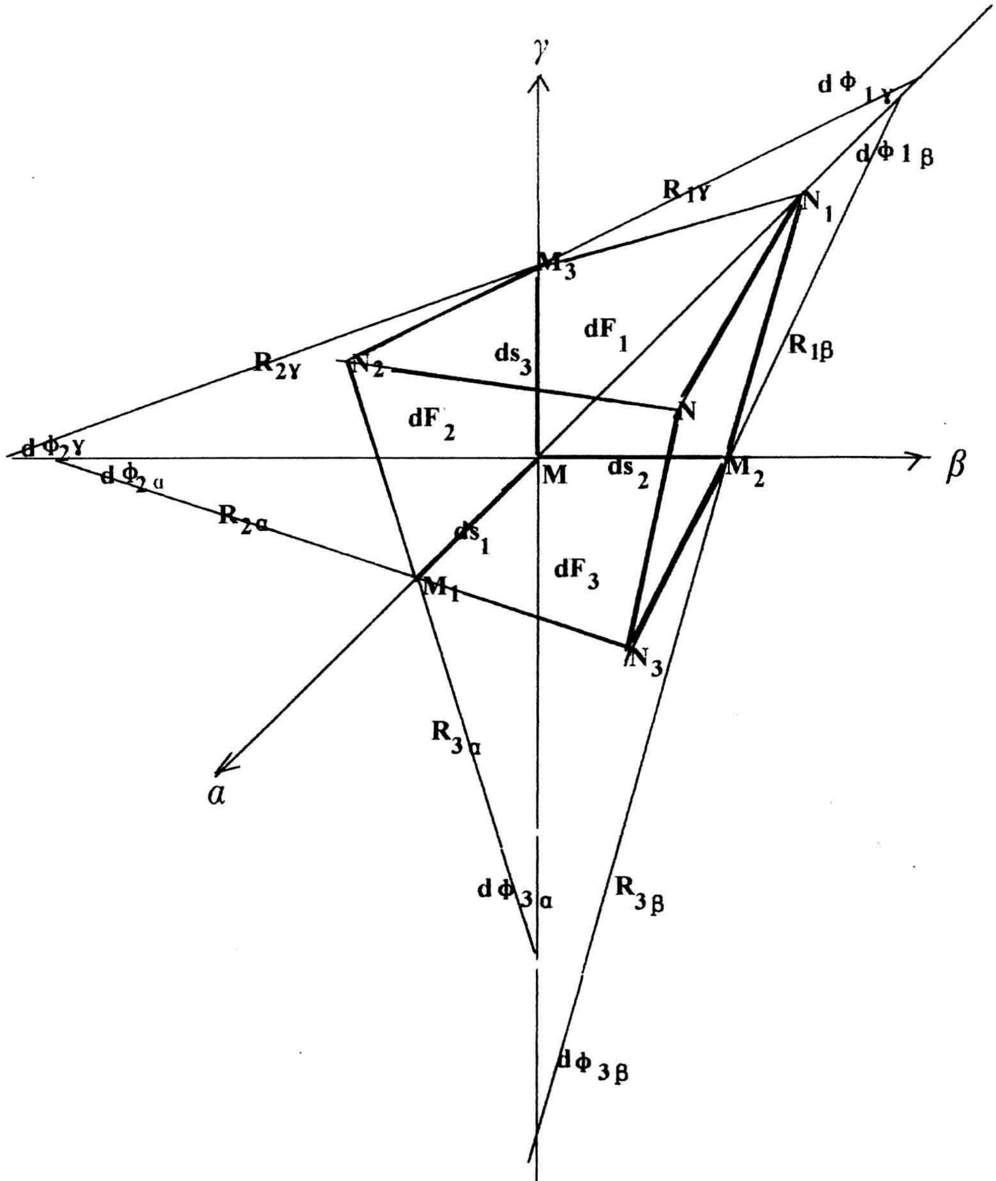

Fig. 8

$$d\varphi_{1\beta} = \frac{M_1N_3 - MM_2}{MM_1} = \frac{\left(H_2 d\beta + \frac{\partial H_2 d\beta}{\partial \alpha} d\alpha\right) - H_2 d\beta}{H_1 d\alpha} = \frac{1}{H_1}\frac{\partial H_2}{d\alpha} d\beta$$

