弹性理论基础习题解答

（一）平面问题

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前　言

本《解答》系原无锡工业专科学校预备教师金毓铨、陆水月、张新秋、严宝珑等十四位同志于今（1961）年暑假转来我校以后进行基础理论补课时编演的，并经他们的授课教师朱泽源同志校订。

扬州工专力学教研组

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翻印说明

一九七七年全国工科院校教材会议决定在机制专业《材料力学》教材中增加“弹性理论的平面问题”一章。为便于教师事先备课的参考，经本教研组部分同志建议，由国营五一一厂工学院于七九年翻印了原扬州工专编印的《弹性理论基础习题解答》的平面问题部分，在本市各职工大学、工科院校以及三机部所属各工学院作为教学参考资料交流。现据外地兄弟院校建议，再一次翻印在全省兄弟院校交流。

本《解答》的题次基本上按徐芝纶编《弹性理论》（人民教育出版社1960年一版）的习题次序。

南京市各职工大学力学教研组

1980.5.10

《弹性理论基础习题解答》

本"解答"系原无锡工业专科学校予备教师金毓铨、陆水月、张新秋、严宝珑等十四位同志于今(一九六一)年暑假转来我校以后进行基础理论补课时编写的；并经他们的授课教师朱泽沅同志校订。

扬州工专力学教研组 1961年11月14日

第二章 平面问题

〔2-1〕试证明，如果体力虽不是常量而却是有势的力

$$X=-\frac{\partial V}{\partial x},\qquad Y=-\frac{\partial V}{\partial y}$$

其中 V 是势函数，那末，应力分量

$$\sigma_x=\frac{\partial^2\varphi}{\partial y^2}+V,\qquad \sigma_y=\frac{\partial^2\varphi}{\partial x^2}+V,\qquad \tau_{xy}=-\frac{\partial^2\varphi}{\partial x\partial y}$$

也能满足平衡微分方程，而平面应力情况下的相容条件是

$$\frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=-(1-\mu)\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)$$

平面应变情况下的相容条件是

$$\frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=-\frac{1-2\mu}{1-\mu}\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)$$

〔证〕① 微分：

$$\sigma_x=\frac{\partial^2\varphi}{\partial y^2}+V,\longrightarrow \frac{\partial\sigma_x}{\partial x}=\frac{\partial^3\varphi}{\partial x\partial y^2}+\frac{\partial V}{\partial x}$$

$$\sigma_y=\frac{\partial^2\varphi}{\partial x^2}+V,\longrightarrow \frac{\partial\sigma_y}{\partial y}=\frac{\partial^3\varphi}{\partial x^2\partial y}+\frac{\partial V}{\partial y}$$

$$\tau_{xy}=-\frac{\partial^2\varphi}{\partial x\partial y},\longrightarrow \frac{\partial\tau_{xy}}{\partial x}=-\frac{\partial^3\varphi}{\partial x^2\partial y}$$

$$\frac{\partial\tau_{xy}}{\partial y}=-\frac{\partial^3\varphi}{\quad}$$

② 代入平衡方程：

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+X=0,\longrightarrow$$

$$\frac{\partial^3\varphi}{\partial x\partial y^2}+\frac{\partial V}{\partial x}-\frac{\partial^3\varphi}{\partial x\partial y^2}-\frac{\partial V}{\partial x}=0,\qquad 0=0$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+Y=0 \longrightarrow$$

$$-\frac{\partial^3 \varphi}{\partial x^2 \partial y}+\frac{\partial^3 \varphi}{\partial x^2 \partial y}+\frac{\partial V}{\partial y}-\frac{\partial V}{\partial y}=0, \qquad 0=0$$

可见能满足平衡微分方程。

③ 再微分：

$$\frac{\partial^2 \sigma_x}{\partial x^2}=\frac{\partial^2 \varphi}{\partial x^2 \partial y^2}+\frac{\partial^2 V}{\partial x^2}, \qquad \frac{\partial^2 \sigma_x}{\partial y^2}=\frac{\partial^4 \varphi}{\partial y^4}+\frac{\partial^2 V}{\partial y^2}$$

$$\frac{\partial^2 \sigma_y}{\partial x^2}=\frac{\partial^4 \varphi}{\partial x^4}+\frac{\partial^2 V}{\partial x^2}, \qquad \frac{\partial^2 \sigma_y}{\partial y^2}=\frac{\partial^4 \varphi}{\partial x^2 \partial y^2}+\frac{\partial^2 V}{\partial y^2}$$

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y}=-\frac{\partial^4 \varphi}{\partial x^2 \partial y^2}$$

④ 代入相容条件：

平面问题的相容条件

$$\frac{\partial^2}{\partial y^2}(\varepsilon_x)+\frac{\partial^2}{\partial x^2}(\varepsilon_y)=\frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

将物理方程代入，得

$$\frac{\partial^2}{\partial y^2}(\sigma_x-\mu\sigma_y)+\frac{\partial^2}{\partial x^2}(\sigma_y-\mu\sigma_x)=2(1+\mu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

$$\therefore \frac{\partial^4 \phi}{\partial y^4}+\frac{\partial^2 V}{\partial y^2}-\mu\frac{\partial^4 \varphi}{\partial x^2 \partial y^2}-\mu\frac{\partial^2 V}{\partial y^2}+\frac{\partial^4 \varphi}{\partial x^4}+\frac{\partial^2 V}{\partial x^2}-\mu\frac{\partial^4 \phi}{\partial x^2 \partial y^2}-\mu\frac{\partial^2 V}{\partial x^2}$$

$$=2\frac{\partial^2 \tau_{xy}}{\partial x \partial y}+2\mu\frac{\partial^2 \tau_{xy}}{\partial x \partial y}=-2\frac{\partial^2 \varphi}{\partial x^2 \partial y^2}-2\mu\frac{\partial^4 \varphi}{\partial x^2 \partial y^2}$$

$$\therefore \frac{\partial^4 \varphi}{\partial x^4}+2\frac{\partial^4 \phi}{\partial x^2 \partial y^2}+\frac{\partial^4 \varphi}{\partial y^4}+\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)(1-\mu)=0$$

即平面应力情况下的相容条件为

$$\frac{\partial^4 \varphi}{\partial x^4}+2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2}+\frac{\partial^4 \varphi}{\partial y^4}=-(1-\mu)\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)$$

