GEOMETRY OF CURVES

JOHN W. RUTTER

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Geometry of Curves

John W. Rutter

Integrating the three main areas of curve geometry—parametric, algebraic, and projective curves—*Geometry of Curves* offers a unique approach that provides a mathematical structure for solving problems—not just a catalog of theorems. The author begins with the basics, then takes readers on a fascinating journey from conics, higher algebraic curves, and transcendental curves, through the properties of parametric curves, the classification of limaçons, and envelopes of curve families, to projective curves, their relationship to algebraic curves, and their application to asymptotes and boundedness.

The uniqueness of this volume lies in its integration of the different types of curves, its use of analytic methods, and its generous number of examples, exercises, and illustrations. The result is a practical work—almost entirely self-contained—that not only imparts a deeper understanding of the theory, but a heightened appreciation of geometry and interest in more advanced studies.

J.W. Rutter is an honorary fellow at the University of Liverpool. He received his Ph.D. from Oxford University and has taught at Oxford, Stanford, and California universities, and has held visiting positions at Cambridge University; Institut Hautes Études Scientifiqes, Paris; and Centre de Recerca Matemàtica, Barcelona. He is the author of more than 30 research papers and monographs.

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To my mother

and father

Preface

This book covers the material for a first full university course in geometry; specifically it is an introduction to the geometry of plane curves, given as parametric curves, algebraic curves, or projective curves. It is intended for students who have previously studied elementary calculus including partial differentiation, the elementary theory of complex numbers, and elementary coordinate geometry. It is developed from a one-semester geometry course I have given over a number of years to undergraduates specialising in mathematics, statistics or computing, or following degree courses involving a substantial study of mathematics.

My aim in writing the book is to make the material covered readily available in one book and in a form suitable for a modern first university course in geometry. Previously in order to cover the material in the book, it was necessary to read isolated sections of a number of other texts of varying levels of sophistication. Topics covered here have been integrated and presented in a manner suitable for a first course, and new elementary proofs have been developed where possible. Most of the material included is what I believe would be termed elementary in modern university terms, and as far as possible the proofs I have given use only elementary ideas. I have starred a small number of conceptually or technically more difficult sections and proofs; these may be left for a second reading. I have also starred a small number of sections and results which can be omitted depending on the time available. A large number of exercises of varying difficulty are included as are many worked examples. I believe that in geometry, as in most areas of mathematics, doing exercises helps the student more quickly to understand and to appreciate the subject. Solutions to exercises are included roughly on an alternate basis. One of my main aims has been to lay out a mathematical structure, understandable to modern students, which can be used for solving problems, rather than to provide a catalogue of theorems; this, I believe, is also a main aim of many modern first courses in calculus or algebra. I have included numerous figures to illustrate the ideas, proofs, and solutions, and to illustrate specific classical curves.

Many students have obtained degrees in mathematics having studied little or no geometry at university or at school, a situation which is, I believe, regrettable. The publication of this book coincides, I believe, with the rising interest in the return to the study of geometry at university level. The book will be suitable for use in many mathematics departments, including those where a complete geometry course is not currently taught and that wish to introduce one, since it provides an elementary introduction to a number of important areas. Students who successfully completed the course on which the book is based developed a heightened appreciation of geometry and many of them went on to study more advanced courses in differential geometry and/or algebraic and projective geometry.

An introductory chapter contains, for clarification and reference, basic material which will already be known by many readers. In the first chapter the basic equations of lines, circles, and conics are given; the relationship between parametric, algebraic, and polar equations is considered. The techniques for classification of conics in general position are given in the second chapter. The third chapter presents examples of some higher algebraic and transcendental curves having features such as cusps, nodes, or isolated points, which do not occur in the case of conics. In the fourth to ninth chapters, the standard properties of parametric curves are obtained, including tangents and normals, inflexions, undulations, cusps, and curvature; some of these properties are applied to give properties of algebraic curves such as tangents, normals, and curvature. In the tenth chapter, features such as cusps, inflexions, and curvature are used to classify limaçons into five classes. In the eleventh, twelfth, and thirteenth chapters, the evolute, parallel, involute, and roulette of parametric curves are considered. The fourteenth chapter gives an account of envelopes of families of parametric, algebraic, and other curves. In the fifteenth chapter tangents and branches of algebraic curves at singular points are investigated. The sixteenth chapter studies projective curves and their relationship with algebraic curves, including applications to asymptotes and boundedness. Throughout the book many classical curves are considered as examples and some are studied in more detail. I have included sections on the history and applications of several classes of curves such as conics, spirals, cubics, trochoids, and Watt's curves.