$$d\varphi_2\gamma = \frac{M_2N_1 - MM_3}{MM_2} = \frac{\left(H_3d\gamma + \frac{\partial H_3 d\gamma}{\partial\beta}\, d\beta\right) - H_3d\gamma}{H_2d\beta} = \frac{1}{H_2}\frac{\partial H_3}{\partial\beta}\, d\gamma$$

$$d\varphi_2\alpha = \frac{M_2N_3 - MM_1}{MM_2} = \frac{\left(H_1d\alpha + \frac{\partial H_1 d\alpha}{\partial\beta}\, d\beta\right) - H_1d\alpha}{H_2d\beta} = \frac{1}{H_2}\frac{\partial H_1}{\partial\beta}\, d\alpha$$

$$d\varphi_3\alpha = \frac{M_3N_2 - MM_1}{MM_3} = \frac{\left(H_1d\alpha + \frac{\partial H_1 d\alpha}{\partial\gamma}\, d\gamma\right) - H_1d\alpha}{H_3d\gamma} = \frac{1}{H_3}\frac{\partial H_1}{\partial\gamma}\, d\alpha$$

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$$d\varphi_3\beta = \frac{M_3N_1 - MM_2}{MM_3} = \frac{\left(H_2d\beta + \frac{\partial H_2 d\beta}{\partial\gamma}\, \partial\gamma\right) - H_2d\beta}{H_3d\gamma} = \frac{1}{H_3}\frac{\partial H_2}{\partial\gamma}\, d\beta$$

The radius of curvature may be defined as

$$\frac{1}{\rho} = \lim_{p \to p} \frac{\Delta\theta}{\Delta s}$$

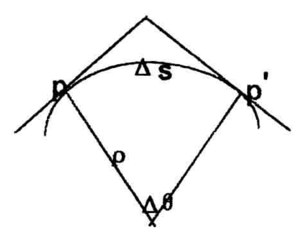

Fig. 9

Hence

$$k_1\gamma = \frac{d\varphi_{1\gamma}}{ds_3} = \frac{1}{H_1H_3}\frac{\partial H_3}{\partial\alpha}$$

$$k_1\beta = \frac{d\varphi_1\beta}{ds_2} = \frac{1}{H_1H_2}\frac{\partial H_2}{\partial\alpha}$$

$$k_2\gamma = \frac{d\varphi_2\gamma}{ds_2} = \frac{1}{H_2H_3}\frac{\partial H_3}{\partial\beta}$$

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$$k_2\alpha = \frac{d\varphi_2\alpha}{ds_1} = \frac{1}{H_1H_2}\frac{\partial H_1}{\partial\beta}$$

$$k_3\alpha = \frac{d\varphi_3\alpha}{ds_1} = \frac{1}{H_1H_3}\frac{\partial H_1}{\partial\gamma}$$

$$k_3\beta = \frac{d\varphi_3\beta}{ds_2} = \frac{1}{H_2H_3}\frac{\partial H_2}{\partial\gamma}$$

According to the geometrical hypothesis of the shell theory, the line normal to the middle surface remains normal after deformation. Its length is unchanged. Therefore in Fig.(8), axis MM_3 is a straight line. Hence in Eq.(7) where $H_3=1$. The radii of curvature which corresponds to Fig.(7) are

$$k_1 = \frac{d\varphi_3\alpha}{ds_1} = \frac{\frac{1}{H_3}\frac{\partial H_1}{\partial\gamma}d\alpha}{H_1 d\alpha} = \frac{1}{H_1H_3}\frac{\partial H_1}{\partial\gamma} = \frac{1}{A}\frac{\partial H_1}{\partial\gamma}$$

$$\partial H_1 = Ak_1\partial\gamma$$

$$H_1 = Ak_1\gamma$$

$$k_2 = \frac{d\varphi_3\beta}{ds_2} = \frac{\frac{1}{H_3}\frac{\partial H_2}{\partial\gamma}d\beta}{H_2 d\beta} = \frac{1}{H_2H_3}\frac{\partial H_2}{\partial\gamma} = \frac{1}{B}\frac{\partial H_2}{\partial\gamma}$$

$$\partial H_2 = Bk_2\partial\gamma$$

$$H_2 = Bk_2\gamma$$

The Lama coefficients of the surface varies with a distance γ apart from the middle plane along the normal line are