在平面应变情况下，将 $\mu \longrightarrow \dfrac{\mu}{1-\mu}$，则 $1-\mu \longrightarrow \dfrac{1-2\mu}{1-\mu}$，则

$$\frac{\partial^4 \varphi}{\partial x^4}+2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2}+\frac{\partial^4 \varphi}{\partial y^4}=-\frac{1-2\mu}{1-\mu}\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)$$

〔2-2〕试用位移分量表明应力分量，并将这些表达式代入平衡微分方程式，从而导出按位移求解问题时所需用的基本方程式。

〔解〕① 用位移分量表应力分量：

$$\frac{\partial u}{\partial x}=\varepsilon_x=\frac{1}{E}(\sigma_x-\mu\sigma_y), \qquad \sigma_x=\frac{E}{1-\mu^2}\left(\frac{\partial u}{\partial x}+\mu\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial v}{\partial y}=\varepsilon_y=\frac{1}{E}(\sigma_y-\mu\sigma_x),\qquad \sigma_y=\frac{E}{1-\mu^2}\left(\mu\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=\frac{2(1+\mu)}{E}\tau_{xy},\qquad \tau_{xy}=\frac{E}{2(1+\mu)}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

②微分：

$$\frac{\partial\sigma_x}{\partial x}=\frac{E}{1-\mu^2}\left[\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 v}{\partial x\partial y}\right]$$

$$\frac{\partial\sigma_y}{\partial y}=\frac{E}{1-\mu^2}\left[\mu\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}\right]$$

$$\frac{\partial\tau_{xy}}{\partial x}=\frac{E}{2(1+\mu)}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}\right]$$

$$\frac{\partial\tau_{xy}}{\partial y}=\frac{E}{2(1+\mu)}\left[\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}\right]$$

③代入平衡微分方程：

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+X=0 \qquad (a)$$

$$\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+Y=0 \qquad (b)$$

将②的微分式代入式(a)，得

$$\frac{2E}{2(1-\mu)(1+\mu)}\left\{\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 v}{\partial x\partial y}+\frac{1-\mu}{2}\left[\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}\right]\right\}+X=0,$$

$$\frac{2G}{1-\mu}\left\{\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 v}{\partial x\partial y}+\frac{1}{2}\cdot\frac{\partial^2 v}{\partial x\partial y}-\frac{\mu}{2}\cdot\frac{\partial^2 v}{\partial x\partial y}+\frac{1}{2}\cdot\frac{\partial^2 u}{\partial y^2}-\frac{\mu}{2}\cdot\frac{\partial^2 u}{\partial y^2}\right\}+X=0$$

即 $$\frac{2G}{1-\mu}\left\{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}-\frac{1}{2}\cdot\frac{\partial^2 v}{\partial x\partial y}+\frac{\mu}{2}\cdot\frac{\partial^2 v}{\partial x\partial y}+\frac{1-\mu}{2}\cdot\frac{\partial^2 u}{\partial y^2}\right\}+X=0$$

即 $$\frac{2}{1-\mu}\left\{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{1-\mu}{2}\left(\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 v}{\partial x\partial y}\right)\right\}+\frac{X}{G}=0$$

或 $$\frac{2}{1-\mu}\cdot\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+\frac{X}{G}=0$$

将②的微分式代入式(b)，得

$$\frac{2E}{2(1-\mu)(1+\mu)}\left\{\frac{\partial^2 v}{\partial y^2}+\mu\frac{\partial^2 u}{\partial x\partial y}+\frac{1-\mu}{2}\cdot\frac{\partial^2 v}{\partial x^2}+\frac{1-\mu}{2}\cdot\frac{\partial^2 u}{\partial x\partial y}\right\}+Y=0;$$

$$\frac{2G}{1-\mu}\left\{\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 u}{\partial x\partial y}-\frac{1}{2}\cdot\frac{\partial^2 u}{\partial x\partial y}+\frac{\mu}{2}\cdot\frac{\partial^2 u}{\partial x\partial y}+\frac{1-\mu}{2}\cdot\frac{\partial^2 v}{\partial x^2}\right\}+Y=0$$

即 $$\frac{2G}{1-\mu}\left\{\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}\right)+\frac{1-\mu}{2}\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 u}{\partial x\partial y}\right)\right\}+Y=0$$

或 $$\frac{2}{1-\mu}\cdot\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{Y}{G}=0$$

④ 令 $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=e$, $\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\omega$,

则得平面应力情况下按位移求解的基本方程:

$$\frac{2}{1-\mu}\cdot\frac{\partial e}{\partial x}-\frac{\partial \omega}{\partial y}+\frac{X}{G}=0$$

$$\frac{2}{1-\mu}\cdot\frac{\partial e}{\partial y}+\frac{\partial \omega}{\partial x}+\frac{Y}{G}=0$$

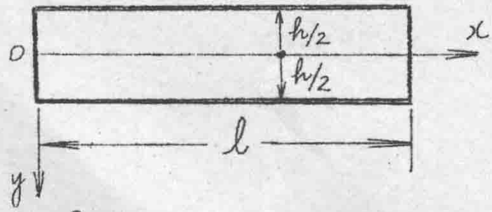
在平面应变情况下: $\mu \to \frac{\mu}{1-\mu}$, $1-\mu \to \frac{1-2\mu}{1-\mu}$, $\frac{2}{1-\mu}=\frac{2(1-\mu)}{1-2\mu}$,

故按位移求解问题的基本方程为

$$\frac{2(1-\mu)}{1-2\mu}\cdot\frac{\partial e}{\partial x}-\frac{\partial \omega}{\partial y}+\frac{X}{G}=0,$$

$$\frac{2(1-\mu)}{1-2\mu}\cdot\frac{\partial e}{\partial y}+\frac{\partial \omega}{\partial x}+\frac{Y}{G}=0.$$

[3-1] 试考察应力函数 $\varphi=\frac{P}{2h^3}xy(3h^2-4y^2)$ 能不能满足相容条件。如果能满足，试求应力分量（体力不计），画出图示矩形板各边上的面力，求出每一边上水平和铅直面力的合成，并指出所能解答的问题。



