復旦大學

第三屆科學討論會

數学分組

题　　目：用法巴多項式的蔡查罗組合在具有光滑境界的連續点集上的近迫理論

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原书空白页

Approximation by Cesàro combination of Faber's polynomials on the continuum having fairly smooth boundary

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1.1. TERMINOLOGY, Denote by $\vartheta(s)$ the angle of the tangent at the point $z = z(s)$ of a plane curve Γ with the arc length s, making with the real axis. Writing

$$\omega(t, \vartheta(s)) = \max_{|s'-s|\leq t} |\vartheta(s') - \vartheta(s)|,$$

the curve Γ is said to be fairly smooth, if the fuction $\frac{1}{t}\log\frac{1}{t}$, $\omega(t, \vartheta(s))$ is integrable on $0 \leq t \leq 1$. A bounded continuum K in the z-plane is said to be ordinary, when the boundary Γ of K is a fairly smooth closed Jordan curve.

Let $\chi(v)$ be differentiable on M and be such that, for any $\varepsilon > 0$, the inequality

$$\left| \frac{\chi(v') - \chi(v)}{v' - v} - \chi'(v) \right| < \varepsilon$$

holds for v' and v satisfying $|v' - v| < \delta$, $\delta = \delta(\varepsilon)$

then we say that $\chi' v$ is the uniform derivative of $\chi(v)$ on M.

Associating to an ordinary continuum K, we have a sequence of Faber's polynomials*
$\Phi_0(z), \Phi(z), \ldots$. If $f(z)$ is absolutely integrable on the boundary Γ of K, then we have the Faber's series

$$f(z) \sim \sum a_k \Phi_k(z)$$

with the Faber's coefficients

$$a_k = \frac{1}{2\pi i} \int_{|\tau|=\rho} \frac{f(\Psi(\tau))}{\tau^{k+1}} d\tau \quad (k=0,1,2,\ldots).$$

where the function $z = \Psi(\omega)$ with $\frac{\Psi(\omega)}{w} \to 1$ $(\omega \to \infty)$ maps $|\omega| > \rho$ onto the complement domain of K.

We write $(r)_n = (r+1)(r+2)\cdots(r+n)/n!$ and call the sums $\sum_{k=0}^{n} (r)_{n-k} u_k / (r)_n$ $(n=0, 1, \ldots)$ the Cesàro combinations of the sequence $\{u_k\}$ with the order r.

The modulus of continuity of a continuous function $\chi(v)$ defining on M will be denoted by

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* [2] V. §§ 4.4—4.6

$$\omega(t,\chi(v)) = \max_{|v'-v|\le t} \left|\chi(v')-\chi(v)\right|,$$

which is increasing in t. We call

$$\limsup_{t\to 0} t\,\frac{\omega'(t,\chi)}{\omega(t,\chi)} \text{ and } \liminf_{t\to 0} t\,\frac{\omega'(t,\chi)}{\omega(t,\chi)}$$

respectively the upper index and the lower index of $\chi(v)$. It should be observed that $\chi(v)$ is reduced to a constant when and only when the upper index of $\chi(v)$ is greater than unity, and that the lower index is

§2. RÉSUMÉ. The purpose of the present paper is to study the degree of approximation when the uniform derivative $f(z)$ of a bounded function on an ordinary domain D the kernel of the ordinary continuum K. is approached by Cesàro means of the Faber's series of $f(z)$. The results obtained may be stated as follows.

The function $f(z)$ possesses limiting boundary values at each point of Γ, and wherewith $f(z)$ becomes a continuous function on the continuum $K = D + \Gamma$. (Theorem 12).