As well as giving the classification of conics in Chapter 2, a classification of cubic algebraic curves is given in Chapter 15 using the results on singular points.

I have followed the analytic method almost exclusively in the sections on algebraic and projective geometry, and have often used the calculus in proofs. Early in the twentieth century, certain purists would have objected to these methods, but with changing fashions, needs, and current school syllabuses, there are, I believe, few now who would. A university course in synthetic geometry would in any case have objectives quite different to the ones of this book.

I give the analytic description of projective space and base, for example, the proofs and techniques for asymptotes and boundedness on that description. Additionally, I have, in Chapter 16, indicated how the projective plane can be obtained by identifying opposite pairs of points on the sphere; this is perhaps the most sophisticated concept in the book, but I believe that understanding this geometrical construction will lead the reader to a fuller appreciation of projective space, the projective method and projective curves. However this geometrical construction could be omitted until a second reading. I have also included a number of ways in which projective curves can be drawn or pictured, since I believe that such representations will aid the reader to achieve a fuller understanding of these projective ideas.

The book is essentially self-contained. I have included in Chapter 2 results on and methods of orthogonal diagonalisation of quadratic forms in two variables. These are used in the classification of conics, in moving a conic to canonical position. I have also included at the end of Chapter 4 results in calculus and analysis which are used in the book.

The lecture course given was supplemented by practical classes. In the practical classes the students, collaborating in small groups, draw curves by hand using a variety of techniques, including rectangular and polar plotting, enveloping, and the methods of conchoids, cissoids, and strophoids. Although not essential to the course, practical classes are particularly popular among students and I recommend their adoption. Completion of the practical work helps and motivates students to understand the theory. The drawing of curves is one of the visual-art forms of mathematics and gives students the opportunity to achieve satisfaction in a non-theoretical part of the subject. I have included in Chapter 17 a list of practical projects suitable for students to share in groups of six, with each student in the group generally drawing a different curve. This can be modified as required by the lecturer. As an alternative to their use in practical classes, a selection of these projects could be used for take-home assignments. Plotting curves using computer packages is also popular, and a number of packages are available including Maple, MATLAB, and Mathematica. The drawing of curves by hand could be partially or wholly replaced by computer drawing in the practical classes and curve-drawing exercises. Some programs for drawing sized curves in MATLAB are given in Chapter 18.

In the practical work and some of the exercises involving curve-drawing, some standard ready-drawn curves are needed. There are many packages which can be used for drawing curves. In Chapter 18 programs are given for use with MATLAB for drawing these standard curves. A program for drawing polar graph paper is included for localities where such paper is not available.

A list of books for further reading is given in Chapter 19.

As well as being suitable for students aiming for degrees having a high content of mathematics, the book is also appropriate for students of mathematically based subjects such as engineering, who also may be required to study plane curves at some depth.

The book could also be used as a supplementary text for courses in calculus, vector calculus, linear algebra, differential geometry, singularity theory, algebraic geometry, and computer graphics.

My thanks are due to a number of people including Ian Porteous for his support in the course, Victor Flynn and several reviewers for reading some of the chapters and suggesting improvements, Peter Giblin for advice on computer graphics, Rachid Chalabi and Steve Downing for advice on the use of LaTeX and for its smooth running, and Dave Alliot of Chapman and Hall/CRC production for his detailed reading and advice. I am also grateful to students of the University of Liverpool who tried out drafts of the manuscript in class; the high satisfaction rating they expressed in student surveys and individually was an incentive.

Relevant documents and developments subsequent to publication may be available on the following linked websites.

www.crcpress.com

www.liv.ac.uk/~jwrutter/curves

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