[解]

$$\frac{\partial\varphi}{\partial x}=\frac{P}{2h^3}y(3h^2-4y^2),\qquad \frac{\partial\varphi}{\partial y}=\frac{P}{2h^3}x(3h^2-12y^2),$$

$$\frac{\partial^2\varphi}{\partial x^2}=0,\qquad \frac{\partial^2\varphi}{\partial y^2}=\frac{P}{2h^3}x(-24y),$$

$$\frac{\partial^4\varphi}{\partial x^4}=0,\qquad \frac{\partial^3\varphi}{\partial y^3}=\frac{P}{2h^3}x(-24),$$

$$\frac{\partial^4\varphi}{\partial x^2\partial y^2}=0,\qquad \frac{\partial^4\varphi}{\partial y^4}=0$$

能满足相容条件 $\frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=0,$

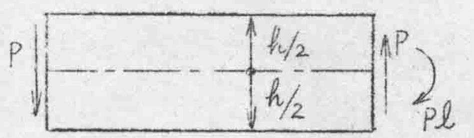
$$\sigma_x=\frac{\partial^2\varphi}{\partial y^2}=\frac{P}{2h^3}x(-24y)=-\frac{12Pxy}{h^3}$$

$$\sigma_y=\frac{\partial^2\varphi}{\partial x^2}=0$$

$$\tau_{xy}=-\frac{\partial^2\varphi}{\partial x\partial y}=-\frac{P}{2h^3}(3h^2-12y^2)$$

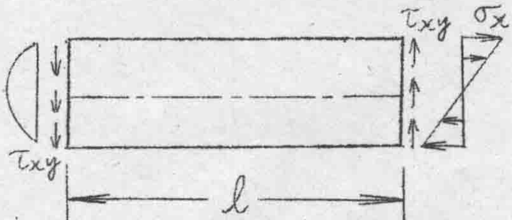
当 $x=0$，$\overline{X}=-(\sigma_x)_{x=0}=0,$

$$\overline{Y}=-(\tau_{xy})_{x=0}=P\left(\frac{3}{2h}-\frac{6}{h^3}y^2\right),$$



当 $x=l$，$\overline{X}=(\sigma_x)_{x=l}=-\frac{12Ply}{h^3},$

$$\overline{Y}=(\tau_{xy})_{x=l}=-\frac{3P}{2h}\left(1-4\frac{y^2}{h^2}\right).$$



τ_{xy}的合力：

$$\int_{-h/2}^{h/2}(\tau_{xy})_{x=0\text{或}l}\,dy=\pm P$$

σ_x的合力和合力矩：

$$\int_{-h/2}^{h/2}(\sigma_x)_{x=l}\,dy=0$$

$$\int_{-h/2}^{h/2}(\sigma_x)_{x=l}\,y\,dy=-Pl$$

上下边界：

当 $y=\pm\frac{h}{2}$，$(\sigma_y)_{y=\pm h/2}=0$

$$(\tau_{xy})_{y=\pm h/2}=-\frac{P}{2h^3}\left[\frac{3}{2h}-\frac{6}{h^3}\left(\frac{h}{2}\right)^2\right]=0$$

$$\therefore\ \overline{X}=\overline{Y}=0$$

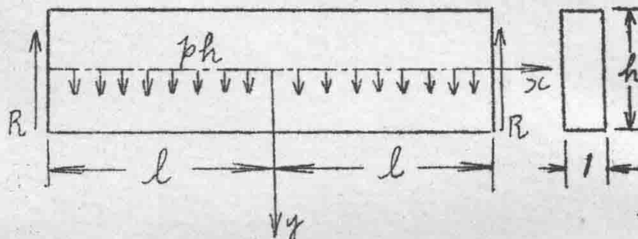
〔3-2〕设图示简支梁只受自重作用，而梁的容重为p，试用应力函数

$$\varphi=\frac{x^2}{2}(Ay^3+By^2+Cy+D)+x(Ey^3+Fy^2+Gy)-\frac{A}{10}y^5-\frac{B}{6}y^4+Hy^3+Ky^2$$

求应力分布。〔提示：体力$X=0$，而$Y=p$.〕

〔解〕考虑整个梁的平衡，可得两端支座反力为

$$R=phl$$



从应力分布公式可得应力分布：

$$\sigma_x=\frac{\partial^2\varphi}{\partial y^2}-Xx=\frac{x^2}{2}(6Ay+2B)+x(6Ey+2F)-2Ay^3-2By^2+6Hy+2K \qquad (1)$$

$$\sigma_y=\frac{\partial^2\varphi}{\partial x^2}-Yy=Ay^3+By^2+Cy+D-py \qquad (2)$$

$$\tau_{xy}=-\frac{\partial^2\varphi}{\partial x\partial y}=-x(3Ay^2+2By+C)-(3Ey^2+2Fy+G) \qquad (3)$$

这些应力分布满足平衡条件和相容条件，还要确定适当的常数A、B、C、D、E、F、G、K，使其在弹性体的所有边界上满足边界条件。

利用对称条件：

由于σ_x对称于y轴，即当$x=-x$时，σ_x值相等，故由式(1)必有$E=0$，$F=0$.
又因τ_{xy}、τ_{yx}对于y轴的反对称，由式(3)知$G=0$.

$$\therefore\ E=F=G=0$$

考虑梁的上、下边界情况：

当 $y=\frac{h}{2}$，$\quad\tau_{xy}=0$，

$$\therefore\ \frac{3}{4}Ah^2+Bh+C=0 \qquad (a)$$

$y=-\frac{h}{2}$，$\quad\tau_{xy}=0$，

$$\therefore\ \frac{3}{4}Ah^2-Bh+C=0 \qquad (b)$$

由式(a)、(b)可解得　$B=0$，　$C=-\frac{3}{4}Ah^2$

当 $y=\frac{h}{2}$，　$\sigma_y=0$，　$\therefore \frac{A}{8}h^3+\frac{1}{2}Ch+D-p\frac{h}{2}=0$　　(c)

$y=-\frac{h}{2}$，　$\sigma_y=0$，　$\therefore -\frac{A}{8}h^3-\frac{1}{2}Ch+D+p\frac{h}{2}=0$　　(d)

由式(c)、(d)可解得　$D=0$

将 $D=0$ 及 $C=-\frac{3}{4}Ah^2$ 代入式(c)可得

$$\frac{A}{8}h^3-\frac{1}{2}\cdot\frac{3}{4}Ah^3-p\frac{h}{2}=0, \qquad \therefore A=-\frac{2p}{h^2}$$