Let α and β denote the upper **index** as **well** as the lower index of $f(z)$ respectively, and let $f(z) \sim \sum a_k \Phi_k(z)$. Suppose either $\alpha < 1$ or $\beta > 0$ when on writing $\sigma_n^{r}(z, f) = \sum_{k=0}^{n} (r)_{n-k}\, a_k\, \Phi_k(z)/(r)_n$

$$\left| \sigma_n^{r}(z, f) - f(z) \right| \leqslant C\omega\left(\frac{1}{n+1}, f\right)$$

holds on K, if $r > 0$, where C is independent of n and z (Theorem 14 and Theorem 16). If $\alpha = 1$, then

$$\left| \sigma_n^{r}(z, f) - f(z) \right| \leqslant C\omega\left(\frac{1}{n+1}, f\right) \log \frac{1}{n+1},$$

Theorem 15).

In particular, we see that $f(z) \in$ Lip $\alpha\ (0 < \alpha \leqslant 1)$ implies

$$\sigma_n^{r}(z, f) - f(z) = O\left(\frac{1}{n^{\alpha}}\right).$$

This is reduced to a theorem of Alper, when $r = 1$*.

The proofs of these theorems of approximation, are based upon the following fundamental inequality

$$\left| f(z) - \sum_{k=0}^{n} c_k \Phi_k(z) \right| \leqslant B \left| f(\Psi(\omega)) - \sum_{0}^{n} c_k \omega^k \right|,$$

depending only upon the nature of the boundary Γ;

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* [1] Theorem 3.

$C_0, C_1, \cdots C_n$ being arbitrary constants.

To establish the fundamental inequality, we have proved that $\omega(t, \Psi'(e^{i\theta})) = O(\Omega(t, \vartheta(s)))$ (Theorem 11). This demands to improve a theorem of Hardy and Littlewood[5] concerning $f(z) \in$ Lip $\alpha(|z| \leq 1)$ into the following form:

Let $g(z)$ be regular in $|z|$, and be continuous on $|z| \leq 1$ with the modulus of continuity $\omega(t)$. If the upper index α of $g(z)$ is less than unity, then

$$|g'(z)| \leq \frac{C}{1-r}\,\omega(1-r)$$

with $r = |z| < 1$, and if $\alpha = 1$, then

$$|g'(z)| \leq \frac{C}{1-r}\,\omega(1-r)\log\frac{1}{1-r}.$$

Conversely, if

$$|g'(z)| \leq B\,\frac{\lambda(1-r)}{1-r},$$

then $\omega(t) \leq 5\lambda H$ (Theorem 8).

Setting $C_k = a_k (r)_{n-k}/(r)_n$, we have to estimate

$$f(\Psi(\omega)) - \sum_0^n a_k (r)_{n-k}\,\omega^k/(r)_n$$

on the circumference $|\omega| = \rho$. This is achieved by establishing the corresponding propositions on the Fourier series of continuous functions of a real variable (Theorem 1 and Theorem 2) - Thence we obtain the degree of approximation of $\sigma_n^{r}(z,f)-f(z)$, as stated above. By the way, we give a new criterion of uniform convergence for the Fourier series (Theorem 3)

II. The Fourier series of continuous founctions.

2.1. APPROXIMATION BY CESÀRO'S MEANS. A matrix

$$(\rho_{n0}, \rho_{n1}, \cdots, \rho_{nn}) \quad (n=0,1,\cdots)$$

satisfying the following two conditions

$$\lim_{n\to\infty} \rho_{nk} = 1 \quad (k=0,1,\cdots)$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\rho_{n0}+2\sum_{k=1}^{n}\rho_{nk}\cos kt\right|dt < K$$

is called of the type (A)*. A class of matrices of the type (A) is given by

$$\rho_{nk} = \frac{(r)_{n-k}}{(r)_n} \quad (r>0,\ k=0,1,\cdots,n;\ n=0,1,\cdots)$$

* Cf. [8] V. § 5.