$$H_1 = A(1+k_1\gamma)$$

$$H_2 = B(1+k_2\gamma)$$

$$H_3 = 1$$

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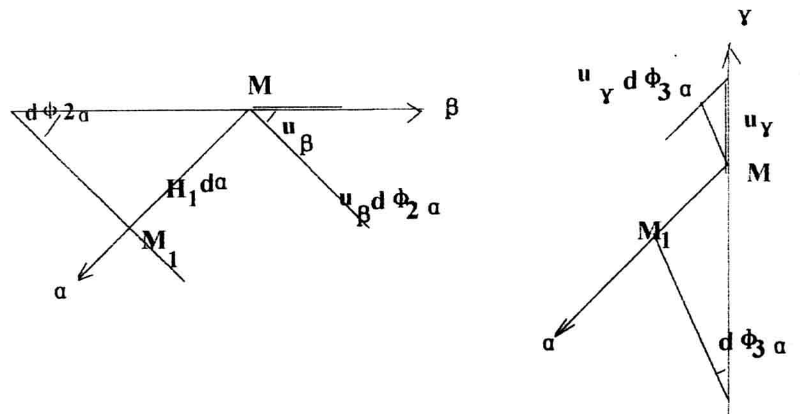
3. The Elongation And Shear Strain Of Parallelepiped In Curved Orthogonal Coordinates

There are six forms of deformation near point M in Fig.(8). That is , the elongation along perpendicular lines MM_1, MM_2, MM_3, and the shear between perpendicular planes dF_1, dF_2, dF_3. Let $e_{\alpha\alpha}$, $e_{\beta\beta}$ and $e_{\gamma\gamma}$ are the relative elongation of MM_1, MM_2, MM_3, respectively and $e_{\alpha\beta}$, $e_{\beta\gamma}$ and $e_{\gamma\alpha}$ represent shear strain. The total elongation in α direction is the sum of elongation in α direction and the induced elongation in that direction caused by the elongation in β and γ direction. Let the elongation of point M in α direction be u_α, then from Fig.(10) we have

$$e_{\alpha\alpha 1} = \frac{\partial u_\alpha}{ds_1} = \frac{\partial u_\alpha}{H_1 \partial\alpha}$$



(a) (b)

Fig. 10

$$e_{\alpha\alpha 2} = \frac{u_\beta\, d\varphi_2\alpha}{H_1 d\alpha} = \frac{u_\beta \frac{1}{H_2} \frac{\partial H_1}{\partial\beta} d\alpha}{H_1 d\alpha} = u_\beta \frac{1}{H_1 H_2} \frac{\partial H_1}{\partial\beta} = u_\beta k_2\alpha$$

$$e_{\alpha\alpha 3} = \frac{u_\gamma\, d\varphi_3\alpha}{H_1 d\alpha} = u_\gamma \frac{\frac{1}{H_3} \frac{\partial H_1}{\partial\gamma} d\alpha}{H_1 d\alpha} = u_\gamma \frac{1}{H_1 H_3} \frac{\partial H_1}{\partial\gamma} = u_\gamma k_3\alpha$$

The total elongation in α direction is

$$e_{\alpha\alpha} = e_{\alpha\alpha 1} + e_{\alpha\alpha 2} + e_{\alpha\alpha 3} = \frac{\partial u_\alpha}{H_1 d\alpha} + \frac{1}{H_1 H_2} \frac{\partial H_1}{\partial\beta} u_\beta + \frac{1}{H_1 H_3} \frac{\partial H_1}{\partial\gamma} u_\gamma$$

Similarly

$$e_{\beta\beta} = \frac{1}{H_2} \frac{\partial u_\beta}{\partial\beta} + \frac{1}{H_2 H_3} \frac{\partial H_2}{\partial\gamma} u_\gamma + \frac{1}{H_1 H_2} \frac{\partial H_2}{\partial\alpha} u_\alpha \qquad \textbf{22}$$

$$e_{\gamma\gamma} = \frac{1}{H_3} \frac{\partial u_\gamma}{\partial\gamma} + \frac{1}{H_1 H_3} \frac{\partial H_3}{\partial\alpha} u_\alpha + \frac{1}{H_2 H_3} \frac{\partial H_3}{\partial\beta} u_\beta$$

The shear strain of dF_1, dF_2 and dF_3 are constituted by two parts in each plane. In $\alpha\beta$ plane (dF_3), the total shear strain is the sum of the relative angular displacements of ds_1 and ds_2, as shown in Fig.(11). The angular displacement increases as the point moves from M to M'(u_α). Its value equals to $\partial u_\alpha / \partial s_2$. For $M_2\ddot{M}$ (u_β), the angular displacement decreases with a value of $\Delta d\varphi$.