以而有　$C=-\frac{3}{4}Ah^2=-\frac{3}{4}\left(-\frac{2p}{h^2}\right)h^2=\frac{3}{2}p$

将 $A=-\frac{2p}{h^2}$，$B=0$，$C=\frac{3p}{2}$，$D=0$，$E=0$，$F=0$，$G=0$ 代入式(1)、(2)、(3)可得

$$\sigma_x=-\frac{6p}{h^2}x^2y+\frac{4p}{h^2}y^3+6Hy+2K \tag{4}$$

$$\sigma_y=-\frac{2p}{h^2}+\frac{1}{2}py \tag{5}$$

$$\tau_{xy}=-\frac{6p}{h^2}x\left(\frac{1}{4}h^2-y^2\right) \tag{6}$$

再考虑左、右端面边界情况：

在梁的左、右端面 $x=\pm l$，将其代入式(4)，可得

$$\sigma_x=-\frac{6p}{h^2}l^2y+\frac{4p}{h^2}y^3+6Hy+2K \tag{4-a}$$

由于在端面处没有水平面力和力偶矩，所以有

$$\int_F \sigma_x dF=0 \tag{e}$$

$$\int_F \sigma_x y\,dF=0 \tag{f}$$

将式(4-a)代入(e)，有

$$\int_{-h/2}^{h/2}\left(-\frac{6p}{h^2}l^2y+\frac{4p}{h^2}y^3+6Hy+2K\right)(1\times dy)=0,$$

即　$$-\frac{6p}{h^2}l^2\int_{-h/2}^{h/2}y\,dy+\frac{4p}{h^2}\int_{-h/2}^{h/2}y^3dy+6H\int_{-h/2}^{h/2}y\,dy+2K\int_{-h/2}^{h/2}dy=0,$$

即　$$0+0+0+2K\left(\frac{h}{2}+\frac{h}{2}\right)=0, \qquad \therefore K=0$$

将式(4-a)代入(f)，有

$$-\frac{6p}{h^2}l^2\int_{-h/2}^{h/2}y^2dy+\frac{4p}{h^2}\int_{-h/2}^{h/2}y^4dy+6H\int_{-h/2}^{h/2}y^2dy=0$$

或　$$-\frac{6p}{h^2}\cdot\frac{l^2}{3}\left(\frac{h^3}{8}+\frac{h^3}{8}\right)+\frac{4p}{5h^2}\left(\frac{h^5}{32}+\frac{h^5}{32}\right)+\frac{6H}{3}\left(\frac{h^3}{8}+\frac{h^3}{8}\right)=0$$

即 $$-\frac{1}{2}phl^2+\frac{1}{20}ph^3+\frac{1}{2}Hh^3=0$$

从而，可得 $$H=\frac{pl^2}{h^2}-\frac{p}{10}$$

再将 $K=0$, $H=\frac{pl^2}{h^2}-\frac{p}{10}$ 代入式(4)、(5)、(6)，可得应力分量的最后结果：

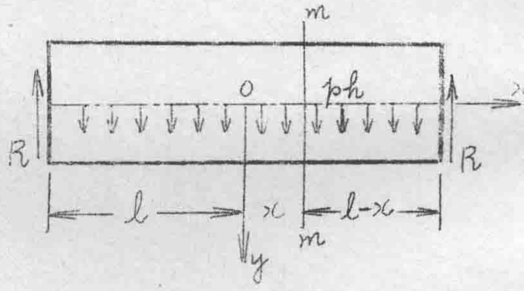
$$\sigma_x=\frac{6p}{h^2}(l^2-x^2)y+py\left(4\frac{y^2}{h^2}-\frac{3}{5}\right)$$

$$\sigma_y=\frac{1}{2}py\left(1-4\frac{y^2}{h^2}\right)$$

$$\tau_{xy}=\tau_{yx}=-\frac{6p}{h^2}x\left(\frac{1}{4}h^2-y^2\right)$$

按材料力学公式：

$$\sigma_x=\frac{My}{J_Z}$$



梁上任一截面 m-m 上的弯矩为

$$M=R(l-x)-\frac{1}{2}ph(l-x)^2$$
$$=\left[phl-\frac{1}{2}ph(l-x)\right](l-x)$$
$$=\frac{1}{2}ph(l+x)(l-x)=\frac{1}{2}ph(l^2-x^2)$$

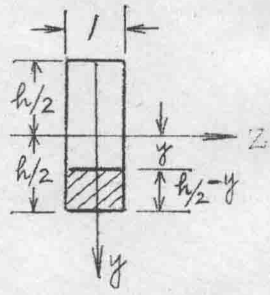
而截面对中性轴的惯矩 $J_Z=\frac{1}{12}bh^3=\frac{1}{12}h^3$ $(\because b=1)$

$$\therefore \sigma_x=\frac{My}{J_Z}=\frac{1}{2}ph(l^2-x^2)y\Big/\frac{1}{12}h^3=\frac{6p}{h^2}(l^2-x^2)y$$

因此，弹性理论中的 σ_x 的第一项与材力中的 $\sigma_x=\frac{My}{J_Z}$ 一样，

$$\therefore\quad \sigma_x=\frac{My}{J_Z}+py\left(4\frac{y^2}{h^2}-\frac{3}{5}\right)$$

按材料力学公式 $\tau=\frac{QS}{J_Zb}$



梁上任一截面 m-m 上的剪力为

$$Q=-R+ph(l-x)=-phl+ph(l-x)=-phx$$

截面上距 Z 轴为 y 处的

$$S=\left(\frac{1}{2}h-y\right)\times1\times\frac{1}{2}\left(\frac{h}{2}+y\right)=\frac{1}{2}\left(\frac{1}{4}h^2-y^2\right)$$

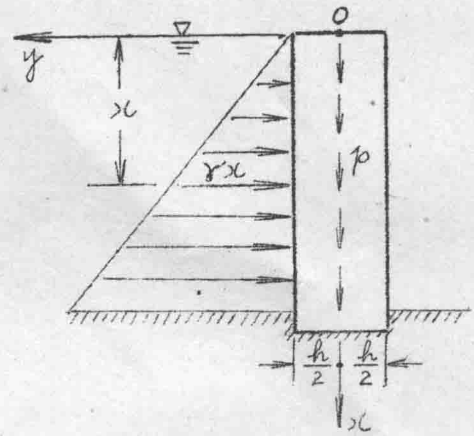
$$\therefore \tau=\frac{QS}{J_Z\times1}=\frac{-phx\cdot\frac{1}{2}\left(\frac{1}{4}h^2-y^2\right)}{h^3/12}=-\frac{6px}{h^2}\cdot\left(\frac{1}{4}h^2-y^2\right)$$