The condition $\rho_{nk} \to 1 \ (n \to \infty)$ is evidently satisfied. The condition on the boundedness of the integrals of $\rho_{n0} + 2\sum_{k=1}^{n} \rho_{nk} \cos kt$ is well-known, and can also be verified by the following

LEMMA 1. Write

$$K_n^r(t) = \sum_{k=0}^{n} \frac{(r)_{n-k}}{(r)_n} \cos kt$$

and let $\frac{r\pi}{n} \leq t \leq \pi$, then for $n > 0$, $r \geq 0$,

$$K_n^{\nu}(t) = \frac{r(2n+r-1)}{2(n+r)(n+r-1)\sin^2\frac{t}{2}} + \frac{\sin(n_r) - \frac{1}{2}\pi r)}{(r)_n\left(2\sin\frac{t}{2}\right)^{r+1}} + \frac{\rho_n}{n^3 t^4},$$

$$\frac{dK_n^r(t)}{dt} = \frac{r(2n+r-1)}{2(n+r)(n+r-1)} \frac{\cos\frac{t}{2}}{\sin^3\frac{t}{2}} + \frac{d}{dt}\frac{\sin(n_r - \frac{1}{2}\pi r)}{(r)_n\left(2\sin\frac{t}{2}\right)^{\nu+1}} + \frac{\sigma_n}{n^2 t+1},$$

where $n_r = n + \frac{1}{2} + \frac{1}{2} r$, ρ_n and σ_n are both numerically less than an absolute constant. If $r < 3$, then these equations hold true in $0 < t \leq \pi$.

PROOF. The function $2(r)_n \sin\frac{t}{2} K_n^{\nu}(t)$ is the imaginary part of $z^{-n-\frac{1}{2}} \sum_{k=0}^{n} (r-1)_k z^k$ with

Abel's transformation gives

$$\sum_{k=0}^{n}(r-1)_k z^k = -\frac{(r-1)_n}{1-z} z^{n+1} + \frac{1}{1-z}\sum_{k=0}^{n}(r-2)_k z^k$$

$$= -\frac{(r-1)_n}{1-z} z^{n+1} - \frac{(r-2)_n}{(1-z)^2} z^{n+1} -$$

$$+ \frac{1}{(1-z)^2}\sum_{k=0}^{n}(r-3)_k z^k$$

Hence we have

$$K_n^{\nu}(t) = \frac{r(2n+2r-1)}{2(n+r)(n+\nu-1)\sin^2\frac{t}{2}} +$$

$$+ \frac{\sin(n_n - \frac{1}{2}\pi r)}{(r)_n (2\sin\frac{t}{2})^{r+1}} + \frac{f_n(r,t)}{n^5+1}$$

for $r < 3$. Indeed, the imaginary part of

$$\frac{z^{-n-\frac{1}{2}}}{(1-z)^2}\sum_{k=0}^{n}(r-3)_k z^k = z^{-n-\frac{1}{2}}(1-z)^{-1} - $$

$$-(1-z)^{-2}\sum_{k=n+1}^{\infty}(r-3)_k z^{k-n-\frac{1}{2}}$$

is equal to

$$\frac{\sin(nr - \frac{1}{2}\pi r)}{(2\sin\frac{t}{2})^{r+1}} + \frac{f_n(r,t)}{n^3 t^4},$$

where

$$f_n(r,t) = \frac{n^3 t^4}{(r)_n 2\sin\frac{t}{2}} \vartheta\left\{\frac{1}{(1-z)^2}\sum_{k=n+1}^{\infty}(r-3) \quad {}^{k-n-\frac{1}{2}}\right\}$$

which is numerically less than $\frac{1}{4}\left(\frac{\pi}{2}\right)^4\left|\frac{(r-3)_{n+1}n^3}{(\nu)_n}\right| < 10.$

If $3 \leq r < m+1$, then we write

$$\sum_{k=0}^{n} (r-1)_k z^k =$$

$$= -\sum_{\nu=1}^{2} \frac{(r-\nu)_n}{(1-z)^{\nu}} z^{n+1} - \sum_{\nu=3}^{m} \frac{(r-\nu)_n}{(1-z)^{\nu}} z^{n+1} + (1-z)^{-\nu} -$$

$$-(1-z)^{-m} \sum_{k=n+1}^{\infty} (r-m-1)_k z^k.$$

The absolute value of the last term is less than

$$\frac{2(r-m-1)_{n+1}}{\left(2\sin\frac{t}{2}\right)^{m+1}} < \frac{2(r-3)_n}{t^3}$$

if $nt \geq \frac{\pi}{2}$, and the sum $\sum_{\nu=3}^{m} (r-\nu)_n (1-z)^{-\nu}$ is numerically less than $16(r-3)_n t^{-3}$ when $nt > \pi r$.