$$\Delta d\varphi = \frac{d\varphi_1\beta}{ds_2} u_\beta = k_1\beta\, u_\beta$$

So the relative angular displacement of ds_2 in $\alpha\beta$ plane may be defined as follows

$$e_{\alpha\beta 1} = \frac{\partial u_\alpha}{\partial s_2} - \Delta d\varphi = \frac{\partial u_\alpha}{\partial s_2} - k_1\beta\, u_\beta$$

Also

$$e_{\alpha\beta 2} = \frac{\partial u_\beta}{\partial s_1} - \Delta d\varphi = \frac{\partial u_\beta}{\partial s_1} - \frac{d\varphi_2\alpha}{ds_1} u_\alpha = \frac{\partial u_\beta}{\partial s_1} - k_2\alpha\, u_\alpha$$

The total angular displacement in $\alpha\beta$ plane is

$$e_{\alpha\beta} = e_{\alpha\beta 1} + e_{\alpha\beta 2} = \frac{\partial u_\alpha}{\partial s_2} - k_1\beta\, u_\beta + \frac{\partial u_\beta}{\partial s_1} - k_2\alpha\, u_\alpha$$

$$= \frac{\partial u_\alpha}{H_2\partial\beta} - \frac{1}{H_1H_2}\frac{\partial H_2}{\partial\alpha} u_\beta + \frac{\partial u_\beta}{H_1\partial\alpha} - \frac{1}{H_1H_2}\frac{\partial H_1}{\partial\beta} u_\alpha$$

$$= \frac{H_2}{H_1}\frac{\partial}{\partial\alpha}\left(\frac{1}{H_2}u_\beta\right) + \frac{H_1}{H_2}\frac{\partial}{\partial\beta}\left(\frac{1}{H_1}u_\alpha\right)$$

Similarly

$$e_{\beta\gamma} = \frac{H_3}{H_2}\frac{\partial}{\partial\beta}\left(\frac{1}{H_3}u_\gamma\right) + \frac{H_2}{H_3}\frac{\partial}{\partial\gamma}\left(\frac{1}{H_2}u_\beta\right)$$

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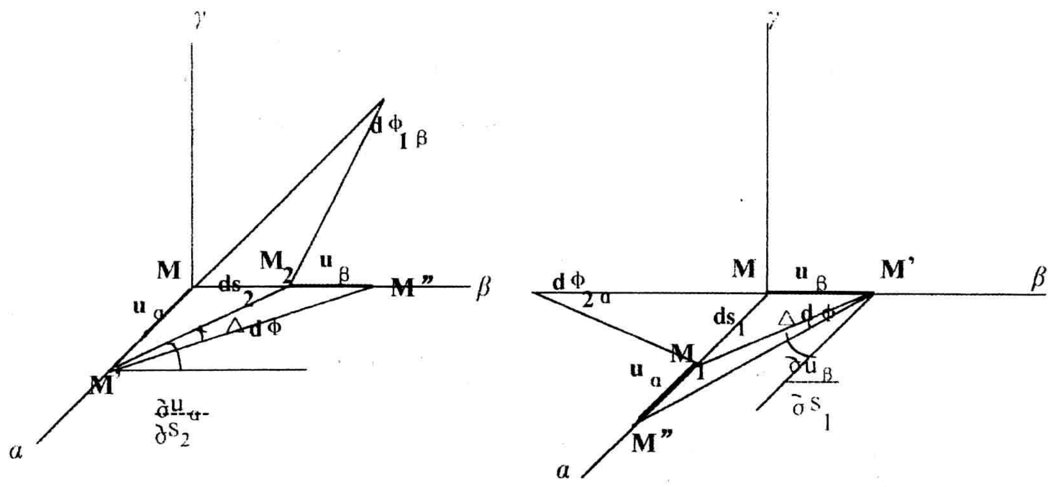
$$e_{\gamma\alpha} = \frac{H_1}{H_3}\frac{\partial}{\partial\gamma}\left(\frac{1}{H_1}u_\alpha\right) + \frac{H_3}{H_1}\frac{\partial}{\partial\alpha}\left(\frac{1}{H_3}u_\gamma\right)$$


γ
dΦ 1β
M
M2
uβ
ds2
M"
β
uα
ΔdΦ
M'
∂uα / ∂s2
α
γ
M
uβ
M'
β
dΦ 2α
ds1
ΔdΦ
M1
uα
∂uβ / ∂s1
M"
α

(a) (b)

Fig. 11