可见 $$\tau_{xy}=\tau=\frac{QS}{J_Zb}$$

$$\therefore\quad \sigma_x=\frac{My}{J_Z}+py\left(4\frac{y^2}{h^2}-\frac{3}{5}h\right),\qquad \sigma_y=\frac{1}{2}py\left(1-4\frac{y^2}{h^2}\right),$$

$$\tau_{xy}=\tau_{yx}=\frac{QS}{Jb}.$$

〔3-3〕挡水墙的容重为p，厚度为h，如图示，而水的容重为γ，试求应力分布。



〔提示〕体力$X=p$，而$Y=0$，可假设$\sigma_y=xf(y)$，上端边界条件如不能完全满足，可应用圣文南原理。

〔解〕(一)确定应力函数φ：

从应力分布公式 $\sigma_y=\frac{\partial^2\varphi}{\partial x^2}-Yy$，

而 $Y=0$， $\therefore\sigma_y=\frac{\partial^2\varphi}{\partial x^2}$

假设 $\sigma_y=xf(y)$， 则 $\frac{\partial^2\varphi}{\partial x^2}=xf(y)$

积分 $\frac{\partial\varphi}{\partial x}=\frac{1}{2}x^2f(y)+f_1(y)$

再积分 $\varphi=\frac{1}{6}x^3f(y)+xf_1(y)+f_2(y)$

确定未知函数$f(y)$、$f_1(y)$、$f_2(y)$，它们应满足相容条件

$$\frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=0$$

对φ求四阶偏导数有

$$\frac{\partial^4\varphi}{\partial x^4}=0,\qquad \frac{\partial^4\varphi}{\partial x^2\partial y^2}=x\frac{d^2f(y)}{dy^2},$$

$$\frac{\partial^4\varphi}{\partial y^4}=\frac{1}{6}x^3\frac{d^4f(y)}{dy^4}+x\frac{d^4f_1(y)}{dy^4}+\frac{d^4f_2(y)}{dy^4}$$

将上三式代入相容条件，得

$$\frac{x^3}{6}\cdot\frac{d^4f(y)}{dy^4}+x\left(\frac{d^4f_1(y)}{dy^4}+2\frac{d^2f(y)}{dy^2}\right)+\frac{d^4f_2(y)}{dy^4}=0$$

上式可看成为x的幂函数，要使上列等式成立，必须使x的系数为零。

$$\frac{d^4f(y)}{dy^4}=0,$$

积分四次，得 $f(x)=Ay^3+By^2+Cy+D$

又 $$\frac{d^4f_1(y)}{dy^4}+2\frac{d^2f(y)}{dy^2}=0,$$

$$\therefore \quad \frac{d^4 f_1(y)}{dy^4} = -2\frac{d^2 f(y)}{dy^2} = -2(6Ay+2B)$$

或 $$\frac{d^4 f_1(y)}{dy^4} = -12Ay-4B$$

等式两边积分四次，可得 $$f_1(y) = -\frac{A}{10}y^5 - \frac{B}{6}y^4 + Ey^3 + Fy^2 + Gy$$

又 $$\frac{d^4 f_2(y)}{dy^4} = 0,$$

积分四次，可得 $$f_2(y) = Hy^3 + Ky^2 \quad (\text{略去后面几项})$$

故得应力函数为

$$\varphi = \frac{1}{6}x^3(Ay^3+By^2+Cy+D) + x\left(-\frac{A}{10}y^5 - \frac{B}{6}y^4 + Ey^3 + Fy^2 + Gy\right) + Hy^3 + Ky^2 \quad (1)$$

(二) 求应力分量：

将上述应力函数代入应力分量公式，可得

$$\sigma_x = \frac{\partial^2\varphi}{\partial y^2} - Xx = \frac{x^3}{6}(6Ay+2B) + x(-2Ay^3-2By^2+6Ey+2F) + 6Hy + 2K - px \quad (2)$$

$$\sigma_y = \frac{\partial^2\varphi}{\partial x^2} - Yy = x(Ay^3+By^2+Cy+D) \quad (3)$$

$$\tau_{xy} = \tau_{yx} = -\frac{\partial^2\varphi}{\partial x \partial y} = -\frac{1}{2}x^2(3Ay^2+2By+C) - \left(-\frac{1}{2}Ay^4 - \frac{2}{3}By^3 + 3Ey^2 + 2Fy + G\right) \quad (4)$$

(三) 从边界条件确定积分常数：

当 $y=\frac{h}{2}$ 时，$\sigma_y = -\gamma x$，代入式(3)可得

$$A\cdot\frac{h^3}{8} + B\cdot\frac{h^2}{4} + C\frac{h}{2} + D = -\gamma,$$

当 $y=-\frac{h}{2}$ 时，$\sigma_y=0$，$-A\cdot\frac{h^3}{8} + B\cdot\frac{h^2}{4} - C\cdot\frac{h}{2} + D = 0$

两式相加，得 $$\frac{1}{2}h^2B + 2D = -\gamma \quad (a)$$

两式相减，得 $$\frac{1}{4}h^3A + hC = -\gamma \quad (b)$$

当 $y=\frac{h}{2}$ 时，$\tau_{xy}=0$，从式(4)可得

$$-\frac{x^2}{2}\left(3A\cdot\frac{h^2}{4} + 2B\cdot\frac{h}{2} + C\right) - \left(-\frac{1}{2}A\cdot\frac{h^4}{16} - \frac{2}{3}B\cdot\frac{h^3}{8} + 3E\cdot\frac{h^2}{4} + 2F\cdot\frac{h}{2} + G\right) = 0$$

即 $$-\frac{x^2}{2}\left(\frac{3}{4}Ah^2 + Bh + C\right) - \left(-\frac{A}{32}h^4 - \frac{B}{12}h^3 + \frac{3}{4}Eh^2 + Fh + G\right) = 0$$

当 $y=-\frac{h}{2}$ 时，$\tau_{xy}=0$，从式(4)可得

$$-\frac{x^2}{2}\left(\frac{3}{4}Ah^2 - Bh + C\right) - \left(-\frac{A}{32}h^4 + \frac{B}{12}h^3 + \frac{3}{4}Eh^2 - Fh + G\right) = 0$$