It follows that

$$\left| f_n(r,t) \right| < \frac{18\pi(r-3)_n n^3}{2(r)_n} = \frac{9\pi n^3 r(r-1)(r-2)}{(n+r)(n+r-1)(n+r-2)} < 30.$$

We have thus established the first part of the lemma. The later part is obtained by differentiation.*

We can now prove the following

THEOREM 1. Let

$$g(\theta) \sim \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \left\{ \alpha_n \cos n\theta + \beta_n \sin n\theta \right\}$$

be the Fouries series of the continuous function

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* Cf. [3] § 3.5.

$g(\theta)$ with the upper index α then, on writing
$\frac{1}{2}\alpha_0 = A_0$, $A_n = \alpha_n \cos n\theta + \beta_n \sin n\theta \ (n>0)$,
the Cesàro means

$$\sigma_n^r(\theta, g) = \sum_{k=0}^{n} \frac{(r)_{n-k}}{(r)_n} A_k(\theta) \quad (r>0)$$

satisfy

$$\left| \sigma_n^r(\theta, g) - g(\theta) \right| \leqq \frac{C_n}{1-\alpha}\, \omega\left(\frac{1}{n+1}; g\right)$$

provided that $\alpha < 1$. If $\alpha = 1$, then

$$\left| \sigma_n^r(\theta, g) - g(\theta) \right| \leqq C_r\, \omega\left(\frac{1}{n+1}; g\right) \log(n+1),$$

the factor is not allowed to be omitted.

Proof. Writing $\gamma(t) = g(\theta+t) + g(\theta-t) - 2g(\theta)$, we have

$$\sigma_n^r(\theta, g) - g(\theta) = \frac{1}{\pi}\int_0^{\pi} \gamma(t) K_n^r(t)\, dt .$$

In virtue of $|\gamma(t)| \leqq 2\omega(t, g)$ and $|K_n^\nu(t)| \leqq 2n$, we see that the integral

$$I_1 = \frac{1}{\pi}\int_0^{\frac{r\pi}{n}} \gamma(t) K_n^r(t)\, dt$$

is numerically less than $\frac{1}{\pi} 2\omega\left(\frac{r\pi}{n}\right) 2n \frac{r\pi}{n}$.

In fact, owing to the product $t\,\omega(t)$ is increasing in t, we have

$$\frac{r\pi}{n}\, \omega\left(\frac{r\pi}{n}\right) \leqq \frac{1}{n}\, \omega\left(\frac{1}{n}\right)$$

when $r\pi \leqq 1$, and otherwise it is less than

$$\frac{r\pi}{r}(r\pi+2)\,\omega\left(\frac{1}{n}\right) \le 2r^2\pi^2\frac{1}{n}\,\omega\left(\frac{1}{n}\right).$$