两式相减，得 $$-\frac{x^2}{2}(2Bh) - \left(-\frac{1}{6}Bh^3 + 2Fh\right) = 0 \quad (c)$$

要使上面等式成立，必须等式两边对应项系数相等。即

$$2Bh=0,\quad \therefore B=0;\qquad 2Fh=0,\quad \therefore F=0.$$

两式相加，得

$$-\frac{x^2}{2}\left(\frac{3}{2}Ah^2+2C\right)-\left(-\frac{1}{16}Ah^4+\frac{3}{2}Eh^2+2G\right)=0 \tag{d}$$

同上理，有 $\frac{3}{2}Ah^2+2C=0$，又从(b)知 $C=-\frac{h^2}{4}A-\frac{\gamma}{h}$，

联解二式可得 $\quad A=\frac{2\gamma}{h^3}$，$\quad C=-\frac{3\gamma}{2h}$.

又将 $B=0$ 代入式(a)，得 $D=-\frac{\gamma}{2}$.

将 $A=\frac{2\gamma}{h^3}$，$B=0$，$C=-\frac{3\gamma}{2h}$，$D=-\frac{\gamma}{2}$，$F=0$ 代入式(2)、(3)、(4)，可得

$$\sigma_x=x^3\left(\frac{2\gamma}{h^3}y\right)+x\left(-\frac{4\gamma}{h^3}y^3+6Ey\right)+6Hy+2K-px \tag{5}$$

$$\sigma_y=x\left(\frac{2\gamma}{h^3}y^3-\frac{3}{2}\cdot\frac{\gamma}{h}y-\frac{\gamma}{2}\right)=\gamma x\left(2\frac{y^3}{h^3}-\frac{3y}{2h}-\frac{1}{2}\right) \tag{6}$$

$$\tau_{xy}=\tau_{yx}=-\frac{1}{2}x^2\left(6\frac{\gamma}{h^3}y^2-\frac{3\gamma}{2h}\right)-\left(-\frac{\gamma}{h^3}y^4+3Ey^2+G\right) \tag{7}$$

再进一步确定积分常数E、G、H、K:

当 $x=0$ 时，$\sigma_x=0$，无法满足式(5)，所以据圣文南原理有

$$\int_F \sigma_x dF=0,\qquad \int \sigma_x\cdot y\cdot dF=0$$

将式(5)代入，得 $\quad \int_F \sigma_x dF=\int_{-h/2}^{h/2}(6Hy+2K)(1\times dy)=0,$

即 $\quad 3H\left[\left(\frac{h}{2}\right)^2-\left(-\frac{h}{2}\right)^2\right]+2K\left[\frac{h}{2}-\left(-\frac{h}{2}\right)\right]=0,\qquad \therefore K=0$

又 $\quad \int_F \sigma_x y\,dF=\int_{-h/2}^{h/2}(6Hy+2K)y(1\times dy)=0$

即 $\quad 2H\left[\left(\frac{h}{2}\right)^3-\left(-\frac{h}{2}\right)^3\right]+K\left[\left(\frac{h}{2}\right)^2-\left(-\frac{h}{2}\right)^2\right]=0,\qquad \therefore H=0$

又当 $x=0$ 时，顶面剪力等于零，故 $\int_F \tau_{xy}dF=0$，将式(7)代入，可得

$$\int_F \tau_{xy}\cdot(1\times dy)=\int_{-h/2}^{h/2}\left(-\frac{\gamma}{h^3}y^4+3Ey^2+G\right)dy=-\frac{1}{80}\gamma h^2+\frac{1}{4}Eh^3+Gh=0$$

或

$$-\frac{1}{40}\gamma h+\frac{1}{2}Eh^2+2G=0 \tag{e}$$

又将 $x=0$ 及 $A=\frac{2\gamma}{h^3}$ 代入式(d)，可得

$$-\frac{1}{8}\gamma h+\frac{3}{2}Eh^2+2G=0 \tag{f}$$

联立解(e)、(f)两式，可得 $\quad E=\frac{1}{10}\cdot\frac{\gamma}{h}$，$\qquad G=-\frac{1}{80}\gamma h$

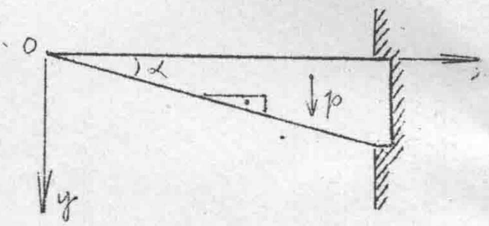
将 $H=0,\ K=0,\ E=\frac{1}{10}\cdot\frac{\gamma}{h},\ G=-\frac{1}{80}\cdot\gamma h$ 代入式(5)、(6)、(7)，可得挡水墙应力分另的最后结果

$$\sigma_x=\frac{2\gamma}{h^3}x^3y+\frac{3\gamma}{5h}xy-\frac{4\gamma}{h^3}xy^3-px$$

$$\sigma_y=\gamma x\left(2\frac{y^3}{h^3}-\frac{3y}{2h}-\frac{1}{2}\right)$$

$$\tau_{xy}=\tau_{yx}=\frac{\gamma y^2}{h^3}(y^3-3x^2)+\frac{3\gamma}{2h}\left(\frac{x^2}{2}-\frac{y^2}{5}\right)-\frac{\gamma h}{80}$$

〔3-4〕设图中三角形悬臂梁只受重力作用而梁的容重为 p，试用纯三次式的应力函数求解。



〔解〕取纯三次应力函数

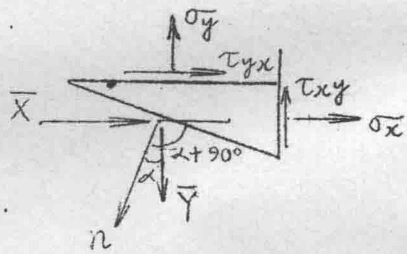
$$\varphi=a_3x^3+b_3x^2y+c_3xy^2+d_3y^3,$$

$\because$ 体积力 $X=0,\quad Y=p$

$$\therefore\ \sigma_x=\frac{\partial^2\varphi}{\partial y^2}-Xx=2c_3x+6d_3y,$$

$$\sigma_y=\frac{\partial^2\varphi}{\partial x^2}-Yy=6a_3x+2b_3y-py$$

$$\tau_{xy}=\tau_{yx}=-\frac{\partial^2\varphi}{\partial x\partial y}=-2b_3x-2c_3y$$