Hence, denote greater of $\frac{4}{\pi}$ and $8r^2$ by $C_1(r)$ we have

$$|I_1| \leqslant C_1(r)\,\omega\left(\frac{1}{n+1}\right).$$

Write

$$K_n^r(t) = k_n^r(t) + j_n^r(t)$$

with

$$K_n^r(t) = k_n^r(t) + j_n^r(t)$$

we have

$$I_2 = \frac{1}{\pi}\int_{\frac{r\pi}{n}}^{\pi}\gamma(t)k_n^r(t)\,dt = -\frac{1}{\pi}\int_{\left(\frac{r}{n}+\frac{1}{n_r}\right)\pi}^{\pi+\frac{\pi}{n_r}}\gamma\left(t+\frac{\pi}{n_r}\right)\frac{\sin(n_r t-\frac{1}{2}\pi r)\,dt}{(r)_n\left[2\sin\left(\frac{t}{2}+\frac{\pi}{2n_r}\right)\right]^{r+1}}$$

and

$$2I_2 = \frac{1}{\pi}\int_{\frac{r\pi}{n}}^{\pi}\left\{\gamma(t)-\gamma\left(t+\frac{\pi}{n_r}\right)\right\}k_n^r(t)\,dt + I_2' + I_2'' + I_2''',$$

where

$$I_2' = \frac{1}{\pi(r)_n}\int_{\frac{r\pi}{n}}^{\pi}\gamma\left(t+\frac{\pi}{n_r}\right)\left\{\frac{\sin(n_r t-\frac{1}{2}\pi r)}{\sin^{r+1}\frac{t}{2}} - \frac{\sin(n_r t-\frac{1}{2}\pi r)}{\sin^{r+1}\left(\frac{t}{2}+\frac{\pi}{2n_r}\right)}\right\}dt,$$

$$I_2'' = \frac{1}{\pi(r)_n}\int_{\frac{r\pi}{n}}^{\left(\frac{r}{n}+\frac{1}{n_r}\right)\pi}\gamma\left(t+\frac{\pi}{n_r}\right)\frac{\sin(n_r t-\frac{1}{2}\pi r)\,dt}{\left[2\sin\left(\frac{t}{2}+\frac{\pi}{2n_r}\right)\right]^{r+1}},$$

$$I_2''' = -\frac{1}{n(r)_n}\int_{\pi}^{\pi+\frac{\pi}{n_r}}\gamma\left(t+\frac{\pi}{n_r}\right)\frac{\sin(n_r t-\frac{1}{2}\pi r)\,dt}{\left[2\sin\left(\frac{t}{2}+\frac{\pi}{2n_r}\right)\right]^{r+1}}$$

The integral I_2'' is numerically less than

$$\frac{1}{\pi(r)_n}\,\omega\left(\frac{r\pi}{n}+\frac{2\pi}{n_r}\right)\left[2\sin\left(\frac{r\pi}{2n}+\frac{\pi}{2n_r}\right)\frac{2}{\pi}\right]^{-r-1}\frac{\pi}{n_r}\left(C_2(r)\,\omega\left(\frac{1}{n}\right)\right),$$

the constant $C_2(r)$ depends only upon r.

Clearly,

$$\left|I_2'''\right|<\frac{1}{\pi(r)_n}\,2\omega(\pi)\left[2\sin\left(\frac{r\pi}{2n}+\frac{\pi}{2n_r}\right)\right]^{-r-1}\frac{\pi}{n_r}<C_3(r)\,n^{-1-r}.$$

On account of

$$\left[\sin\frac{t}{2}\right]^{-1-r}-\left[\sin\left(\frac{t}{2}+\frac{\pi}{2n_r}\right)\right]^{-1-r}=O\left(n^{-1}t^{-r-2}\right),$$

we see that

$$\left|I\right|\leqslant\frac{C'(r)}{\pi n(r)_n}\int_{\frac{r\pi}{n}}^{\pi}\frac{\omega(2t)}{2t}\,\frac{dt}{t^{1+r}}<$$

$$<\frac{C''(r)}{n^{1+r}}\cdot n\,\omega\left(\frac{1}{n}\right)\int_{\frac{r\pi}{n}}^{\pi}\frac{dt}{t^{1+r}}<C_3(r)\,\omega\left(\frac{1}{n}\right),$$

because $\omega(t)t^{-1}$ is decreasing in t. Finally, the integral $2I_2-I_2'-I_2''-I_2'''$ is numerically less than

$$\frac{1}{\pi}\int_{\frac{r\pi}{n}}^{\pi}\omega\left(\frac{\pi}{n_r},\gamma\right)\frac{dt}{(r)_n\left(2\sin\frac{t}{2}\right)^{1+r}}<C_4(r)\,\omega\left(\frac{1}{n}\right).$$