由边界条件确定积分常数 a_3、b_3、c_3、d_3：

当 $y=0,\quad \tau_{yx}=0,\quad \therefore -2b_3x-2c_3(0)=0,\quad \therefore b_3=0$

$y=0,\quad \sigma_y=0,\quad \therefore 6a_3x+2b_3(0)-p(0)=0,\quad \therefore a_3=0$

将 $a_3=0,\quad b_3=0$ 代入应力分另表达式中，得

$$\sigma_x=2c_3x+6d_3y,$$

$$\sigma_y=-py$$

$$\tau_{xy}=\tau_{yx}=-2c_3y$$

由图可知，在下边界 $x=y\operatorname{ctg}\alpha$，在边界条件方程

$$l\sigma_x+m\tau_{yx}=\overline{X},\qquad m\sigma_y+l\cdot\tau_{xy}=\overline{Y}$$

中，因面力 $\overline{X}=0,\ \overline{Y}=0$ 而成为

$$l\sigma_x+m\tau_{yx}=0,\qquad m\sigma_y+l\tau_{xy}=0$$

又 $\quad l=\cos(N,x)=\cos(\alpha+90^\circ)=-\sin\alpha$

$\quad m=\cos(N,x)=\cos\alpha$

故 $-\sin\alpha(2c_3x+6d_3y)-\cos\alpha(2c_3y)=0$ (1)

$-py\cos\alpha+2c_3y\sin\alpha=0$ (2)

将 $x=y\,ctg\,\alpha$ 及由式(2)所得的 $c_3=\frac{1}{2}p\,ctg\,\alpha$ 代入式(1)，得

$$-\sin\alpha\cdot2\cdot\frac{1}{2}p\,ctg\,\alpha\cdot y\,ctg\,\alpha-\sin\alpha\cdot6d_3y-\cos\alpha\cdot2\cdot\frac{1}{2}p\,ctg\,\alpha\,y=0,$$

或 $-py\,ctg^2\alpha\sin\alpha-py\,ctg^2\alpha\sin\alpha=6d_3y\sin\alpha$

$$\therefore d_3=-\frac{1}{3}p\,ctg^2\alpha$$

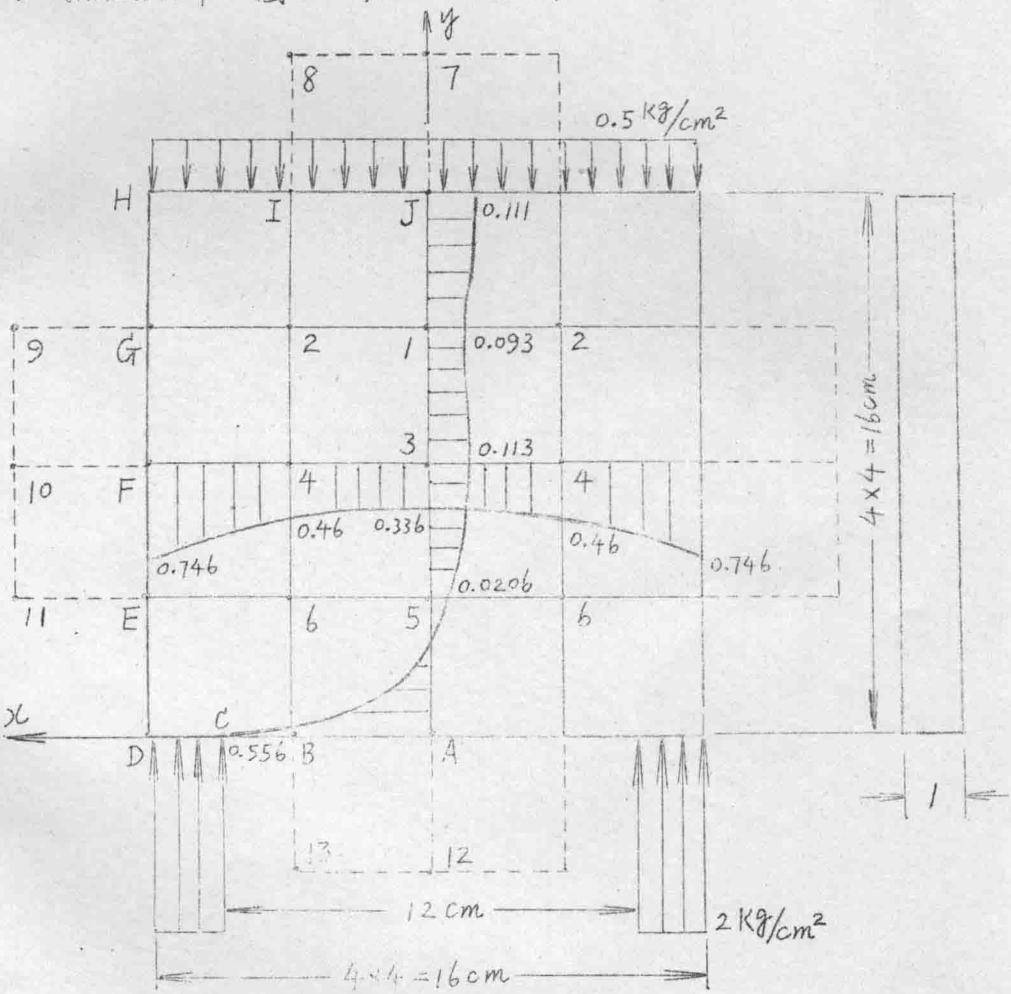
再将 $c_3=\frac{1}{2}p\,ctg\,\alpha$，$d_3=-\frac{1}{3}p\,ctg^2\alpha$ 代入应力分量表达式中，得最后结果

$$\sigma_x=px\,ctg\,\alpha-2py\,ctg^2\alpha$$

$$\sigma_y=-py$$

$$\tau_{xy}=\tau_{yx}=-py\,ctg\,\alpha$$

〔3-5〕试用差分法计算图示矩形梁内的应力。取 $h=4\,cm$.



〔解〕求σ_x（跨度中央截面）：

①计算边界上节点的$\frac{\partial\varphi}{\partial y}$、$\frac{\partial\varphi}{\partial x}$、$\varphi$值。

由于对称，只须计算左半梁（x，y都是正值）。

设 $\left(\frac{\partial\varphi}{\partial y}\right)_A=\left(\frac{\partial\varphi}{\partial x}\right)_A=\varphi_A=0$

由于从A到C无任何外力，故$\frac{\partial\varphi}{\partial y}$、$\frac{\partial\varphi}{\partial x}$、$\varphi$都等于零。

D点：$\left(\frac{\partial\varphi}{\partial x}\right)_D=-\int_A^D\bar{Y}\cdot ds$，式中右边积分表示AD间负$y$方向面力的总和，故

$$\left(\frac{\partial\varphi}{\partial x}\right)_D=-2\cdot2=-4$$

$\varphi_D=\int_A^D(y_D-y)\bar{X}\cdot ds-\int_A^D(X-X_D)\bar{Y}ds$，式中右边积分表示AD间面力对D点的力矩总和，以↷为正，故

$$\phi_D=-2\cdot2\cdot1=-4$$

E、F、G、H点：DH间无任何外力，故$\frac{\partial\varphi}{\partial x}$、$\frac{\partial\varphi}{\partial y}$、$\varphi$值均无变化。

I点：$\phi_I=2\cdot2\cdot3-0.5\cdot4\cdot2=12-4=8$

J点：$\phi_J=2\cdot2\cdot7-0.5\cdot8\cdot4=28-16=12$

节点	A	B	C	D、E、F、G、H	I	J
$\frac{\partial\varphi}{\partial y}$	0	0	0		0	0
$\frac{\partial\varphi}{\partial x}$	0			-4		
φ	0	0	0	-4	8	12

②用边界内的φ值表示边界外各点的φ值。

上、下两边 $\bar{X}=0$，$\frac{\partial\varphi}{\partial y}=0$

$\left(\frac{\partial\varphi}{\partial y}\right)_J=\frac{\varphi_7-\varphi_1}{2h}=0$，　　$\therefore\varphi_7=\varphi_1$；

同样　$\varphi_{12}=\varphi_5$，　$\varphi_{13}=\varphi_6$，　$\varphi_8=\varphi_2$

左边，$\left(\frac{\partial\varphi}{\partial x}\right)_G=\frac{\varphi_3-\varphi_2}{2h}=-4$，　$\therefore\varphi_9=-32+\varphi_2$

同样可得　$\varphi_{10}=\varphi_4-32$，　$\varphi_{11}=\varphi_6-32$

③边界内各点的差分方程。

节点1：

$$20\varphi_1-8(2\varphi_2+\varphi_3+\varphi_J)+2(2\varphi_4+2\varphi_I)+(2\varphi_G+\varphi_5+\varphi_7)=0,$$

即 $$21\varphi_1-16\varphi_2-8\varphi_3+4\varphi_4+\varphi_5=72 \qquad (a)$$

节点2:

$$20\varphi_2-8(\varphi_1+\varphi_4+\varphi_G+\varphi_I)+2(\varphi_3+\varphi_F+\varphi_H+\varphi_J)+(\varphi_8+\varphi_2+\varphi_6+\varphi_9)=0,$$

即 $$-8\varphi_1+23\varphi_2+2\varphi_3-8\varphi_4+\varphi_6=56 \qquad (b)$$

节点3:

$$20\varphi_3-8(2\varphi_4+\varphi_1+\varphi_5)+2(2\varphi_2+2\varphi_6)+(\varphi_J+2\varphi_F+\varphi_A)=0,$$

即 $$-8\varphi_1+4\varphi_2+20\varphi_3-16\varphi_4-8\varphi_5+4\varphi_6+8=0$$

或 $$-2\varphi_1+\varphi_2+5\varphi_3-4\varphi_4-2\varphi_5+\varphi_6=-2 \qquad (c)$$

节点4:

$$20\varphi_4-8(\varphi_2+\varphi_3+\varphi_6+\varphi_F)+2(\varphi_1+\varphi_5+\varphi_E+\varphi_G)+(\varphi_I+\varphi_4+\varphi_B+\varphi_{10})=0$$

即 $$2\varphi_1-8\varphi_2-8\varphi_3+22\varphi_4+2\varphi_5-8\varphi_6-8=0$$

或 $$\varphi_1-4\varphi_2-4\varphi_3+11\varphi_4+\varphi_5-4\varphi_6=4 \qquad (d)$$

节点5:

$$20\varphi_5-8(\varphi_3+2\varphi_6+\varphi_A)+2(2\varphi_4+2\varphi_B)+(\varphi_1+2\varphi_E+\varphi_{12})=0$$

即 $$\varphi_1-8\varphi_3+4\varphi_4+21\varphi_5-16\varphi_6=8 \qquad (e)$$

节点6:

$$20\varphi_6-8(\varphi_4+\varphi_5+\varphi_B+\varphi_E)+2(\varphi_3+\varphi_A+\varphi_D+\varphi_F)+(\varphi_2+\varphi_6+\varphi_{13}+\varphi_{11})=0$$

即 $$\varphi_2+2\varphi_3-8\varphi_4-8\varphi_5+23\varphi_6=16 \qquad (f)$$

联立解(a)、(b)、(c)、(d)、(e)、(f)六式,可得

$$\varphi_1=11.11 \qquad \varphi_2=7.50 \qquad \varphi_3=8.73$$
$$\varphi_4=6.04 \qquad \varphi_5=4.53 \qquad \varphi_6=3.29$$

[校核] 式(a)左 $=21\times11.11-16\times7.5-8\times8.73+4\times6.04+4.53$
$=233-120-69.7+24.16+4.53=72=$右

式(b)左 $=-88.9+23\times7.5+2\times8.73-8\times6.04+3.29$
$=-88.9+172.5+17.5-48.3+3.29=56=$右

式(c)左 $=-22.2+7.5+5\times8.73-4\times6.02-2\times4.53+3.3$
$=54.5-55.4=-1=$右

式(d)左 $=11.11-30-34.9+66.4+4.53-13.2=82-78.1\doteq4=$右

式(e)左 $=11.11-8\times8.73+4\times6.04+21\times4.53-16\times3.3$
$=11.11-69.6+24.16+95.1-52.8=130.3-122.4=8=$右

式(f)左 $=7.5+2\times8.73-48.3-36.24+74=100.9-84.5=16=$右