Therefore we obtain

$$\left| I_1 \right| + \left| I_2 \right| \leqslant \left(C_1(r) + C_2(r) + C_3(r) + C_4(r)\, \omega\left(\frac{1}{n}\right) \right.$$

This result holds good for $\alpha \leqslant 1$. We have to consider the integral

$$I_3 = \frac{1}{\pi} \int_{\frac{r\pi}{n}}^{\pi} \gamma(r)\, \bar{\gamma}_n^r(t)\, dt$$

Whose numerical value is less than a constant multiple of

$$\frac{1}{n} \int_{\frac{r\pi}{n}}^{\pi} \frac{\omega(t)}{t^2}\, dt,$$

by respecting Lemma 1.

If the index α is less than unity, then there is a positive number ε such that $\alpha + \varepsilon < 1$, so that the function $t^{-\alpha-\varepsilon}\, \omega(t)$ is decreasin t, as

$$t\, \frac{\omega'(t)}{\omega(t)} < \alpha + \varepsilon .$$

Accordingly, we have

$$\int_{\frac{r\pi}{n}}^{\pi} \frac{\omega(t)}{t^2}\, dt =$$

$$= \int_{\frac{r\pi}{n}}^{\pi} \frac{\omega(t)}{t^{\alpha+\varepsilon}} \frac{dt}{t^{2-\alpha-\varepsilon}} \leqslant \left(\frac{n}{r\pi}\right)^{\alpha+\varepsilon} \omega\left(\frac{r\pi}{n}\right) \int_{\frac{r\pi}{n}}^{\pi} \frac{dt}{t^{2-\alpha-\varepsilon}} <$$

$$< \frac{n}{1-\alpha-\varepsilon} \left(\frac{1}{r\pi}\right)^{\alpha+\varepsilon} \omega\left(\frac{r\pi}{n}\right).$$

Letting $\varepsilon \to 0$, we obtain

$$|I_3| \leq \frac{1}{1-\alpha} \frac{1}{(r\pi)^\alpha} \omega\left(\frac{1}{n}\right) \max(1, 2r\pi).$$

Therefore

$$|I_1 + I_2 + I_3| < \frac{C_r}{1-\alpha} \omega\left(\frac{1}{n+1}\right)$$

In general, we have

$$\frac{1}{n}\int_{\frac{r\pi}{n}}^{\pi} \frac{\omega(t)}{t^2}\,dt \leq \left[\frac{\omega(\tau)}{t}\right]_{t=\frac{r\pi}{n}} \frac{1}{n}\int_{\frac{r\pi}{n}}^{\pi} \frac{dt}{t} < C_5(r)\omega\left(\frac{1}{n}\right)\log n$$

Hence the relation

$$|I_1 + I_2 + I_3| < C(r)\,\omega\left(\frac{1}{n}\right) \log n \qquad (n > 0)$$

holds true universally.

The importance of the factor $\log(n+1)$ may be seen from the continuous even function

$$g(\theta) = \theta \log\frac{1}{\theta}, \quad 0 \leq \theta \leq \pi.$$

$$g(\theta \pm 2m\pi) = g(\theta), \quad m = 1, 2, \ldots.$$

For $0 < t < 1$, we have $\omega(t, g) = t \log\frac{1}{t}$. It follows from the foregoing proof that

$$\sigma_n^r(0) - g(0) =$$

$$= \frac{r(2n+r-1)}{2(n+r)(n+r-1)} \frac{1}{\pi}\int_{\frac{1}{n}}^{\pi} \frac{-t\log t}{\sin^2\frac{t}{2}}\,dt + O\left(\omega\left(\frac{1}{n}\right)\right) =$$